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ON STOCHASTIC QUASILINEAR EVOLUTION EQUATIONS IN HILBERT SPACES

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Abstract. In this paper, the existence and uniqueness of local mild solution to quasilinear equation with additive cylindrical Wiener process in a separable Hilbert space are established using contraction mapping principle. Here we employ the technique used by Pazy to treat homogeneous quasilinear evolution equations. We first show the existence of mild solution of the corresponding linear part, using which we define a contraction map on a suitable complete metric space. The fixed point then obtained using contraction mapping principle is the mild solution of the quasilinear equation.

1. Introduction

In recent decades there has been a tremendous emphasis on understanding and modeling nonlinear processes which are often governed by nonlinear stochastic differential equations. The reason for this growing interest is because the field of stochastic processes forms a bridge between central mathematical issues and practical applications. It arose initially from the study of Brownian motion by Wiener and was extended by Itô, Levy, Kolmogorov and others,

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to a general theory of stochastic differential equations (SDEs). In the case of ordinary differential equations (ODEs), this type of stochastic equations has been well developed since Itô introduced the stochastic integral equations in the mid-1940s (see [9],[8]). Until the 1960s, most of the work on stochastic differential equations had been confined to stochastic ODEs. Since then, spurred by the modern applications, partial differential equations (PDEs) with random parameters, such as the coefficients or the forcing term, have begun to attract the attention of many researchers.

Before the 1970s, there was no general framework for the study of stochastic PDEs. Later, by recasting stochastic partial differential equations (SPDEs) as stochastic evolution equations or stochastic ODEs in Hilbert or Banach spaces, a more coherent theory of SPDEs, under the cover of stochastic evolution equations, began to develop steadily. For study of ODEs one can refer [13] (see also [20]). Since then the SPDEs are more or less, synonymous with stochastic evolution equations. The study of SPDEs in Hilbert space goes back to Baklan [1]. He proved the existence of solutions for a stochastic parabolic equation or parabolic Itô equation with the aid of the associated Green's function. This technique is a precursor to what is now known as the semigroup method and the solution in the Itô sense is called a mild solution. For further reading on this, one can refer to [3] and[4].

Semigroup theory is an important part in mathematics having several connections with theory of partial differential equations. Semigroups have been successfully applied to treat both semilinear and quasilinear equations. A systematic study of deterministic quasilinear equations of evolution can be seen in [10], [11], [12] and [14]. The assumption that the linear part of the equation is an infinitesimal generator of a linear semigroup, is equivalent to the minimal requirement that the studied equation, in its simplest form, has a unique solution continuously depending on the initial data. Semigroup formulation allows a uniform treatment of parabolic, hyperbolic and also delay equations. In numerous situations results obtained by more specialized SPDE method can be recovered by semigroup approach.

The Cauchy problem for stochastic semilinear equation in Hilbert space $\mathbb{X}$ is given by,

$$\begin{aligned} \mathrm{d}x(t) + A\,(t)\,x(t)\mathrm{d}t &= f(t,x(t))\mathrm{d}t + \sigma(t,x(t))\mathrm{d}W(t), \qquad t \in (0,T], \\ x(0) &= x_0. \end{aligned} \tag{1.1}$$

where $(\Omega, \mathbb{P}, \mathcal{F})$ is a complete probabiltiy space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. The operators $\{A(t)\}_{t\in[0,T]}$ are unbounded and with domains which may be time dependent. The functions $f : [0,T]\times\mathbb{X} \to \mathbb{X}$ and $\sigma : [0,T]\times\mathbb{X} \to L(\mathbb{Y},\mathbb{X})$

($\mathbb{Y}$ is another Hilbert space) are measurable and adapted. $W(t)$ is cylindrical Wiener process on $\mathbb{Y}$ with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$.

Dawson [6] used semigroup methods to study (1.1) in the autonomous case (A is constant). This work has been further developed by Da Prato and Zabczyk and their collaborators (see [4], [5] and references therein). In [17], Seidler considered the non-autonomous case with $D(A(t))$ constant in time. An extension to non-autonomous setting with $D(A(t))$ depending on time was given in [15] and [21]. In [7], Fernando studied stochastic counterpart of Kato's quasilinear partial differential equations seen in [12] using semigroup approach with A depending on time and space and taking $S^{-\alpha}$ in the place of $\sigma(t,)$, $\alpha > 2$ and proved the existence and uniqueness of mild solutions using Banach fixed point theorem. Fixed point theorems play a vital role in proving existence (see [18] and [19]) and also controllability of solutions of various differential equations (see [2]). In this paper, we consider Lipschitz perturbation to stochastic quasilinear partial differential equation with additive cylindrical Wiener process considered in [7].

2. Preliminaries

We consider $\mathbb{X}$ and $\mathbb{Y}$ to be separable Hilbert spaces with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$ respectively. $L(\mathbb{X},\mathbb{Y})$ denotes the space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$ with norm $\|\cdot\|_{L(\mathbb{X},\mathbb{Y})}$.
For $\beta, M \in \mathbb{R}$, let $G(\mathbb{X}, M, \beta)$ denote the set of all linear operators A in $\mathbb{X}$, such that $-A$ generates a C_0 semigroup $\{e^{-tA}\}_{t\geq 0}$ with

$$\left\|e^{-tA}\right\|_{L(\mathbb{X},\mathbb{X})} \leq Me^{\beta t}, \;\; t \in [0,\infty).$$

In particular, A is m-accretive if $A \in G(\mathbb{X}, 1, 0)$, in which case $\{e^{-tA}\}_{t\geq 0}$ is a contraction semigroup. A is quasi-m-accretive if $A \in G(\mathbb{X}, 1, \beta)$. Moreover if A is quasi-m-accretive, then A is stable with stability index $1, \beta$ (see [10]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. $W(t)$ be a cylindrical Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ on $\mathbb{X}$. Consider the Cauchy problem for stochastic quasilinear equation with additive cylindrical Wiener noise

$$\begin{aligned} \mathrm{d}x(t) + A\left(t, x(t)\right)x(t)\mathrm{d}t &= f(t, x(t))\mathrm{d}t + \Phi \mathrm{d}W(t), \quad\quad t \in (0,T], \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

in $\mathbb{X}$. Here $x(t,\omega)$ is the unknown taking values in $\mathbb{X}$ and x_0 is $\mathcal{F}_0$ measurable. $\{A\left(t, x(t), \omega\right)\}_{t\in[0,T],\omega\in\Omega}$ is a family of possibly unbounded operators on $\mathbb{X}$ and f is a semilinear nonlinearity. From now on, we do not mention the dependence on the probability space explicitly unless necessary.

For a fixed $\mathbb{X}$-valued stochastic process $z \in E$, we consider the corresponding linear equation to (2.1),

$$\begin{aligned} \mathrm{d}x(t) + A\left(t, z(t)\right) x(t)\mathrm{d}t &= f\left(t, z(t)\right)\mathrm{d}t + \Phi \mathrm{d}W(t), \qquad t \in (0, T], \\ x(0) &= x_0. \end{aligned} \tag{2.2}$$

Here we make the following assumptions,

(H1) Let B be an open ball in $\mathbb{Y}$ with center a and containing x_0 a.s. Therefore, it is possible to choose r_0 such that $\|x_0 - a\|_{\mathbb{Y}}^2 < r_0$ a.s. For some $\tilde{T} \leq T$ (to be determined later) we denote E to be the set of all strongly measurable functions $z : \Omega \times [0, \tilde{T} \wedge \tau_N] \to \mathbb{Y}$, such that for all $t \in [0, \tilde{T} \wedge \tau_N]$, $\|z(t) - a\|_{\mathbb{Y}}^2 \leq r_0$ a.s., and $t \to z(t)$ is continuous as $\mathbb{X}$-valued functions a.s., where τ_N is stopping time defined later.

(H2) $\mathbb{X}$ be a separable Hilbert space. $\mathbb{Y}$ is another separable Hilbert space which is densely and continuously embedded in $\mathbb{X}$ and there exists an isomorphism $S : \mathbb{Y} \to \mathbb{X}$. The norm of $\mathbb{Y}$ is chosen such that S is an isometry from $\mathbb{Y}$ to $\mathbb{X}$, *i.e.*, we have $\|S\|_{L(\mathbb{Y},\mathbb{X})} = \|S^{-1}\|_{L(\mathbb{X},\mathbb{Y})} = 1$.

(H3) $A(t, z)$ is a linear operator on $\mathbb{X}$ and $\forall (t, z) \in [0, T] \times E$, $A(t, z) \in G(\mathbb{X}, 1, \beta)$ and

$$SA(t, z)S^{-1} = A(t, z) + B(t, z)$$

where $B(t, z) \in L(\mathbb{X}, \mathbb{X})$ with $\|B(t, z)\|_{L(\mathbb{X},\mathbb{X})} \leq C_1$. The above relation is satisfied in the strict sense, including the domain relation. Therefore, $x \in D(A(t, z))$ if and only if $S^{-1}x \in D(A(t, z))$ and $A(t, z)S^{-1}x \in \mathbb{Y}$.

(H4) $\mathbb{Y} \subset D(A(t, z))$ for $(t, z) \in [0, T] \times E$ so that $\forall (t, z) \in [0, T] \times E$, $A(t, z)$ restricted to $\mathbb{Y}$ belongs to $L(\mathbb{Y}, \mathbb{X})$ with $\|A(t, z)\|_{L(\mathbb{Y},\mathbb{X})} \leq C_0$. Also, $\forall z \in E$, $t \to A(t, z)$ is continuous in $L(\mathbb{Y}, \mathbb{X})$ norm and $\forall t \in [0, T]$, $z \to A(t, z)$ is Lipschitz continuous in $L(\mathbb{Y}, \mathbb{X})$ norm,

$$\|A(t, z_1) - A(t, z_2)\|_{L(\mathbb{Y},\mathbb{X})} \leq L_1 \|z_1 - z_2\|_{\mathbb{X}}.$$

(H5) $\forall (t, z) \in [0, T] \times E$, we have $A(t, z)a \in \mathbb{Y}$ with $\|A(t, z)a\|_{\mathbb{Y}} \leq C_2$.

(H6) f be a bounded function on $[0, T] \times E \to \mathbb{Y}$ with $\|f(t, z)\|_{\mathbb{Y}} \leq C_3$. Also, $\forall z \in E, t \to f(t, z)$ is continuous in $\mathbb{X}$-norm and $\forall t \in [0, T], z \to f(t, z)$ is $\mathbb{X}$-Lipschitz continuous

$$\|f(t, z_1) - f(t, z_2)\|_{\mathbb{X}} \leq L_2 \|z_1 - z_2\|_{\mathbb{X}}.$$

(H7) The operator Φ satisfies $\|S^n \Phi\|_{L_0^2}^2 \leq M < \infty$, for $n = 1, 2$. L_0^2 is Hilbert-Schmidt space, same as defined in [4].

The constants $\beta, C_0, C_1, C_2, C_3, L_1, L_2$ in the assumptions depend on $x(t)$. Let us define the stopping time τ_N by,

$$\tau_N := \inf_{t \geq 0} \{t : \beta \vee C_0 \vee C_1 \vee C_2 \vee C_3 \vee L_1 \vee L_2 \geq N\}.$$

Remark 2.1. Let $X_1(t,s)$ and $X_2(t,s)$ be evolution operators on $0 \leq s \leq t \leq T$, associated with operators $A_1(t)$ and $A_2(t)$ on $0 \leq t \leq T$ (satisfying assumptions (H3) and (H4)) respectively. Assume that $\mathbb{Y} \subset D(A_1(t)) \cap D(A_2(t))$, for $0 \leq t \leq T$. Then $\forall\, y \in \mathbb{Y}$ we have,

$$\begin{aligned} X_1(t,s)y - X_2(t,s)y &= -\int_s^t \frac{\partial}{\partial r} X_1(t,r) X_2(r,s) y \mathrm{d}r \\ &= \int_s^t X_1(t,r)\left[A_2(r) - A_1(r)\right] X_2(r,s) y \mathrm{d}r. \quad (2.3) \end{aligned}$$

Remark 2.2. We have the following elementary inequality for any $m, n \in \mathbb{N}$ and $a_i \geq 0$

$$\left(\sum_{i=1}^n a_i\right)^m \leq n^{m-1} \sum_{i=1}^n a_i^m. \quad (2.4)$$

3. Existence theorem

In this section, we prove the the existence of mild solution to the linear equation (2.2) corresponding to the quasilinear equation (2.1). We denote $A(t, z(t))$ by $A^z(t)$ and similarly for f and B and rewrite (2.2) as follows,

$$\begin{aligned} \mathrm{d}x(t) + A^z(t)\, x(t)\mathrm{d}t &= f^z(t)\mathrm{d}t + \Phi \mathrm{d}W(t), \qquad t \in (0,T], \\ x(0) &= x_0. \quad (3.1) \end{aligned}$$

Lemma 3.1. *For $t \in [0, \tilde{T}]$, $t \to A^z(t)$ is continuous in $L(\mathbb{Y}, \mathbb{X})$ norm a.s.*

Proof. Consider,

$$\begin{aligned} &\|A^z(t) - A^z(t')\|_{L(\mathbb{Y},\mathbb{X})} \\ &= \|A(t,z(t)) - A(t', z(t'))\|_{L(\mathbb{Y},\mathbb{X})} \\ &\leq \|A(t,z(t)) - A(t,z(t'))\|_{L(\mathbb{Y},\mathbb{X})} + \|A(t,z(t')) - A(t',z(t'))\|_{L(\mathbb{Y},\mathbb{X})} \\ &\leq L_1 \|z(t) - z(t')\|_{\mathbb{X}} + \|A(t,z(t')) - A(t',z(t'))\|_{L(\mathbb{Y},\mathbb{X})}. \end{aligned}$$

Using hypothesis (H4) and the assumption $t \to z(t)$ is continuous a.s. we obtain the desired result. □

Lemma 3.2. *For $t \in [0, \tilde{T}]$, $t \to B^z(t)$ is weakly continuous and hence strongly measurable as an operator valued function a.s.*

Proof. By hypothesis (H3), we can write

$$S^{-1}B^z(t)y = A^z(t)S^{-1}y - S^{-1}A^z(t)y, \quad \forall\, y \in \mathbb{Y}.$$

Since, $S^{-1}y \in \mathbb{Y}$ we have the right hand side of the above equation is continuous in $\mathbb{X}$-norm a.s., using Lemma 3.1. Hence the same is true for left member and therefore we have, $\forall y \in \mathbb{Y},\ t \to S^{-1}B^z(t)y$ is continuous in $\mathbb{X}$-norm a.s. Using hypothesis (H3), we obtain $\|S^{-1}B^z(t)\|_{L(\mathbb{X},\mathbb{X})} \leq C_1\|S^{-1}\|_{L(\mathbb{X},\mathbb{X})}$ and further since $\mathbb{Y}$ is dense in $\mathbb{X}$ we have $\forall x \in \mathbb{X},\ t \to S^{-1}B^z(t)x$ is continuous in $\mathbb{X}$-norm a.s. Since,

$$\|S^{-1}B^z(t)x\|_{\mathbb{Y}} \leq \|S^{-1}\|_{L(\mathbb{X},\mathbb{Y})}\|B^z(t)\|_{L(\mathbb{X},\mathbb{X})}\|x\|_{\mathbb{X}} \leq C_1\|x\|_{\mathbb{X}} \leq C_1\|x\|_{\mathbb{Y}},$$

we have $t \to S^{-1}B^z(t)$ is bounded in $\mathbb{Y}$-norm a.s. By using Lemma 7.4 in [12], we obtain $\forall x \in \mathbb{X},\ t \to S^{-1}B^z(t)x$ is weakly continuous as $\mathbb{Y}$-valued function a.s. Consequently, since S^{-1} is an isometry from $\mathbb{X}$ to $\mathbb{Y}$ we obtain $\forall x \in \mathbb{X}$, $t \to B^z(t)x$ is strongly measurable as $\mathbb{X}$-valued function a.s. □

Theorem 3.3. *If hypothesis* (H1)-(H7) *are satisfied and* $\mathbf{E}\|x_0\|^2_{\mathbb{Y}} < \infty$*, then* (3.1) *has a unique mild solution,*

$$x \in L_2(\Omega; C([0, T \wedge \tau_N], \mathbb{Y})).$$

Proof. The above two lemmas which are direct consequences of the corresponding deterministic results in [12] ensures all the condition of Theorem 1 in [12] are fulfilled by the family $\{A^z(t)\}$. This guarantees that there exists a unique family of evolution operators $\{X^z(t,s)\}$ defined on $0 \leq s \leq t \leq T \wedge \tau_N$ for (3.1) with the properties described in Theorem 1 in [12]. Also we can obtain the following estimates as given in [12],

$$\sup_{0\leq s\leq t\leq T\wedge\tau_N} \|X^z(t,s)\|_{L(\mathbb{X},\mathbb{X})} \leq e^{\beta T} \leq e^{NT}, \tag{3.2}$$

$$\sup_{0\leq s\leq t\leq T\wedge\tau_N} \|X^z(t,s)\|_{L(\mathbb{Y},\mathbb{Y})} \leq e^{(\beta+C_1)T} \leq e^{2NT}. \tag{3.3}$$

The stochastic mild solution for the linear problem (3.1) in terms of the family of evolution operators $\{X^z(t,s)\}$ on $0 \leq s \leq t \leq T \wedge \tau_N$ is usually represented as follows:

$$x(t) = X^z(t,0)x_0 + \int_0^t X^z(t,s)f^z(s)\mathrm{d}s + \int_0^t X^z(t,s)\Phi \mathrm{d}W(s).$$

Since $X^z(t,s)$ is generally not $\mathcal{F}_s$ measurable, the stochastic integral above is not defined in Itô sense. To rectify this issue, we apply integration by parts and obtain a new representation for the mild solution (as in [15]) which does not involve the stochastic integration of non-adapted integrands as seen below.

$$\begin{aligned} x(t) &= X^z(t,0)x_0 + \int_0^t X^z(t,s)f^z(s)\mathrm{d}s + X^z(t,0)\int_0^t \Phi\mathrm{d}W(\theta) \\ &\quad + \int_0^t X^z(t,s)A^z(s)\left(\int_s^t \Phi\mathrm{d}W(\theta)\right)\mathrm{d}s. \end{aligned} \tag{3.4}$$

We now show that $x(t)$ given in (3.4) is in the space $L_2(\Omega; C([0,T\wedge\tau_N],\mathbb{Y}))$ for which we consider

$$\begin{aligned} &\mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\|x(t)\|^2_{\mathbb{Y}} \\ &= \mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\left\|X^z(t,0)x_0 + \int_0^t X^z(t,s)f^z(s)\mathrm{d}s + X^z(t,0)\int_0^t \Phi\mathrm{d}W(\theta)\right. \\ &\quad \left. + \int_0^t X^z(t,s)A^z(s)\left(\int_s^t \Phi\mathrm{d}W(\theta)\right)\mathrm{d}s\right\|^2_{\mathbb{Y}} \\ &\le 4\left(\mathbf{E}\sup_{0\le t\le T}\|X^z(t,0)x_0\|^2_{\mathbb{Y}} + \mathbf{E}\sup_{0\le t\le T}\left\|\int_0^t X^z(t,s)f^z(s)\mathrm{d}s\right\|^2_{\mathbb{Y}}\right. \\ &\quad + \mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\left\|X^z(t,0)\int_0^t \Phi\mathrm{d}W(\theta)\right\|^2_{\mathbb{Y}} \\ &\quad \left. + \mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\left\|\int_0^t X^z(t,s)A^z(s)\left(\int_s^t \Phi\mathrm{d}W(\theta)\right)\mathrm{d}s\right\|^2_{\mathbb{Y}}\right) \\ &= 4\left(A+B+C+D\right), \end{aligned}$$

where we have used the inequality (2.4). Now, using (3.3) we estimate the first term above as follows,

$$\begin{aligned} A &= \mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\|X^z(t,0)x_0\|^2_{\mathbb{Y}} \\ &\le \mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\|X^z(t,0)\|^2_{L(\mathbb{Y},\mathbb{Y})}\|x_0\|^2_{\mathbb{Y}} \;\le\; e^{4NT}\mathbf{E}\|x_0\|^2_{\mathbb{Y}}. \end{aligned}$$

Using (3.3) and hypothesis (H6) we obtain,

$$\begin{aligned} B &= \mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\left\|\int_0^t X^z(t,s)f^z(s)\mathrm{d}s\right\|^2_{\mathbb{Y}} \\ &\le \mathbf{E}\sup_{0\le t\le T\wedge\tau_N} t\int_0^t \|X^z(t,s)\|^2_{L(\mathbb{Y},\mathbb{Y})}\|f^z(s)\|^2_{\mathbb{Y}}\mathrm{d}s \\ &\le T^2N^2e^{4NT}. \end{aligned}$$

Using the fact that S is an isometry (*i.e.*, hypothesis (H2)), (3.3), Theorem 3.8(ii) and Lemma 7.7 in [4] we obtain the following estimate,

$$\begin{aligned}
C &= \mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\| X^z(t,0)\int_0^t \Phi \mathrm{d}W(\theta)\right\|_{\mathbb{Y}}^2 \\
&= \mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\| SX^z(t,0)S^{-1}\int_0^t S\Phi \mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2 \\
&\le \mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \|S\|_{L(\mathbb{Y},\mathbb{X})}^2 \|X^z(t,0)\|_{L(\mathbb{Y},\mathbb{Y})}^2 \left\|S^{-1}\right\|_{L(\mathbb{X},\mathbb{Y})}^2 \left\|\int_0^t S\Phi \mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2 \\
&\le e^{4NT}\mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\|\int_0^t S\Phi \mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2 \\
&\le 4e^{4NT} \sup_{0\le t\le T} \mathbf{E} \left\|\int_0^t S\Phi \mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2 \\
&\le 4MTe^{4NT}.
\end{aligned}$$

Using hypothesis (H3) and (2.4) we get,

$$\begin{aligned}
D &= \mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\|\int_0^t X^z(t,s)A^z(s)\left(\int_s^t \Phi \mathrm{d}W(\theta)\right)\mathrm{d}s\right\|_{\mathbb{Y}}^2 \\
&= \mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\|\int_0^t X^z(t,s)A^z(s)S^{-1}\left(\int_s^t S\Phi \mathrm{d}W(\theta)\right)\mathrm{d}s\right\|_{\mathbb{Y}}^2 \\
&= \mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\|\int_0^t X^z(t,s)\left(S^{-1}A^z(s)+S^{-1}B^z(s)\right)\left(\int_s^t S\Phi \mathrm{d}W(\theta)\right)\mathrm{d}s\right\|_{\mathbb{Y}}^2 \\
&\le 2\mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\|\int_0^t SX^z(t,s)S^{-1}A^z(s)\left(\int_s^t S\Phi \mathrm{d}W(\theta)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2 \\
&\quad + 2\mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\|\int_0^t SX^z(t,s)S^{-1}B^z(s)\left(\int_s^t S\Phi \mathrm{d}W(\theta)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2 \\
&= D_1 + D_2.
\end{aligned}$$

Using hypothesis (H2), (3.3) and hypothesis (H4) the first term is estimated as follows,

$$\begin{aligned}
D_1 &= 2\mathbf{E} \sup_{0\le t\le T\wedge\tau_N} \left\|\int_0^t S\mathbb{X}^z(t,s)S^{-1}A^z(s)\left(\int_s^t S\Phi \mathrm{d}W(\theta)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2 \\
&\le 2Te^{4NT}\mathbf{E} \sup_{0\le t\le T\wedge\tau_N} C_0^2 \int_0^t \left\|\int_s^t S^2\Phi \mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2 \mathrm{d}s
\end{aligned}$$

$$= 2Te^{4NT}N^2\mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\int_0^t \left\| \left(\int_0^t - \int_0^s\right) S^2\Phi \mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2 \mathrm{d}s$$
$$\le 8T^2e^{4NT}N^2\mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\left\|\int_0^t S^2\Phi\mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2$$
$$\le 32T^2e^{4NT}N^2\int_0^{T\wedge\tau_N}\left\|S^2\Phi\right\|_{L_2^0}^2 \mathrm{d}\theta$$
$$\le 32T^3N^2Me^{4NT}.$$

Applying hypothesis (H3) we obtain the following result for the second term,

$$D_2 = 2\mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\left\|\int_0^t SX^z(t,s)S^{-1}B^z(s)\left(\int_s^t S\Phi\mathrm{d}W(\theta)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2$$
$$\le 2Te^{4NT}\mathbf{E}\sup_{0\le t\le T\wedge\tau_N}C_1^2\int_0^t\left\|\int_s^t S\Phi\mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2\mathrm{d}s$$
$$= 2Te^{4NT}N^2\mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\int_0^t\left\|\left(\int_0^t - \int_0^s\right)S\Phi\mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2\mathrm{d}s$$
$$\le 8T^2e^{4NT}N^2\mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\left\|\int_0^t S\Phi\mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2$$
$$\le 32T^2e^{4NT}N^2\sup_{0\le t\le T\wedge\tau_N}\mathbf{E}\left\|\int_0^t S\Phi\mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2$$
$$\le 32T^2e^{4NT}N^2\int_0^{T\wedge\tau_N}\|S\Phi\|_{L_2^0}^2\mathrm{d}\theta$$
$$\le 32T^3N^2Me^{4NT}.$$

Therefore we have, using the fact that $\mathbf{E}\|x_0\|_{\mathbb{Y}}^2 < \infty$ and hypothesis (H7),

$$\mathbf{E}\sup_{0\le t\le T\wedge\tau_N}\|x(t)\|_{\mathbb{Y}}^2$$
$$\le 4e^{4NT}\left(\mathbf{E}\|x_0\|_{\mathbb{Y}}^2 + T^2N^2 + 4TM + 64T^3N^2M\right)$$
$$= K < \infty. \tag{3.5}$$

This completes the proof of the theorem. □

4. Local mild solution

In this section, we obtain the mild solution of (2.1) by first defining a map Λ on the set E and choosing time small enough so that Λ becomes a contraction.

Then using contraction mapping principle we obtain a fixed point for Λ in E which is the mild solution of (2.1).

Theorem 4.1. *If hypothesis* (H1)-(H7) *are satisfied and* $\mathbf{E}\|x_0 - a\|^2_{\mathbb{Y}} < \infty$, *then* (2.1) *has a unique mild solution*

$$x \in L_2(\Omega; C([0, \tilde{T} \wedge \tau_N], \mathbb{Y})) \text{ for some } \tilde{T} > 0 \text{ with } \tilde{T} \leq T.$$

Proof. We first show that the solution obtained in Theorem 3.3 belongs to E a.s. for some $T_1 \leq T$. For the same, we first set $\tilde{x}(t) = x(t) - a$ and see that,

$$\begin{aligned} \mathrm{d}\tilde{x}(t) + A^z(t)\tilde{x}(t)\mathrm{d}t &= f^z(t)\mathrm{d}t + \Phi\mathrm{d}W(t) - A^z(t)a\,\mathrm{d}t, \quad t \in (0, T], \\ \tilde{x}(0) &= x_0 - a. \end{aligned}$$

Then,

$$\begin{aligned} x(t) - a = X^z(t,0)(x_0 - a) &+ \int_0^t X^z(t,s)f^z(s)\mathrm{d}s \\ &+ X^z(t,0)\int_0^t \Phi\mathrm{d}W(\theta) + \int_0^t X^z(t,s)A^z(s)\left(\int_s^t \Phi\mathrm{d}W(\theta)\right)\mathrm{d}s \\ &- \int_0^t X^z(t,s)A^z(s)a\,\mathrm{d}s. \end{aligned}$$

We now consider for some $T_1 \leq T$ (to be fixed later), and using (2.4) we get,

$$\begin{aligned} &\mathbf{E} \sup_{0 \leq t \leq T_1 \wedge \tau_N} \|x(t) - a\|^2_{\mathbb{Y}} \\ &\leq 5\Bigg(\mathbf{E} \sup_{0 \leq t \leq T_1} \|X^z(t,0)(x_0 - a)\|^2_{\mathbb{Y}} \\ &\quad + \mathbf{E} \sup_{0 \leq t \leq T_1 \wedge \tau_N} \left\|\int_0^t X^z(t,s)f^z(s)\mathrm{d}s\right\|^2_{\mathbb{Y}} \\ &\quad + \mathbf{E} \sup_{0 \leq t \leq T_1 \wedge \tau_N} \left\|X^z(t,0)\int_0^t \Phi\mathrm{d}W(\theta)\right\|^2_{\mathbb{Y}} \\ &\quad + \mathbf{E} \sup_{0 \leq t \leq T_1 \wedge \tau_N} \left\|\int_0^t X^z(t,s)A^z(s)\left(\int_s^t \Phi\mathrm{d}W(\theta)\right)\mathrm{d}s\right\|^2_{\mathbb{Y}} \\ &\quad + \mathbf{E} \sup_{0 \leq t \leq T_1 \wedge \tau_N} \left\|\int_0^t X^z(t,s)A^z(s)a\mathrm{d}s\right\|^2_{\mathbb{Y}}\Bigg). \end{aligned}$$

The first four terms in the above inequality can be estimated as in Theorem 3.3. We now use hypothesis (H5) along with (3.3) to obtain the following estimate for the fifth term.

$$\begin{aligned}
&\mathbf{E}\sup_{0\le t\le T_1\wedge\tau_N}\left\|\int_0^t X^z(t,s)A^v(t)a\,\mathrm{d}t\right\|_{\mathbb{Y}}^2\\
&\le T_1\mathbf{E}\sup_{0\le t\le T_1\wedge\tau_N}\int_0^t\|X^z(t,s)A^v(s)a\|_{\mathbb{Y}}^2\,\mathrm{d}s\\
&\le T_1e^{4NT_1}\mathbf{E}\sup_{0\le t\le T_1}\int_0^t\|A^v(s)a\|_{\mathbb{Y}}^2\,\mathrm{d}s\le T_1^2N^2e^{4NT_1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\sup_{0\le t\le T_1\wedge\tau_N}\|x(t)-a\|_{\mathbb{Y}}^2\\
&\le 5e^{4NT_1}\Big(\mathbf{E}\,\|x_0-a\|_{\mathbb{Y}}^2+2T_1^2N^2+4T_1M+64T_1^3N^2M\Big)\le r_0,
\end{aligned}$$

by choosing $T_1\le T$ sufficiently small. Using Kolmogorov's continuous theorem, there exists a continuous modification of x which belongs to E a.s.
Now let us define a distance function on E as,

$$d(v,w)=\mathbf{E}\sup_{0\le t\le T_1\wedge\tau_N}\|v(t)-w(t)\|_{\mathbb{X}}^2\,.$$

Then we see that (E,d) forms a complete metric space, since closed ball in $\mathbb{Y}$ is a closed subset of $\mathbb{X}$ (see Lemma 7.3 in [12]) and also using the completeness of the space $\mathbb{X}$. We now define a map,

$$\Lambda:E\to E\quad\ni\quad\Lambda(z)=x,$$

where x is the solution of equation (3.1) given by Theorem 3.3. Next, we show that Λ is a strict contraction if we choose $\tilde{T}\le T_1\le T$. Consider,

$$\begin{aligned}
&d\left(\Lambda(z_1),\Lambda(z_2)\right)\\
&=\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\|\Lambda(z_1)-\Lambda(z_2)\|_{\mathbb{X}}^2\\
&\le 5\Bigg(\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\|\left(X^{z_1}(t,0)-X^{z_2}(t,0)\right)(x_0-a)\|_{\mathbb{X}}^2\\
&\quad+\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\int_0^t\left(X^{z_1}(t,s)f^{z_1}(s)-X^{z_2}(t,s)f^{z_2}(s)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2\\
&\quad+\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\left(X^{z_1}(t,0)-X^{z_2}(t,0)\right)\int_0^t\Phi\mathrm{d}W(\theta)\right\|_{\mathbb{X}}^2\\
&\quad+\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\int_0^t\left(X^{z_1}(t,s)A^{z_1}(s)-X^{z_2}(t,s)A^{z_2}(s)\right)\left(\int_s^t\Phi\mathrm{d}W(\theta)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2
\end{aligned}$$

$$
\left.+\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\int_0^t\left(X^{z_1}(t,s)A^{z_1}(s)a-X^{z_2}(t,s)A^{z_2}(s)\right)a\,\mathrm{d}s\right\|_{\mathbb{X}}^2\right)
$$
$$
=5(F+G+H+I+J).
$$

Using (2.3), (3.2), (3.3) and hypothesis (H4) we obtain the following estimate,

$$
\begin{aligned}
F&=\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\left(X^{z_1}(t,0)-X^{z_2}(t,0)\right)(x_0-a)\right\|_{\mathbb{X}}^2\\
&\le\tilde{T}\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\int_0^t\left\|X^{z_1}(t,r)\left(A^{z_2}(r)-A^{z_1}(r)\right)X^{z_2}(r,0)(x_0-a)\right\|_{\mathbb{X}}^2\mathrm{d}r\\
&\le\tilde{T}^2e^{6N\tilde{T}}\mathbf{E}\left(\|x_0-a\|_{\mathbb{Y}}^2\sup_{0\le t\le\tilde{T}}\|A^{z_1}(t)-A^{z_2}(t)\|_{L(\mathbb{Y},\mathbb{X})}^2\right)\\
&\le r_0N^2\tilde{T}^2e^{6N\tilde{T}}\mathbf{E}\sup_{0\le t\le\tilde{T}}\|z_1(t)-z_2(t)\|_{\mathbb{X}}^2.
\end{aligned}
$$

Using (2.3), (2.4), Minkowski's inequality for integrals, hypothesis (H4) and (H6),

$$
\begin{aligned}
G&=\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\int_0^t\left(X^{z_1}(t,s)f^{z_1}(s)-X^{z_2}(t,s)f^{z_2}(s)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2\\
&\le2\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\int_0^t\left[X^{z_1}(t,s)-X^{z_2}(t,s)\right]f^{z_1}(s)\mathrm{d}s\right\|_{\mathbb{X}}^2\\
&\quad+2\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\int_0^tX^{z_2}(t,s)\left[f^{z_1}(s)-f^{z_2}(s)\right]\mathrm{d}s\right\|_{\mathbb{X}}^2\\
&=2\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\int_0^t\int_s^tX^{z_1}(t,r)\left[A^{z_2}(r)-A^{z_1}(r)\right]X^{z_2}(r,s)f^{z_1}(s)\mathrm{d}r\mathrm{d}s\right\|_{\mathbb{X}}^2\\
&\quad+2\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left\|\int_0^tX^{z_2}(t,s)\left[f^{z_1}(s)-f^{z_2}(s)\right]\mathrm{d}s\right\|_{\mathbb{X}}^2\\
&\le2\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\left[\int_0^t\int_0^r\left\|X^{z_1}(t,r)\left[A^{z_2}(r)-A^{z_1}(r)\right]X^{z_2}(r,s)f^{z_1}(s)\right\|_{\mathbb{X}}\mathrm{d}s\mathrm{d}r\right]^2\\
&\quad+2\tilde{T}\mathbf{E}\sup_{0\le t\le\tilde{T}\wedge\tau_N}\int_0^t\left\|X^{z_2}(t,s)\left[f^{z_1}(s)-f^{z_2}(s)\right]\right\|_{\mathbb{X}}^2\mathrm{d}s\\
&\le2\left(\tilde{T}^4N^4e^{6N\tilde{T}}+\tilde{T}^2N^2e^{2N\tilde{T}}\right)\mathbf{E}\sup_{0\le t\le\tilde{T}}\|z_1(t)-z_2(t)\|_{\mathbb{X}}^2.
\end{aligned}
$$

Using (2.3), (3.2), (3.3), Hölder's inequality, hypothesis (H7) and (H4) we obtain the following inequalities,

$$
\begin{aligned}
H =&\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| (X^{z_1}(t,0) - X^{z_2}(t,0)) \int_0^t \Phi \mathrm{d}W(\theta) \right\|_{\mathbb{X}}^2 \\
\le&\tilde{T}\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \int_0^t \left\| X^{z_1}(t,r)\left(A^{z_2}(r) - A^{z_1}(r)\right) X^{z_2}(r,0) \left(\int_0^t \Phi \mathrm{d}W(\theta) \right) \right\|_{\mathbb{X}}^2 \mathrm{d}r \\
\le&\tilde{T}e^{6N\tilde{T}}\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \int_0^t \|A^{z_1}(r) - A^{z_2}(r)\|_{L(\mathbb{Y},\mathbb{X})}^2 \left\| \int_0^t S\Phi \mathrm{d}W(\theta) \right\|_{\mathbb{X}}^2 \mathrm{d}r \\
\le&\tilde{T}^3 M e^{6N\tilde{T}}\mathbf{E} \sup_{0\le t\le \tilde{T}} \|z_1(t) - z_2(t)\|_{\mathbb{X}}^2 .
\end{aligned}
$$

We now consider,

$$
\begin{aligned}
I =&\ \mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| \int_0^t \left(X^{z_1}(t,s)A^{z_1}(s) - X^{z_2}(t,s)A^{z_2}(s) \right) \left(\int_s^t \Phi \mathrm{d}W(\theta) \right) \mathrm{d}s \right\|_{\mathbb{X}}^2 \\
\le&\ 2\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| \int_0^t X^{z_1}(t,s)\left(A^{z_1}(s) - A^{z_2}(s)\right) \left(\int_s^t \Phi \mathrm{d}W(\theta) \right) \mathrm{d}s \right\|_{\mathbb{X}}^2 \\
&+ 2\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| \int_0^t \left(X^{z_1}(t,s) - X^{z_2}(t,s) \right) A^{z_2}(s) \left(\int_s^t \Phi \mathrm{d}W(\theta) \right) \mathrm{d}s \right\|_{\mathbb{X}}^2 \\
=&\ I_1 + I_2.
\end{aligned}
$$

The first term in the above inequality is estimated using (3.2), hypothesis (H7) and Hölder's inequality as follows,

$$
\begin{aligned}
I_1 =&\ 2\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| \int_0^t X^{z_1}(t,s)\left(A^{z_1}(s) - A^{z_2}(s)\right) \left(\int_s^t \Phi \mathrm{d}W(\theta) \right) \mathrm{d}s \right\|_{\mathbb{X}}^2 \\
\le&\ 2\tilde{T}e^{2N\tilde{T}}\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \int_0^t \|A^{z_1}(s) - A^{z_2}(s)\|_{L(\mathbb{Y},\mathbb{X})}^2 \left\| \int_s^t S\Phi \mathrm{d}W(\theta) \right\|_{\mathbb{X}}^2 \mathrm{d}s \\
\le&\ 2\tilde{T}^2 M e^{2N\tilde{T}}\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \|z_1(t) - z_2(t)\|_{\mathbb{X}}^2 .
\end{aligned}
$$

The second term in estimating I is now considered. Using (2.3), Minkowski's inequality for integral, (3.2), (3.3), hypothesis (H7) and hypothesis (H3)

$$
\begin{aligned}
I_2 =& 2\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| \int_0^t \left(X^{z_1}(t,s) - X^{z_2}(t,s)\right) A^{z_2}(s) \left(\int_s^t \Phi \mathrm{d}W(\theta)\right) \mathrm{d}s \right\|_{\mathbb{X}}^2 \\
\le& 2\tilde{T}^2 \mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \int_0^t \int_0^r \left\| X^{z_1}(t,r) \left(A^{z_1}(r) - A^{z_2}(r)\right) X^{z_2}(r,s) A^{z_2}(s) \right. \\
& \qquad \left. \left(\int_s^t \Phi \mathrm{d}W(\theta)\right) \right\|_{\mathbb{X}}^2 \mathrm{d}s\mathrm{d}r \\
\le& 2\tilde{T}^2 e^{6N\tilde{T}} \mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \|A^{z_1}(t) - A^{z_2}(t)\|_{L(\mathbb{Y},\mathbb{X})}^2 \\
& \qquad \int_0^t \int_0^r \left\| S A^{z_2}(s) S^{-1} \left(\int_s^t S\Phi \mathrm{d}W(\theta)\right) \right\|_{\mathbb{X}}^2 \mathrm{d}s\mathrm{d}r \\
\le& 2\tilde{T}^2 e^{6N\tilde{T}} \mathbf{E} \sup_{0\le t\le \tilde{T}} \|A^{z_1}(t) - A^{z_2}(t)\|_{L(\mathbb{Y},\mathbb{X})}^2 \\
& \qquad \int_0^t \int_0^r \left\| \left(A^{z_2}(s) + B^{z_2}(s)\right) \left(\int_s^t S\Phi \mathrm{d}W(\theta)\right) \right\|_{\mathbb{X}}^2 \mathrm{d}s\mathrm{d}r \\
\le& 4\tilde{T}^2 e^{6N\tilde{T}} \mathbf{E} \sup_{0\le t\le \tilde{T}} \|A^{z_1}(t) - A^{z_2}(t)\|_{L(\mathbb{Y},\mathbb{X})}^2 \\
\le& 2\tilde{T}^4 e^{6N\tilde{T}} \mathbf{E} \sup_{0\le t\le \tilde{T}} \|A^{z_1}(t) - A^{z_2}(t)\|_{L(\mathbb{Y},\mathbb{X})}^2 \\
\le& 4\tilde{T}^4 N^2 M e^{6N\tilde{T}} \mathbf{E} \sup_{0\le t\le \tilde{T}} \|z_1(t) - z_2(t)\|_{\mathbb{X}}^2 .
\end{aligned}
$$

The last term is estimated as earlier and we obtain,

$$
\begin{aligned}
J =& \ \mathbf{E} \sup_{0\le t\le \tilde{T}} \left\| \int_0^t \left(X^{z_1}(t,s) A^{z_1}(s) - X^{z_2}(t,s) A^{z_2}(s)\right) a \mathrm{d}s \right\|_{\mathbb{X}}^2 \\
\le& \ 2\mathbf{E} \sup_{0\le t\le \tilde{T}} \left\| \int_0^t X^{z_1}(t,s) \left(A^{z_1}(s) - A^{z_2}(s)\right) a \mathrm{d}s \right\|_{\mathbb{X}}^2 \\
& + 2\mathbf{E} \sup_{0\le t\le \tilde{T}} \left\| \int_0^t \left(X^{z_1}(t,s) - X^{z_2}(t,s)\right) A^{z_2}(s) a \mathrm{d}s \right\|_{\mathbb{X}}^2 \\
=& \ J_1 + J_2.
\end{aligned}
$$

The first term in the above inequality is estimated using (3.2) and Hölder's inequality as follows,

$$\begin{aligned}J_1 &= 2\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| \int_0^t X^{z_1}(t,s)\left(A^{z_1}(s)-A^{z_2}(s)\right) a\mathrm{d}s \right\|_{\mathbb{X}}^2 \\ &\le 2\tilde{T}e^{2N\tilde{T}}\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \int_0^t \left\|A^{z_1}(s)-A^{z_2}(s)\right\|^2_{L(\mathbb{Y},\mathbb{X})} \|a\|^2_{\mathbb{Y}} \,\mathrm{d}s \\ &\le 2\|a\|^2_{\mathbb{Y}}\tilde{T}^2 e^{2N\tilde{T}}\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\|A^{z_1}(t)-A^{z_2}(t)\right\|^2_{L(\mathbb{Y},\mathbb{X})} \\ &\le 2\tilde{T}^2N^2e^{2N\tilde{T}}\|a\|^2_{\mathbb{Y}}\,\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \|z_1(t)-z_2(t)\|^2_X .\end{aligned}$$

The second term in estimating J is now considered. Using (2.3), Minkowski's inequality for integral, (3.2), (3.3) and hypothesis (H3)

$$\begin{aligned}J_2 &= 2\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| \int_0^t \left(X^{z_1}(t,s)-X^{z_2}(t,s)\right) A^{z_2}(s)a\,\mathrm{d}s \right\|_{\mathbb{X}}^2 \\ &= 2\mathbf{E} \sup_{0\le t\le \tilde{T}\wedge\tau_N} \left\| \int_0^t\int_s^t X^{z_1}(t,r)\left(A^{z_1}(r)-A^{z_2}(r)\right) X^{z_2}(r,s)A^{z_2}(s)a\;\mathrm{d}r\mathrm{d}s \right\|_{\mathbb{X}}^2 \\ &\le \tilde{T}^4e^{6N\tilde{T}}\left\|A^{z_2}(s)a\right\|^2_{\mathbb{Y}}\,\mathbf{E}\sup_{0\le t\le \tilde{T}} \left\|A^{z_1}(t)-A^{z_2}(t)\right\|^2_{L(\mathbb{Y},\mathbb{X})} \\ &\le \tilde{T}^4N^4e^{6N\tilde{T}}\mathbf{E}\sup_{0\le t\le \tilde{T}} \|z_1(t)-z_2(t)\|^2_{\mathbb{X}} .\end{aligned}$$

Therefore,

$$\begin{aligned}&d\left(\Lambda(z_1),\Lambda(z_2)\right) \\ &\le \tilde{T}^2\Big[r_0N^2e^{6N\tilde{T}} + 2\tilde{T}^2N^4e^{6N\tilde{T}} + 2N^2e^{2N\tilde{T}} + \tilde{T}Me^{6N\tilde{T}} + 2Me^{2N\tilde{T}} \\ &\quad + 4\tilde{T}^2N^2Me^{6N\tilde{T}} + 2N^2e^{2N\tilde{T}}\|a\|^2_{\mathbb{Y}} + \tilde{T}^2N^4e^{6N\tilde{T}}\Big]\,\mathbf{E}\sup_{0\le t\le\tilde{T}}\|z_1(t)-z_2(t)\|^2_{\mathbb{X}} \\ &= \rho d(z_1,z_2),\end{aligned}$$

where ρ can be made strictly less than 1 by choosing $\tilde{T}$ small enough. Hence we can now apply contraction mapping principle to show that Λ has an unique fixed point which is the solution of the quasilinear delay stochastic differential equation given by (2.1). □

Remark 4.2. We can now apply the same technique as above to the corresponding stochastic quasilinear delay differential equation to (2.1) in a separable Hilbert space $\mathbb{X}$,

$$\begin{aligned} &\mathrm{d}x(t) + A\left(t, x(t), x(t-r)\right) x(t)\mathrm{d}t \\ &= f(t, x(t), x(t-r))\mathrm{d}t + \Phi \mathrm{d}W(t), \quad t \in (0, T], \\ &x(t) = \phi(t), \quad t \in [-r, 0], \end{aligned} \tag{4.1}$$

where for all $t \in [-r, 0]$, $t \to \phi(t)$ is continuous and $\mathcal{F}_0$ measurable.

We alter the definition of the set E to be the set of all strongly measurable functions $z : \Omega \times [-r, \tilde{T} \wedge \tau_N] \to \mathbb{Y}$, such that for all $t \in [-r, 0]$ we have $z(t) = \phi(t)$ a.s., and for all $t \in [-r, \tilde{T} \wedge \tau_N]$, $\|z(t) - a\|_{\mathbb{Y}}^2 \le r_0$ a.s., along with the condition, $t \to z(t)$ is continuous as $\mathbb{X}$-valued functions a.s. Now, the corresponding linear equation is given as follows,

$$\begin{aligned} \mathrm{d}x(t) + A\left(t, z(t), z(t-r)\right) x(t)\mathrm{d}t &= f(t, z(t), z(t-r))\mathrm{d}t + \Phi \mathrm{d}W(t), t \in (0, T], \\ x(t) &= \phi(t), \quad t \in [-r, 0]. \end{aligned}$$

Here we replace the assumptions (H3)-(H6) by (H3)$'$-(H6)$'$ given below.

(H3)$'$ $A(t, z, w)$ is a linear operator on $\mathbb{X}$ and $\forall (t, z, w) \in [0, T \wedge \tau_N] \times E^2$, $A(t, z, w) \in G(\mathbb{X}, 1, \beta)$ and

$$SA(t, z, w)S^{-1} = A(t, z, w) + B(t, z, w)$$

where $B(t, z, w) \in L(\mathbb{X}, \mathbb{X})$ with $\|B(t, z, w)\|_{L(\mathbb{X},\mathbb{X})} \le C_1$.The above relation is satisfied in the strict sense, including the domain relation. Therefore, $x \in D(A(t, z, w))$ if and only if $S^{-1}x \in D(A(t, z, w))$ and $A(t, z, w)S^{-1}x \in \mathbb{Y}$.

(H4)$'$ $\mathbb{Y} \subset D(A(t, z, w))$ for $(t, z, w) \in [0, T \wedge \tau_N] \times E^2$ so that, $\forall (t, z, w) \in [0, T \wedge \tau_N] \times E^2$, $A(t, z, w)$ restricted to $\mathbb{Y}$ belongs to $L(\mathbb{Y}, \mathbb{X})$ with $\|A(t, z, w)\|_{L(\mathbb{Y},\mathbb{X})} \le C_0$. Also, $\forall (z, w) \in E^2$, $t \to A(t, z, w)$ is continuous in $L(\mathbb{Y}, \mathbb{X})$ norm and $\forall t \in [0, T \wedge \tau_N]$, $(z, w) \to A(t, z, w)$ is Lipschitz continuous in $L(\mathbb{Y}, \mathbb{X})$ norm,

$$\|A(t, z_1, w_1) - A(t, z_2, w_2)\|_{L(\mathbb{Y},\mathbb{X})} \le L_1 \left(\|z_1 - z_2\|_{\mathbb{X}} + \|w_1 - w_2\|_{\mathbb{X}}\right).$$

(H5)$'$ $\forall (t, z, w) \in [0, T \wedge \tau_N] \times E^2$ we have $A(t, z, w)a \in \mathbb{Y}$ with

$$\|A(t, z, w)a\|_{\mathbb{Y}} \le C_2.$$

(H6)$'$ f be a bounded function on $[0, T \wedge \tau_N] \times E \to \mathbb{Y}$ with

$$\|f(t, z, w)\|_{\mathbb{Y}} \le C_3.$$

Also, $\forall(z,w) \in E^2$, $t \to f(t,z,w)$ is continuous in $\mathbb{X}$-norm and $\forall t \in [0, T \wedge \tau_N]$, $(z,w) \to f(t,z,w)$ is $\mathbb{X}$-Lipschitz continuous

$$\|f(t,z_1,w_1) - f(t,z_2,w_2)\|_{\mathbb{X}} \leq L_2 \left(\|z_1 - z_2\|_{\mathbb{X}} + \|w_1 - w_2\|_{\mathbb{X}}\right).$$

Denoting $A(t, z(t), z(t-r))$ by $A^z(t)$ and using similar notation for f we can proceed with the same method used for (2.1) to show that there exists unique mild solution for (4.1).

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