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WEAK CONVERGENCE THEOREMS FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE NON-SELF MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper, we establish some weak convergence results of modified S-iteration process to converge to common fixed points for two total asymptotically nonexpansive non-self mappings in the framework of uniformly convex Banach spaces under the following conditions (i) the Banach space E satisfying Opial condition and (ii) the dual E^* of E has the Kadec-Klee property. Our results extend and improve the previous works from the current existing literature.

1. Introduction

Let C be a nonempty subset of a Banach space E and $T\colon C \to C$ a nonlinear mapping. We denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $\mathcal{F} = F(S) \cap F(T)$.

A mapping $T\colon C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x,\, y \in C$.

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An important generalization of the class of nonexpansive mappings and the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [8]. They proved that every asymptotically nonexpansive self mappings of a nonempty closed convex subset of a real uniformly convex Banach space has a fixed point.

T is called asymptotically nonexpansive if there exists a positive sequence $k_n \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n \, \|x - y\|, \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$.

A mapping $T\colon C \to C$ is called asymptotically noneexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x - y\|\big) \leq 0. \tag{1.3}$$

Observe that if we define

$$a_n = \limsup_{n\to\infty} \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x - y\|\big) \quad \text{and} \quad \mu_n = \max\{0, a_n\},$$

then $\mu_n \to 0$ as $n \to \infty$. It follows that (1.3) is reduced to

$$\|T^n x - T^n y\| \quad \leq \quad \|x - y\| + \mu_n, \tag{1.4}$$

for all $x, y \in C$ and $n \geq 1$.

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck, Kuczumow and Reich [4]. It is known [14] that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense mapping, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate contains properly the class of asymptotically nonexpansive mappings.

Albert *et al.* [2] introduced the concept of total asymptotically nonexpansive mappings in 2006. Recall that T is said to be total asymptotically nonexpansive if

$$\|T^n x - T^n y\| \quad \leq \quad \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n, \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$, where $\{\mu_n\}$ and $\{\nu_n\}$ are nonnegative real sequences such that $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$. From the definition, we see that the class of total asymptotically nonexpansive mappings include the class of asymptotically nonexpansive mappings as a special case; see also [6] for more details.

Remark 1.1. From the above definition, it is clear that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\nu_n = 0$, $\mu_n = k_n - 1$ for all $n \geq 1$, $\psi(t) = t$, $t \geq 0$.

A subset C of E is called a retract of E if there exists a continuous map $P\colon E \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P\colon E \to E$ is said to be a retraction if $P^2 = P$. It follows that if P is a retraction then $Py = y$ for all y in the range of P.

In 2003, Chidume *et al.* [5] defined non-self asymptotically nonexpansive mappings as follows:

Let $P\colon E \to C$ be a nonexpansive retraction of E into C. A non-self mapping $T\colon C \to E$ is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \quad \leq \quad k_n\|x - y\|, \tag{1.6}$$

for all x, $y \in C$ and $n \in \mathbb{N}$.

Also T is called uniformly L-Lipschitzian if for some $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \quad \leq \quad L\|x - y\|,$$

for all x, $y \in C$ and $n \in \mathbb{N}$.

Recently, Yolacan and Kiziltune [28] (J. Nonlinear Sci. Appl. 5(2012), 389-402) defined the following:

Let C be a nonempty closed and convex subset of a Banach space E. Let $P\colon E \to C$ be the nonexpansive retraction of E onto C. A non-self map $T\colon C \to E$ is said to be total asymptotically nonexpansive if there exist sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $[0, \infty)$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \quad \leq \quad \|x - y\| + \mu_n\psi(\|x - y\|) + \nu_n, \tag{1.7}$$

for all x, $y \in C$ and $n \geq 1$.

(i) **Modified S-iteration for self mapping (**[1]**).**

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

(ii) **Modified Mann iteration for non-self mapping ([5]).**

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P(\alpha_n T(PT)^{n-1}x_n + (1-\alpha_n)x_n), \quad n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Recently, Khan [12] introduced and studied the following iteration scheme for non-self mappings.

(iii) **Modified S-iteration for non-self mapping ([12]).**

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P((1-\alpha_n)T(PT)^{n-1}x_n + \alpha_n T(PT)^{n-1}y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \quad n \geq 1, \end{cases} \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Motivated by the above works we introduce and study the following iteration scheme for two total asymptotically nonexpansive non-self mappings $S, T: C \to E$ defined as follows.

(iv) **Modified S-iteration for two non-self mappings.**

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P((1-\alpha_n)T(PT)^{n-1}x_n + \alpha_n S(PS)^{n-1}y_n), \\ y_n = P((1-\beta_n)S(PS)^{n-1}x_n + \beta_n T(PT)^{n-1}x_n), \quad n \geq 1, \end{cases} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $(0,1)$.

Remark 1.2. If we take $S = I$, where I is the identity mapping and $\beta_n = 0$ for all $n \geq 1$, then (1.11) reduces to the modified Mann iteration process for non-self mapping $T: C \to E$.

The asymptotic fixed point theory has a fundamental role in nonlinear functional analysis (see, [3]). A branch of this theory related to asymptotically nonexpansive self and non-self mappings have been developed by many authors (see, e.g., [4], [5], [8]-[11], [13], [15]-[16], [18]-[25], [27]) in Banach spaces with suitable geometrical structure.

The purpose of this paper is to prove some weak convergence theorems of iteration scheme (1.11) for two total asymptotically nonexpansive non-self mappings in the framework of uniformly convex Banach spaces.

2. Preliminaries

For the sake of convenience, we restate the following concepts and results.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

We recall the following:

Let $\mathcal{S} = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E.

The space E has Opial condition [17] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p (1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial condition.

A mapping $T\colon C \to C$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition $\{x_n\}$ converges weakly to $x \in C$ and $\{Tx_n\}$ converges strongly to 0 imply $Tx = 0$.

A Banach space E has the Kadec-Klee property [24] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Proposition 2.1. *Let C be a nonempty subset of a Banach space E which is also a nonexpansive retract of E and $S, T\colon C \to E$ be two total asymptotically nonexpansive non-self mappings. Then there exist nonnegative real sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $[0,\infty)$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that*

$$\|S(PS)^{n-1}x - S(PS)^{n-1}y\| \leq \|x-y\| + \mu_n\psi(\|x-y\|) + \nu_n, \quad (2.1)$$

and

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq \|x-y\| + \mu_n\psi(\|x-y\|) + \nu_n, \quad (2.2)$$

for all $x, y \in C$ and $n \geq 1$.

Proof. Since $S, T\colon C \to E$ are two total asymptotically nonexpansive non-self mappings, there exist nonnegative real sequences $\{\mu'_n\}$, $\{\mu''_n\}$, $\{\nu'_n\}$ and $\{\nu''_n\}$ in $[0,\infty)$ with $\mu'_n, \mu''_n \to 0$ and $\nu'_n, \nu''_n \to 0$ as $n \to \infty$ and strictly increasing continuous functions $\psi_1, \psi_2\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi_1(0) = 0$ and $\psi_2(0) = 0$ such that

$$\|S(PS)^{n-1}x - S(PS)^{n-1}y\| \leq \|x-y\| + \mu'_n\psi_1(\|x-y\|) + \nu'_n, \quad (2.3)$$

and

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq \|x-y\| + \mu_n''\psi_2(\|x-y\|) + \nu_n'', \quad (2.4)$$

for all $x, y \in C$ and $n \geq 1$.

Setting

$$\mu_n = \max\{\mu_n', \mu_n''\}, \quad \nu_n = \max\{\nu_n', \nu_n''\}$$

and

$$\psi(a) = \max\{\psi_1(a), \psi_2(a), \text{ for } a \geq 0\},$$

then we get that there exist nonnegative real sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$\begin{aligned}\|S(PS)^{n-1}x - S(PS)^{n-1}y\| &\leq \|x-y\| + \mu_n'\psi_1(\|x-y\|) + \nu_n' \\ &\leq \|x-y\| + \mu_n\psi(\|x-y\|) + \nu_n,\end{aligned}$$

and

$$\begin{aligned}\|T(PT)^{n-1}x - T(PT)^{n-1}y\| &\leq \|x-y\| + \mu_n''\psi_2(\|x-y\|) + \nu_n'' \\ &\leq \|x-y\| + \mu_n\psi(\|x-y\|) + \nu_n,\end{aligned}$$

for all $x, y \in C$ and $n \geq 1$. This completes the proof. □

Next we state the following useful lemmas to prove our main results.

Lemma 2.2. ([26]) *Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \quad \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 2.3. ([22]) *Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.4. ([24]) *Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $p, q \in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. Then $p = q$.*

Lemma 2.5. ([24]) *Let K be a nonempty convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing continuous convex* *function* $\phi\colon [0,\infty) \to [0,\infty)$ *with* $\phi(0) = 0$ *such that for each Lipschitzian mapping* $T\colon C \to C$ *with the Lipschitz constant* L,

$$\|tTx + (1-t)Ty - T(tx + (1-t)y)\| \le L\phi^{-1}\Big(\|x-y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all $x, y \in K$ *and all* $t \in [0,1]$.

3. Main results

In this section, we prove some weak convergence theorems for two total asymptotically nonexpansive non-self mappings in the framework of uniformly convex Banach spaces. First, we shall need the following lemmas.

Lemma 3.1. *Let* E *be a real Banach space and* C *be a nonempty closed convex subset of* E. *Let* $P\colon E \to C$ *be a nonexpansive retraction of* E *into* C *and* $S, T\colon C \to E$ *be two total asymptotically nonexpansive non-self mappings with sequences* $\{\mu_n\}$ *and* $\{\nu_n\}$ *as defined in Proposition* 2.1 *and* $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be the iteration scheme defined by* (1.11) *and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) *there exists a constant* $K > 0$ *such that* $\psi(t) \le K\,t$, $t \ge 0$.

Then $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in \mathcal{F}$.

Proof. Let $p \in \mathcal{F}$. For the sake of simplicity, set

$$A_n x = P((1-\beta_n)S(PS)^{n-1}x + \beta_n T(PT)^{n-1}x)$$

and

$$B_n x = P((1-\alpha_n)T(PT)^{n-1}x + \alpha_n S(PS)^{n-1}A_n x).$$

Then $y_n = A_n x_n$ and $x_{n+1} = B_n x_n$. Moreover, it is clear that p is a fixed point of B_n for all n.

Consider

$$\begin{aligned}\|A_n x - A_n y\| &= \|P((1-\beta_n)S(PS)^{n-1}x + \beta_n T(PT)^{n-1}x) \\ &\quad - P((1-\beta_n)S(PS)^{n-1}y + \beta_n T(PT)^{n-1}y)\| \\ &\le \|(1-\beta_n)S(PS)^{n-1}x + \beta_n T(PT)^{n-1}x \\ &\quad - (1-\beta_n)S(PS)^{n-1}y + \beta_n T(PT)^{n-1}y\| \\ &= \|(1-\beta_n)(S(PS)^{n-1}x - S(PS)^{n-1}y) \\ &\quad + \beta_n(T(PT)^{n-1}x - T(PT)^{n-1}y)\| \\ &\le (1-\beta_n)[\|x-y\| + \mu_n\psi(\|x-y\|) + \nu_n] \\ &\quad + \beta_n[\|x-y\| + \mu_n\psi(\|x-y\|) + \nu_n]\end{aligned}$$

$$\begin{aligned}
&\le (1-\beta_n)[\|x-y\| + \mu_n K\|x-y\| + \nu_n] \\
&\quad + \beta_n[\|x-y\| + \mu_n K\|x-y\| + \nu_n] \\
&= (1-\beta_n)[(1+\mu_n K)\|x-y\| + \nu_n] \\
&\quad + \beta_n[(1+\mu_n K)\|x-y\| + \nu_n] \\
&\le (1+\mu_n K)\|x-y\| + \nu_n. \qquad (3.1)
\end{aligned}$$

Choosing $x = x_n$ and $y = p$, we get

$$\|y_n - p\| \le (1+\mu_n K)\|x_n - p\| + \nu_n. \qquad (3.2)$$

Now, we consider

$$\begin{aligned}
\|B_n x - B_n y\| &= \|P((1-\alpha_n)T(PT)^{n-1}x + \alpha_n S(PS)^{n-1}A_n x) \\
&\qquad - P((1-\alpha_n)T(PT)^{n-1}y + \alpha_n S(PS)^{n-1}A_n y)\| \\
&\le \|(1-\alpha_n)T(PT)^{n-1}x + \alpha_n S(PS)^{n-1}A_n x \\
&\qquad - (1-\alpha_n)T(PT)^{n-1}y + \alpha_n S(PS)^{n-1}A_n y\| \\
&= \|(1-\alpha_n)(T(PT)^{n-1}x - T(PT)^{n-1}y) \\
&\qquad + \alpha_n(S(PS)^{n-1}A_n x - S(PS)^{n-1}A_n y)\| \\
&\le (1-\alpha_n)[\|x-y\| + \mu_n\psi(\|x-y\|) + \nu_n] \\
&\qquad + \alpha_n[\|A_n x - A_n y\| + \mu_n\psi(\|A_n x - A_n y\|) + \nu_n] \\
&\le (1-\alpha_n)[\|x-y\| + \mu_n K\|x-y\| + \nu_n] \\
&\qquad + \alpha_n[\|A_n x - A_n y\| + \mu_n K\|A_n x - A_n y\| + \nu_n] \\
&= (1-\alpha_n)[(1+\mu_n K)\|x-y\| + \nu_n] \\
&\qquad + \alpha_n[(1+\mu_n K)\|A_n x - A_n y\| + \nu_n] \\
&= (1-\alpha_n)(1+\mu_n K)\|x-y\| \\
&\qquad + \alpha_n(1+\mu_n K)\|A_n x - A_n y\| + \nu_n. \qquad (3.3)
\end{aligned}$$

Now using (3.1) in (3.3), we get

$$\begin{aligned}
\|B_n x - B_n y\| &\le (1-\alpha_n)(1+\mu_n K)\|x-y\| \\
&\qquad + \alpha_n(1+\mu_n K)[(1+\mu_n K)\|x-y\| + \nu_n] + \nu_n \\
&\le (1-\alpha_n)(1+\mu_n K)^2\|x-y\| \\
&\qquad + \alpha_n(1+\mu_n K)^2\|x-y\| + \alpha_n(1+\mu_n K)\nu_n + \nu_n \\
&\le (1+\mu_n K)^2\|x-y\| + (2+\mu_n K)\nu_n \\
&= (1+M_1\mu_n)\|x-y\| + M_2\nu_n, \qquad (3.4)
\end{aligned}$$

for some $M_1, M_2 > 0$. Choosing $x = x_n$ and $y = p$ in (3.4), we get

$$\|x_{n+1} - p\| = \|B_n x_n - p\| \le (1+M_1\mu_n)\|x_n - p\| + M_2\nu_n. \qquad (3.5)$$

Since by hypothesis $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$, so by Lemma 2.2, we have $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

Lemma 3.2. *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $P\colon E \to C$ be a nonexpansive retraction of E into C and $S, T\colon C \to E$ be two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in Proposition 2.1 and $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (1.11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \nu_n < \infty$*;*
(ii) *there exists a constant $K > 0$ such that $\psi(t) \le K\,t$, $t \ge 0$.*

If $\|x - S(PS)^{n-1}x\| \le \|T(PT)^{n-1}x - S(PS)^{n-1}x\|$ for all $x \in C$, then $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$ and therefore $\{x_n\}$ is bounded. Thus there exists a real number $r > 0$ such that $\{x_n\} \subseteq K' = \overline{B_r(0)} \cap C$, so that K' is a closed convex subset of C. Let $\lim_{n\to\infty} \|x_n - p\| = c$. Then $c > 0$ otherwise there is nothing to prove.Now (3.2) implies that

$$\limsup_{n\to\infty} \|y_n - p\| \le c. \tag{3.6}$$

Also

$$\begin{aligned} \|T(PT)^{n-1}x_n - p\| &\le \|x_n - p\| + \mu_n \psi(\|x_n - p\|) + \nu_n \\ &\le \|x_n - p\| + \mu_n K \|x_n - p\| + \nu_n \\ &= (1 + \mu_n K)\|x_n - p\| + \nu_n \end{aligned}$$

for all $n = 1, 2, \ldots$, and

$$\begin{aligned} \|S(PS)^{n-1}x_n - p\| &\le \|x_n - p\| + \mu_n \psi(\|x_n - p\|) + \nu_n \\ &\le \|x_n - p\| + \mu_n M \|x_n - p\| + \nu_n \\ &= (1 + \mu_n M)\|x_n - p\| + \nu_n \end{aligned}$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty} \|T(PT)^{n-1}x_n - p\| \le c. \tag{3.7}$$

and

$$\limsup_{n\to\infty} \|S(PS)^{n-1}x_n - p\| \le c. \tag{3.8}$$

Next,

$$\begin{aligned}\|S(PS)^{n-1}y_n - p\| &\le \|y_n - p\| + \mu_n\psi(\|y_n - p\|) + \nu_n \\ &\le \|y_n - p\| + \mu_n K\|y_n - p\| + \nu_n \\ &= (1+\mu_n K)\|y_n - p\| + \nu_n\end{aligned}$$

gives by virtue of (3.6) that

$$\limsup_{n\to\infty} \|S(PS)^{n-1}y_n - p\| \le c. \tag{3.9}$$

Since

$$c = \|x_{n+1} - p\| = \|(1-\alpha_n)(T(PT)^{n-1}x_n - p) + \alpha_n(S(PS)^{n-1}y_n - p)\|.$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|T(PT)^{n-1}x_n - S(PS)^{n-1}y_n\| = 0. \tag{3.10}$$

From (1.11) and (3.8), we have

$$\begin{aligned}\|x_{n+1} - T(PT)^{n-1}x_n\| &= \alpha_n\|S(PS)^{n-1}y_n - T(PT)^{n-1}x_n\| \\ &\le \|S(PS)^{n-1}y_n - T(PT)^{n-1}x_n\| \\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.11}$$

Hence

$$\begin{aligned}\|x_{n+1} - S(PS)^{n-1}y_n\| &\le \|x_{n+1} - T(PT)^{n-1}x_n\| \\ &\quad + \|T(PT)^{n-1}x_n - S(PS)^{n-1}y_n\| \\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.12}$$

Now

$$\begin{aligned}\|x_{n+1} - p\| &\le \|x_{n+1} - S(PS)^{n-1}y_n\| + \|S(PS)^{n-1}y_n - p\| \\ &\le \|x_{n+1} - S(PS)^{n-1}y_n\| + |y_n - p\| + \mu_n\psi(\|y_n - p\|) + \nu_n \\ &\le \|x_{n+1} - S(PS)^{n-1}y_n\| + |y_n - p\| + \mu_n K\|y_n - p\| + \nu_n \\ &= \|x_{n+1} - S(PS)^{n-1}y_n\| + (1+\mu_n K)\|y_n - p\| + \nu_n, \end{aligned} \tag{3.13}$$

which gives from (3.12) that

$$c \le \liminf_{n\to\infty} \|y_n - p\|. \tag{3.14}$$

From (3.6) and (3.14), we obtain

$$c = \|y_n - p\| = \|(1-\beta_n)(T(PT)^{n-1}x_n - p) + \beta_n(S(PS)^{n-1}x_n - p)\|.$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|T(PT)^{n-1}x_n - S(PS)^{n-1}x_n\| = 0. \tag{3.15}$$

Now

$$\begin{aligned}
&\|T(PT)^{n-1}x_n - x_n\| \\
&\leq \|T(PT)^{n-1}x_n - S(PS)^{n-1}x_n\| + \|S(PS)^{n-1}x_n - x_n\| \\
&\leq \|T(PT)^{n-1}x_n - S(PS)^{n-1}x_n\| + \|S(PS)^{n-1}x_n - T(PT)^{n-1}x_n\| \\
&= 2\|T(PT)^{n-1}x_n - S(PS)^{n-1}x_n\| \\
&\to 0 \quad \text{as} \quad n \to \infty, \qquad (3.16)
\end{aligned}$$

Again note that

$$\begin{aligned}
\|y_n - x_n\| &= \|P((1-\beta_n)S(PS)^{n-1}x_n + \beta_n T(PT)^{n-1}x_n) - Px_n\| \\
&\leq \|(1-\beta_n)S(PS)^{n-1}x_n + \beta_n T(PT)^{n-1}x_n - x_n\| \\
&= \beta_n\|T(PT)^{n-1}x_n - S(PS)^{n-1}x_n\| \\
&\leq (1-\delta)\,\|T(PT)^{n-1}x_n - S(PS)^{n-1}x_n\|.
\end{aligned}$$

Hence by (3.15), we obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \qquad (3.17)$$

Also note that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|P((1-\alpha_n)T(PT)^{n-1}x_n + \alpha_n S(PS)^{n-1}y_n) - Px_n\| \\
&\leq \|(1-\alpha_n)T(PT)^{n-1}x_n + \alpha_n S(PS)^{n-1}y_n - x_n\| \\
&= \|(T(PT)^{n-1}x_n - x_n) + \alpha_n(T(PT)^{n-1}x_n - S(PS)^{n-1}y_n)\| \\
&\leq \|T(PT)^{n-1}x_n - x_n\| + \alpha_n\|T(PT)^{n-1}x_n - S(PS)^{n-1}y_n\| \\
&\to 0 \quad \text{as} \quad n \to \infty, \qquad (3.18)
\end{aligned}$$

so that

$$\begin{aligned}
\|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|x_n - y_n\| \\
&\to 0 \quad \text{as} \quad n \to \infty. \qquad (3.19)
\end{aligned}$$

Furthermore, from

$$\begin{aligned}
\|x_n - T(PT)^{n-2}y_{n-1}\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\| \\
&\quad + \|T(PT)^{n-2}x_{n-1} - T(PT)^{n-2}y_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\| \\
&\quad + \|x_{n-1} - y_{n-1}\| + \mu_{n-1}\psi(\|x_{n-1} - y_{n-1}\|) \\
&\quad + \nu_{n-1}
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\| \\
&\quad + \|x_{n-1} - y_{n-1}\| + \mu_{n-1}K\|x_{n-1} - y_{n-1}\| \\
&\quad + \nu_{n-1} \\
&= \|x_n - x_{n-1}\| + \|x_{n-1} - T(PT)^{n-2}x_{n-1}\| \\
&\quad + (1 + \mu_{n-1}K)\|x_{n-1} - y_{n-1}\| + \nu_{n-1}. \qquad (3.20)
\end{aligned}$$

Using (3.16)-(3.18) in (3.20), we obtain

$$\lim_{n\to\infty} \|x_n - T(PT)^{n-2}y_{n-1}\| = 0. \qquad (3.21)$$

Since T is continuous and P is nonexpansive retraction, it follows from (3.21) that

$$\begin{aligned}
\|T(PT)^{n-1}y_{n-1} - Tx_n\| &= \|TP(T(PT)^{n-2})y_{n-1} - TPx_n\| \\
&\to 0 \quad \text{as} \quad n \to \infty. \qquad (3.22)
\end{aligned}$$

Now, from

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - T(PT)^{n-1}x_n\| + \|T(PT)^{n-1}x_n - T(PT)^{n-1}y_{n-1}\| \\
&\quad + \|T(PT)^{n-1}y_{n-1} - Tx_n\| \\
&\leq \|x_n - T(PT)^{n-1}x_n\| + \|x_n - y_{n-1}\| + \mu_n\psi(\|x_n - y_{n-1}\|) \\
&\quad + \nu_n + \|T(PT)^{n-1}y_{n-1} - Tx_n\| \\
&\leq \|x_n - T(PT)^{n-1}x_n\| + \|x_n - y_{n-1}\| + \mu_nK\|x_n - y_{n-1}\| \\
&\quad + \nu_n + \|T(PT)^{n-1}y_{n-1} - Tx_n\| \\
&= \|x_n - T(PT)^{n-1}x_n\| + (1 + \mu_nK)\|x_n - y_{n-1}\| + \nu_n \\
&\quad + \|T(PT)^{n-1}y_{n-1} - Tx_n\|. \qquad (3.23)
\end{aligned}$$

Using (3.16), (3.19) and (3.22) in (3.23), we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \qquad (3.24)$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \qquad (3.25)$$

This completes the proof. □

Theorem 3.3. *Let E be a uniformly convex Banach space satisfying Opial's condition and C be a nonempty closed convex subset of E. Let $P\colon E \to C$ be a nonexpansive retraction of E into C and $S, T\colon C \to E$ be two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in Proposition* 2.1 *and $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by* (1.11)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in*

$[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:

(i) *$\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$;*

(ii) *there exists a constant $K > 0$ such that $\psi(t) \leq K\,t$, $t \geq 0$.*

If the mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.

Proof. Let $q \in \mathcal{F}$, from Lemma 3.1 the sequence $\{\|x_n - q\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q^* \in C$. From Lemma 3.2, we have

$$\lim_{k\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

Since the mappings $I-S$ and $I-T$ are demiclosed at zero, therefore $Sq^* = q^*$ and $Tq^* = q^*$, which means $q^* \in \mathcal{F}$. Finally, let us prove that $\{x_n\}$ converges weakly to q^*. Suppose on contrary that there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $p^* \in C$ and $q^* \neq p^*$. Then by the same method as given above, we can also prove that $p^* \in \mathcal{F}$. From Lemma 3.1 the limits $\lim_{n\to\infty} \|x_n - q^*\|$ and $\lim_{n\to\infty} \|x_n - p^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q^*\| &= \lim_{n_k\to\infty} \|x_{n_k} - q^*\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - p^*\| = \lim_{n\to\infty} \|x_n - p^*\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - p^*\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - q^*\| = \lim_{n\to\infty} \|x_n - q^*\|,
\end{aligned}$$

which is a contradiction so $q^* = p^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T. This completes the proof. □

It is well known that there exist classes of uniformly convex Banach spaces with out the Opial condition (e.g., L_p spaces, $p \neq 2$). Therefore, Theorem 3.3 is not true for such Banach spaces. We now show that Theorem 3.3 is valid if the assumption that E satisfies the Opial condition is replaced by the dual E^* of E has the Kadec-Klee property (KK-property).

Lemma 3.4. *Under the conditions of Lemma 3.2 and for any $p, q \in \mathcal{F}$, $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in \mathcal{F}$ and therefore $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|tx_n + (1-t)p - q\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p-q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Lemma 3.1. It, therefore, remains to prove the Lemma 3.4 for $t \in (0,1)$. For all $x \in C$, we define the mapping $B_n\colon C \to C$ by:

$$A_n x = P((1-\beta_n)S(PS)^{n-1}x + \beta_n T(PT)^{n-1}x)$$

and

$$B_n x = P((1-\alpha_n)T(PT)^{n-1}x + \alpha_n S(PS)^{n-1}A_n x).$$

Then it follows that $x_{n+1} = B_n x_n$, $B_n(p) = p$ for all $p \in \mathcal{F}$ and we have shown earlier in Lemma 3.1 that

$$\begin{aligned} \|B_n x - B_n y\| &\le (1+M_1\mu_n)\|x-y\| + M_2\nu_n \\ &= h_n\,\|x-y\| + g_n, \end{aligned} \tag{3.26}$$

for some $M_1, M_2 > 0$ and for all $x, y \in C$, where $h_n = 1 + M_1\mu_n$, and $g_n = M_2\nu_n$ with $\sum_{n=1}^{\infty} h_n < \infty$, $\sum_{n=1}^{\infty} g_n < \infty$ and $h_n \to 1$ as $n \to \infty$. Setting

$$W_{n,m} = B_{n+m-1}B_{n+m-2}\dots B_n, \quad m \ge 1 \tag{3.27}$$

and

$$b_{n,m} = \|W_{n,m}(tx_n + (1-t)p) - (tW_{n,m}x_n + (1-t)W_{n,m}q)\|.$$

From (3.26) and (3.27), we have

$$\begin{aligned} &\|W_{n,m}x - W_{n,m}y\| \qquad (3.28)\\ &= \|B_{n+m-1}B_{n+m-2}\dots B_n x - B_{n+m-1}B_{n+m-2}\dots B_n y\| \\ &\le h_{n+m-1}\|B_{n+m-2}\dots B_n x - B_{n+m-2}\dots B_n y\| + g_{n+m-1} \\ &\le h_{n+m-1}h_{n+m-2}\|B_{n+m-3}\dots B_n x - B_{n+m-3}\dots B_n y\| \\ &\quad + g_{n+m-1} + g_{n+m-2} \\ &\ \vdots \\ &\le \Big(\prod_{k=n}^{n+m-1} h_k\Big)\|x-y\| + \sum_{k=n}^{n+m-1} g_k \\ &= V_n\|x-y\| + \sum_{k=n}^{n+m-1} g_k, \end{aligned} \tag{3.29}$$

for all $x, y \in C$, where $V_n = \prod_{k=n}^{n+m-1} h_k$ and $W_{n,m} x_n = x_{n+m}$, $W_{n,m} p = p$ for all $p \in \mathcal{F}$. Thus

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &\leq b_{n,m} + \|W_{n,m}(tx_n + (1-t)p) - q\| \\ &\leq b_{n,m} + V_n a_n(t) + \sum_{k=n}^{n+m-1} g_k. \end{aligned} \tag{3.30}$$

By using Theorem 2.3 in [7], we have

$$\begin{aligned} b_{n,m} &\leq \varphi^{-1}(\|x_n - u\| - \|W_{n,m}x_n - W_{n,m}u\|) \\ &\leq \varphi^{-1}(\|x_n - u\| - \|x_{n+m} - u + u - W_{n,m}u\|) \\ &\leq \varphi^{-1}(\|x_n - u\| - (\|x_{n+m} - u\| - \|W_{n,m}u - u\|)) \end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0, *i.e.*, $b_{n,m} \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} V_n = 1$ and $\sum_{n=1}^{\infty} g_n < \infty$, that is, $g_n \to 0$ as $n \to \infty$, therefore from (3.30), we have

$$\limsup_{n\to\infty} a_n(t) \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t) + 0 = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is, $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. This completes the proof. □

Now, we prove a weak convergence theorem for the spaces whose dual have Kadec-Klee property (KK-property).

Theorem 3.5. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and C be a nonempty closed convex subset of E. Let $P\colon E \to C$ be a nonexpansive retraction of E into C and $S, T\colon C \to E$ be two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in Proposition 2.1 and $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (1.11), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \nu_n < \infty$*;*
(ii) *there exists a constant $K > 0$ such that $\psi(t) \leq K\,t$, $t \geq 0$.*

If the mappings $I - S$ and $I - T$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.

Proof. By Lemma 3.1, $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in C$. By Lemma

3.2, we have

$$\lim_{j\to\infty}\|x_{n_j} - Sx_{n_j}\| = 0 \quad \text{and} \quad \lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = 0.$$

Since by hypothesis the mappings $I - S$ and $I - T$ are demiclosed at zero, therefore $Sp = p$ and $Tp = p$, which means $p \in \mathcal{F}$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in C$. By the same method as above, we have $q \in \mathcal{F}$ and $p, q \in w_w(x_n)$. By Lemma 3.4, the limit

$$\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$$

exists for all $t \in [0, 1]$ and so $p = q$ by Lemma 2.4. Thus, the sequence $\{x_n\}$ converges weakly to $p \in \mathcal{F}$. This completes the proof. □

Example 3.6. ([28], Example 3.10) Let E be the real line with the usual norm $|.|$, $C = [0, \infty)$ and P be the identity mapping. Assume that $S(x) = x$ and $T(x) = \sin x$ for all $x \in C$. Let ψ be the strictly increasing continuous function such that $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Let $\{\mu_n\}_{n\geq 1}$ and $\{\nu_n\}_{n\geq 1}$ be two nonnegative real sequences defined by $\mu_n = \frac{1}{n^2}$ and $\nu_n = \frac{1}{n^3}$ for all $n \geq 1$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$. Then S and T are total asymptotically nonexpansive mappings with common fixed point 0, that is, $\mathcal{F} = F(S) \cap F(T) = \{0\}$.

4. Concluding remarks

In this paper, we establish some weak convergence theorems for newly defined two-step iteration scheme for two total asymptotically nonexpansive nonself mappings in the framework of uniformly convex Banach spaces. The results presented in this paper extend, generalize and improve several results from the current existing literature by means of more general class of mappings, spaces and iteration schemes considered in this paper.

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