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AN N-ORDER ITERATIVE SCHEME FOR A NONLINEAR LOVE EQUATION ASSOCIATED WITH MIXED HOMOGENEOUS CONDITIONS

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Abstract. In this paper, a high-order iterative scheme is established in order to get a convergent sequence at a rate of order N $(N \geq 1)$ to a local unique weak solution of a nonlinear Love equation associated with mixed homogeneous conditions.

1. Introduction

In this paper, we consider the following Love equation with initial conditions and mixed homogeneous conditions

$$u_{tt} - u_{xx} - u_{xxtt} + \lambda u_t = f(x,t,u), \quad 0 < x < 1, \quad 0 < t < T, \tag{1.1}$$

$$u_x(0,t) + u_{xtt}(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $\tilde{u}_0$, $\tilde{u}_1$, f, are given functions and $\lambda \neq 0$ is a given function.

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When $f = 0$, $\lambda = 0$, Eq.(1.1) is related to the Love equation

$$u_{tt} - \tfrac{E}{\rho} u_{xx} - 2\mu^2 k^2 u_{xxtt} = 0, \tag{1.4}$$

presented by V. Radochová in 1978 (see [12]). This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy function

$$\int_0^T dt \int_0^L \left[\tfrac{1}{2} F \rho \left(u_t^2 + \mu^2 k^2 u_{tx}^2 \right) - \tfrac{1}{2} F \left(E u_x^2 + \rho \mu^2 k^2 u_x u_{xtt} \right) \right] dx, \tag{1.5}$$

the parameters in (1.5) have the following meanings: u is the displacement, L is the length of the rod, F is the area of cross-section, k is the cross-section radius, E is the Young modulus of the material and ρ is the mass density. By using the Fourier method, Radochová [12] obtained a classical solution of Prob. (1.4) associated with initial conditions (1.3) and boundary conditions

$$u(0,t) = u(L,t) = 0, \tag{1.6a}$$

or

$$\begin{cases} u(0,t) = 0, \\ \varepsilon u_{xtt}(L,t) + c^2 u_x(L,t) = 0, \end{cases} \tag{1.6b}$$

where $c^2 = \frac{E}{\rho}$, $\varepsilon = 2\mu^2 k^2$. On the other hand, the asymptotic behaviour of solutions for Prob. (1.3), (1.4), (1.6) as $\varepsilon \to 0_+$ was also established by the method of small parameters.

Equations of Love waves or Love type waves have been studied by many authors, we refer to [3], [5], [6], [10], [15], [16] and references therein.

On the other hand, in [13], a symmetric version of the regularized long wave equation (SRLW)

$$\begin{cases} u_{xxt} - u_t = \rho_x + u u_x, \\ \rho_t + u_x = 0, \end{cases} \tag{1.7}$$

has been proposed to describe weakly nonlinear ion acoustic and space - charge waves. Eliminating ρ from (1.7), a class of SRLW is obtained as follows

$$u_{tt} - u_{xx} - u_{xxtt} = -u u_{xt} - u_x u_t. \tag{1.8}$$

Eq.(1.8) is explicitly symmetric in the x and t derivatives and it is very similar to the regularized long wave equation that describes shallow water waves and plasma drift waves [1], [2]. The SRLW equation also arises in many other areas of mathematical physics [4], [9], [11].

In this paper, we associate with Eq.(1.1) a recurrent sequence $\{u_m\}$ defined by

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial t^2} - \frac{\partial^2 u_m}{\partial x^2} - \frac{\partial^4 u_m}{\partial t^2 \partial x^2} + \lambda \frac{\partial u_m}{\partial t} \\ &= \sum_{i=0}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x, t, u_{m-1})(u_m - u_{m-1})^i, \quad 0 < x < 1, \quad 0 < t < T, \end{aligned} \tag{1.9}$$

with u_m satisfying (1.2), (1.3). The first term u_0 is chosen as $u_0 \equiv 0$. If $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, we prove that the sequence $\{u_m\}$ converges at rate of order N to a weak unique solution of Prob.(1.1)–(1.3). The main result is given in Theorems 2.2 and 2.6. In our proofs, the fixed point method and Faedo-Galerkin method are used.

2. A high-order iterative scheme

We put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X.

We denote by $L^p(0,T;X)$, $1 \le p \le \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty \quad \text{for} \quad 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \ \text{ for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $f = f(x,t,u)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$ and $D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} f$; $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = k$, $D^{(0,0,0)} f = f$.

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{1/2}. \tag{2.1}$$

Then the following lemma is known.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \sqrt{2} \|v\|_{H^1} \ \textit{for all } v \in H^1. \tag{2.2}$$

We put

$$V = \{v \in H^1 : v(1) = 0\}.$$

Then V is a closed subspace of H^1 and on V, $v \longmapsto ||v||_{H^1}$ and $v \longmapsto ||v_x||$ are equivalent norms. Furthermore,

$$\|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \quad \text{for all} \quad v \in V. \tag{2.3}$$

We remark that the weak formulation of the initial-boundary value problem (1.1)–(1.3) can be given in the following manner: Find $u \in L^\infty(0,T; V \cap H^2)$ with u_t, $u_{tt} \in L^\infty(0,T; V \cap H^2)$ such that u satisfies the following variational equation

$$\langle u_{tt}(t), w\rangle + \langle u_{xtt}(t) + u_x(t), w_x\rangle + \lambda\langle u_t(t), w\rangle = \langle f(x,t,u), w\rangle, \tag{2.4}$$

for all $w \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ u_t(0) = \tilde{u}_1. \tag{2.5}$$

Next, we need the following assumptions:

(A_1) $\tilde{u}_0$, $\tilde{u}_1 \in V \cap H^2$,
(A_2) $f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R})$ such that
  (i) $D_3^i f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $\ 1 \leq i \leq N-1$,
  (ii) $D_3^N f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R})$,
  (iii) $f(1,t,0) = 0$, $\ \forall\, t \geq 0$.

Consider $T^* > 0$ fixed, let $M > 0$, we put

$$\begin{aligned}
&\|f\|_{C^0(A_M)} = \sup_{(x,t,u)\in A_M} |f(x,t,u)|\,, \text{ with } A_M = [0,1]\times[0,T^*]\times[-M,M],\\
&\|f\|_{C^1(A_M)} = \|f\|_{C^0(A_M)} + \|D_1 f\|_{C^0(A_M)} + \|D_2 f\|_{C^0(A_M)} + \|D_3 f\|_{C^0(A_M)}\,,\\
&K_M(f) = \sum_{i=0}^{N-1} \left\|D_3^i f\right\|_{C^1(A_M)} + \left\|D_3^N f\right\|_{C^0(A_M)}\,.
\end{aligned} \tag{2.6}$$

For each $T \in (0,T^*]$ and $M > 0$, we put

$$\begin{cases}
W(M,T) = \Big\{v \in L^\infty(0,T; V\cap H^2) : v_t \in L^\infty(0,T; V\cap H^2),\\
\qquad\qquad\qquad v_{tt} \in L^\infty(0,T;V),\\
\text{with } \|v\|_{L^\infty(0,T;V\cap H^2)}\,,\ \|v_t\|_{L^\infty(0,T;V\cap H^2)}\,,\ \|v_{tt}\|_{L^\infty(0,T;V)} \leq M\Big\},\\
W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;V\cap H^2)\},
\end{cases} \tag{2.7}$$

where $Q_T = \Omega \times (0,T)$.

We establish the linear recurrent sequence $\{u_m\}$ as follows.

We choose the first term $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{2.8}$$

and associate with problem (1.1)-(1.3) the following problem:

Find $u_m \in W_1(M,T)$ $(m \geq 1)$ which satisfies the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + \langle u_{mx}(t) + u_{mx}''(t), w_x\rangle + \lambda\langle u_m'(t), w\rangle = \langle F_m(t), w\rangle, \ \forall w \in V, \\ u_m(0) = \tilde{u}_0, \ u_m'(0) = \tilde{u}_1, \end{cases} \tag{2.9}$$

where

$$F_m(x,t) = \sum_{i=0}^{N-1} \tfrac{1}{i!}\tfrac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i. \tag{2.10}$$

Then we have the following theorem.

Theorem 2.2. *Let $(A_1)-(A_2)$ hold. Then there exist constants $M>0, T>0$ (M depending on $\tilde{u}_0, \tilde{u}_1$ and T depending $\tilde{u}_0, \tilde{u}_1, f$) such that for $u_0 \equiv 0$, there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (2.9) *and* (2.10).

Proof. The proof consists of several steps.

Step 1. Consider the basis in $V: w_j(x) = \sqrt{\frac{2}{1+\lambda_j^2}}\cos(\lambda_j x)$, $\lambda_j = (2j-1)\frac{\pi}{2}$, $j \in \mathbb{N}$, constructed by the eigenfunctions of the Laplace operator $-\Delta = -\frac{\partial^2}{\partial x^2}$. Put $u_m^{(k)}(t) = \sum_{j=1}^k c_{mj}^{(k)}(t) w_j$, where $c_{mj}^{(k)}$ sastisfy the following system of nonlinear differential equations

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + \left\langle u_{mx}^{(k)}(t) + \ddot{u}_{mx}^{(k)}(t), w_{jx} \right\rangle + \lambda \left\langle \dot{u}_m^{(k)}(t), w_j \right\rangle \\ = \left\langle F_m^{(k)}(t), w_j \right\rangle, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \quad j = 1,2,...,k, \end{cases} \tag{2.11}$$

where

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \to \tilde{u}_0 \quad \text{strongly} \quad V \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^k \beta_j^{(k)} w_j \to \tilde{u}_1 \quad \text{strongly} \quad V \cap H^2, \end{cases} \tag{2.12}$$

and

$$F_m^{(k)}(x,t) = \sum_{i=0}^{N-1} \tfrac{1}{i!}\tfrac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m^{(k)} - u_{m-1})^i. \tag{2.13}$$

We rewrite (2.13) as follows

$$F_m^{(k)}(x,t) = \sum_{i=0}^{N-1} \Psi_i(x,t,u_{m-1})(u_m^{(k)})^i \tag{2.14}$$

with

$$\Psi_i(x,t,u_{m-1}) = \sum_{j=i}^{N-1} \tfrac{(-1)^{j-i}}{i!(j-i)!} \tfrac{\partial^j f}{\partial u^j}(x,t,u_{m-1}) u_{m-1}^{j-i}. \tag{2.15}$$

Let us suppose that u_{m-1} sastifies (2.8). Then we have following lemma.

Lemma 2.3. *Let* (A_1)-(A_2) *hold. For fixed* $M > 0$ *and* $T > 0$, *then the system* (2.11) *has unique solution* $u_m^{(k)}(t)$ *on an interval* $[0, T_m^{(k)}] \subset [0, T]$.

Proof. System (2.11) can be written in form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \sigma_j \dot{c}_{mj}^{(k)}(t) + \mu_j^2 c_{mj}^{(k)}(t) = f_{mj}^{(k)}(t), \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)},\ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)},\ 1 \le j \le k, \end{cases} \tag{2.16}$$

where

$$\begin{aligned} & f_{mj}^{(k)}(t) = \frac{1}{1+\lambda_j^2} \left\langle F_m^{(k)}(t), w_j \right\rangle, \\ & \mu_j^2 = \tfrac{\lambda_j^2}{1+\lambda_j^2},\ \sigma_j = \tfrac{\lambda}{1+\lambda_j^2},\ \lambda_j = (2j-1)\tfrac{\pi}{2},\ j \in \mathbb{N},\ 1 \le j \le k. \end{aligned} \tag{2.17}$$

System (2.16) is equivalent to system of intergal equations

$$\begin{aligned} c_{mj}^{(k)}(t) = \alpha_j^{(k)} + \frac{\beta_j^{(k)}}{\sigma_j}\left(1 - e^{-\sigma_j t}\right) - \mu_j^2 \textstyle\int_0^t d\tau \int_0^\tau e^{-\sigma_j(\tau - s)} c_{mj}^{(k)}(s) ds \\ + \textstyle\int_0^t d\tau \int_0^\tau e^{-\sigma_j(\tau-s)} f_{mj}^{(k)}(s) ds, \quad 1 \le j \le k. \end{aligned} \tag{2.18}$$

Omitting the index m, it is written as follows

$$c = F[c], \tag{2.19}$$

where $F[c] = (F_1[c], ..., F_k[c])$, $c = (c_1, .., c_k)$,

$$\begin{cases} F_j[c](t) = q_j(t) - \mu_j^2 \int_0^t d\tau \int_0^\tau e^{-\sigma_j(\tau-s)} c_j(s) ds \\ \qquad + \dfrac{1}{1+\lambda_j^2} \displaystyle\sum_{i=1}^{N-1} \int_0^t d\tau \int_0^\tau e^{-\sigma_j(\tau-s)} \left\langle \Psi_i(s, u_{m-1})(u(s))^i, w_j \right\rangle ds, \\ u(t) = \sum_{j=1}^k c_j(t) w_j, \\ q_j(t) = \alpha_j^{(k)} + \dfrac{\beta_j^{(k)}}{\sigma_j}\left(1 - e^{-\sigma_j t}\right) \\ \qquad + \dfrac{1}{1+\lambda_j^2} \int_0^t d\tau \int_0^\tau e^{-\sigma_j(\tau-s)} \langle \Psi_0(s, u_{m-1}), w_j \rangle ds,\ 1 \le j \le k. \end{cases} \tag{2.20}$$

For every $T_m^{(k)} \in (0, T]$ and $\rho > 0$ chosen later, we put

$$X = C^0\left([0, T_m^{(k)}]; \mathbb{R}^k\right), \quad S = \{c \in X : \|c\|_X \le \rho\},$$

where

$$\|c\|_X = \sup_{0\le t\le T_m^{(k)}} |c(t)|_1, \quad |c(t)|_1 = \textstyle\sum_{j=1}^k |c_j(t)|,$$

for each $c = (c_1, ..., c_k) \in X$. Clearly S is a closed nonempty subset in X and $F : X \to X$. In what follows, we shall choose $\rho > 0$ and $T_m^{(k)} > 0$ such that $F : S \to S$ is contractive.

(i) First we note that, for all $c = (c_1, .., c_k) \in S$,

$$\begin{aligned}
&\|u(t)\| \le \sum_{j=1}^k |c_j(t)|\,\|w_j\| = \sum_{j=1}^k |c_j(t)|\,\tfrac{1}{\sqrt{1+\lambda_j^2}} \le |c(t)|_1 \le ||c||_X \le \rho,\\
&\|u(t)\|_{C^0(\overline{\Omega})} \le \|u_x(t)\| \le \sum_{j=1}^k |c_j(t)|\,\|w_{jx}\|\\
&\qquad\qquad = \sum_{j=1}^k |c_j(t)| \sqrt{\tfrac{\lambda_j^2}{1+\lambda_j^2}} \le |c(t)|_1 \le ||c||_X \le \rho.
\end{aligned} \tag{2.21}$$

We have

$$|\Psi_i(x,t,u_{m-1})| \le K_M(f) \sum_{j=i}^{N-1} \tfrac{1}{i!(j-i)!} M^{j-i} \equiv \eta_i(M,\rho),\ i = \overline{0, N-1}, \tag{2.22}$$

so

$$\begin{aligned}
\left|\left\langle \Psi_i(s,u_{m-1})(u(s))^i, w_j\right\rangle\right| &\le \|\Psi_i(s,u_{m-1})\|\,\|u(s)\|^i_{C^0(\overline{\Omega})}\,\|w_j\|\\
&\le \eta_i(M,\rho)\rho^i \equiv \bar{\eta}_i(M,\rho),\ \ i = \overline{0, N-1}.
\end{aligned} \tag{2.23}$$

It follows that

$$\begin{aligned}
|F_j[c](t)| &\le |q_j(t)| + \mu_j^2 e^{|\sigma_j|T} \textstyle\int_0^t d\tau \int_0^\tau |c_j(s)|\,ds\\
&\quad + \frac{1}{1+\lambda_j^2} e^{|\sigma_j|T} \sum_{i=1}^{N-1} \textstyle\int_0^t d\tau \int_0^\tau \bar{\eta}_i(M,\rho) ds\\
&\le |q_j(t)| + \mu_k^2 e^{|\sigma_1|T} \textstyle\int_0^t d\tau \int_0^\tau |c_j(s)|\,ds\\
&\quad + \frac{1}{1+\lambda_1^2} e^{|\sigma_1|T} \sum_{i=1}^{N-1} \bar{\eta}_i(M,\rho) \frac{1}{2} \left(T_m^{(k)}\right)^2.
\end{aligned} \tag{2.24}$$

Thus

$$\begin{aligned}
|F[c](t)|_1 &\le |q(t)|_1 + e^{|\sigma_1|T}\left[\rho\mu_k^2 + \frac{k}{1+\lambda_1^2} \sum_{i=1}^{N-1} \bar{\eta}_i(M,\rho)\right] \frac{1}{2}\left(T_m^{(k)}\right)^2\\
&\le ||q||_T + \overline{D}_\rho(M)\left(T_m^{(k)}\right)^2,\ \ \forall\, t \in [0, T_m^{(k)}],
\end{aligned} \tag{2.25}$$

in which

$$\begin{aligned}
&||q||_T = \sup_{t\in[0,T]} |q(t)|_1,\\
&\overline{D}_\rho(M) = \frac{1}{2} e^{|\sigma_1|T}\left[\rho\mu_k^2 + \frac{k}{1+\lambda_1^2} \textstyle\sum_{i=1}^{N-1} \bar{\eta}_i(M,\rho)\right].
\end{aligned} \tag{2.26}$$

Hence

$$\|F[c]\|_X \leq \|q\|_T + \overline{D}_\rho(M)\left(T_m^{(k)}\right)^2. \tag{2.27}$$

(ii) We prove below that

$$\|F[c]-F[d]\|_X \leq G_\rho(M)\left(T_m^{(k)}\right)^2 \|c-d\|_X \tag{2.28}$$

with

$$G_\rho(M) = \frac{1}{2}e^{|\sigma_1|T}\left[\mu_k^2 + \frac{k}{1+\lambda_1^2}\sum_{i=1}^{N-1} i\rho^{i-1}\eta_i(M,\rho)\right]. \tag{2.29}$$

For all $j = 1, ..., k$ and $t \in [0, T_m^{(k)}]$, put

$$u(t) = \sum_{j=1}^{k} c_j(t)w_j, \quad v(t) = \sum_{j=1}^{k} d_j(t)w_j,$$

we have

$$\begin{aligned}
&|F_j[c](t) - F_j[d](t)| \\
&\leq \mu_j^2 e^{|\sigma_j|T}\int_0^t d\tau \int_0^\tau |c_j(s)-d_j(s)|\,ds \\
&\quad + \frac{1}{1+\lambda_j^2}e^{|\sigma_j|T}\sum_{i=1}^{N-1}\int_0^t d\tau\int_0^\tau \|\Psi_i(s,u_{m-1})\|\left\|u^i(s)-v^i(s)\right\|_{C^0(\overline{\Omega})}ds \\
&\leq \mu_k^2 e^{|\sigma_1|T}\int_0^t d\tau \int_0^\tau |c_j(s)-d_j(s)|\,ds \\
&\quad + \frac{1}{1+\lambda_1^2}e^{|\sigma_1|T}\sum_{i=1}^{N-1}\int_0^t d\tau\int_0^\tau \eta_i(M,\rho)\left\|u^i(s)-v^i(s)\right\|_{C^0(\overline{\Omega})}ds.
\end{aligned} \tag{2.30}$$

On the other hand

$$\begin{aligned}
&\left\|u^i(s)-v^i(s)\right\|_{C^0(\overline{\Omega})} \\
&\leq \sum_{j=0}^{i-1}\|u(s)\|_{C^0(\overline{\Omega})}^j \|v(s)\|_{C^0(\overline{\Omega})}^{i-j-1}\|u(s)-v(s)\|_{C^0(\overline{\Omega})} \\
&\leq \sum_{j=0}^{i-1}\rho^j\rho^{i-j-1}|c(s)-d(s)|_1 \leq i\rho^{i-1}\|c-d\|_X.
\end{aligned} \tag{2.31}$$

The result is

$$
\begin{aligned}
&|F[c](t)-F[d](t)|_1 \\
&\le \mu_k^2 e^{|\sigma_1|T}\|c-d\|_X \frac{1}{2}\left(T_m^{(k)}\right)^2 \\
&\quad + \frac{k}{1+\lambda_1^2} e^{|\sigma_1|T} \sum_{i=1}^{N-1} \eta_i(M,\rho) i\rho^{i-1}\|c-d\|_X \frac{1}{2}\left(T_m^{(k)}\right)^2 \\
&= \frac{1}{2}\left(T_m^{(k)}\right)^2 e^{|\sigma_1|T}\left[\mu_k^2 + \frac{k}{1+\lambda_1^2}\sum_{i=1}^{N-1} i\rho^{i-1}\eta_i(M,\rho)\right]\|c-d\|_X \\
&\equiv G_\rho(M)\left(T_m^{(k)}\right)^2\|c-d\|_X
\end{aligned}
\tag{2.32}
$$

with $G_\rho(M)$ as in (2.29). Thus, (2.28) holds. Choosing $\rho > ||q||_T$ and $T_m^{(k)} \in (0,T]$ such that

$$
0 < T_m^{(k)} \le \sqrt{\frac{\rho - ||q||_T}{\overline{D}_\rho(M)}} \quad \text{and} \quad G_\rho(M)\left(T_m^{(k)}\right)^2 < 1. \tag{2.33}
$$

Combining (2.27), (2.28) and (2.33), $F : S \longrightarrow S$ is contractive. We deduce that F has a unique fixed point in S, *i.e.*, system (2.11) has a unique solution $u_m^{(k)}(t)$ in $[0, T_m^{(k)}]$. The proof of Lemma 2.3 is completed. □

The following estimates allow one to take $T_m^{(k)} = T$ independent of m and k.

Step 2. *A priori estimates.*

Put $S_m^{(k)}(t) = p_m^{(k)}(t) + q_m^{(k)}(t) + r_m^{(k)}(t)$, where

$$
\begin{cases}
p_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|u_{mx}^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2, \\
q_m^{(k)}(t) = \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|u_{mxx}^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mxx}^{(k)}(t)\right\|^2, \\
r_m^{(k)}(t) = \left\|\ddot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(t)\right\|^2.
\end{cases}
\tag{2.34}
$$

Then, it follows from (2.11), (2.34) that

$$
\begin{aligned}
S_m^{(k)}(t) &= S_m^{(k)}(0) - 2\lambda\int_0^t \left[\left\|\dot{u}_m^{(k)}(s)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\ddot{u}_m^{(k)}(s)\right\|^2\right] ds \\
&\quad + 2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle ds + 2\int_0^t \langle F_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s)\rangle ds \\
&\quad + 2\int_0^t \langle \dot{F}_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\rangle ds \\
&= S_m^{(k)}(0) + \sum_{j=1}^{4} I_j.
\end{aligned}
\tag{2.35}
$$

We shall estimate, respectively, $S_m^{(k)}(0)$ and the following integrals on the right-hand side of (2.35).

Estimate $S_m^{(k)}(0)$. First, we estimate $\xi_m^{(k)} = \left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2$.

Letting $t \to 0_+$ in (2.11), multiplying the result by $\ddot{c}_{mj}^{(k)}(0)$, we get

$$\left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2 + \left\langle u_{mx}^{(k)}(0), \ddot{u}_{mx}^{(k)}(0)\right\rangle = \left\langle F_m^{(k)}(0), \ddot{u}_{mx}^{(k)}(0)\right\rangle.$$

This implies that

$$\begin{aligned}
\xi_m^{(k)} &= \left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2 \\
&\le \left(\left\|u_{mx}^{(k)}(0)\right\| + \left\|F_m^{(k)}(0)\right\|\right)\left\|\ddot{u}_{mx}^{(k)}(0)\right\| \\
&\le \frac{1}{2}\left(\left\|u_{mx}^{(k)}(0)\right\| + \left\|F_m^{(k)}(0)\right\|\right)^2 + \frac{1}{2}\left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2 \\
&\le \frac{1}{2}\xi_m^{(k)} + \frac{1}{2}\left(\left\|u_{mx}^{(k)}(0)\right\| + \left\|F_m^{(k)}(0)\right\|\right)^2 \\
&= \frac{1}{2}\xi_m^{(k)} + \frac{1}{2}\left(\|\tilde{u}_{0kx}\| + \left\|\sum_{i=0}^{N-1}\tfrac{1}{i!}D_3^i f(\cdot,0,\tilde{u}_0)(\tilde{u}_{0k}-\tilde{u}_0)^i\right\|\right)^2 \\
&= \frac{1}{2}\xi_m^{(k)} + \frac{1}{2}\Bigg[\|\tilde{u}_{0kx}\| \\
&\quad + \sum_{i=0}^{N-1}\tfrac{1}{i!}\sup_{0\le x\le 1,\ |z|\le\|\tilde{u}_{0x}\|}\left|D_3^i f(x,0,z)\right|\left(\|\tilde{u}_{0kx}\| + \|\tilde{u}_{0x}\|\ \right)^i\Bigg]^2.
\end{aligned} \tag{2.36}$$

Thus

$$\begin{aligned}
\xi_m^{(k)} &\le \left[\|\tilde{u}_{0kx}\| + \sum_{i=0}^{N-1}\tfrac{1}{i!}\sup_{0\le x\le 1,\ |z|\le\|\tilde{u}_{0x}\|}\left|D_3^i f(x,0,z)\right|\left(\|\tilde{u}_{0kx}\| + \|\tilde{u}_{0x}\|\ \right)^i\right]^2 \\
&\le \overline{X}_0, \quad \forall\, m,\, k \in \mathbb{N},
\end{aligned} \tag{2.37}$$

where $\overline{X}_0$ is a constant depending only on f, $\tilde{u}_0$, $\tilde{u}_1$. By (2.12), (2.34) and (2.37), we obtain

$$\begin{aligned}
&S_m^{(k)}(0) \\
&= \|\tilde{u}_{1k}\|^2 + \|\tilde{u}_{0kx}\|^2 + 3\|\tilde{u}_{1kx}\|^2 + \|\tilde{u}_{0kxx}\|^2 + \|\tilde{u}_{1kxx}\|^2 + \xi_m^{(k)} \le S_0,
\end{aligned} \tag{2.38}$$

for all m, where S_0 is a constant depending only on f, $\tilde{u}_0$, $\tilde{u}_1$.

First integral $I_1 = -2\lambda\int_0^t \left[\left\|\dot{u}_m^{(k)}(s)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\ddot{u}_m^{(k)}(s)\right\|^2\right] ds.$

We have

$$\begin{aligned} I_1 &= -2\lambda \int_0^t \left[\left\| \dot{u}_m^{(k)}(s) \right\|^2 + \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 + \left\| \ddot{u}_m^{(k)}(s) \right\|^2 \right] ds \\ &\le 2\left|\lambda\right| \int_0^t S_m^{(k)}(s) ds. \end{aligned} \tag{2.39}$$

Next, the following estimates are need.

Lemma 2.4. *We have*

$$\begin{aligned} &\text{(i)} \left\| F_m^{(k)}(t) \right\| \le \sum_{i=0}^{N-1} \tilde{b}_i \left(\sqrt{S_m^{(k)}(t)} \right)^i, \\ &\text{(ii)} \left\| F_{mx}^{(k)}(t) \right\| \le \sum_{i=0}^{N-1} \tilde{b}_i \left(\sqrt{S_m^{(k)}(t)} \right)^i, \\ &\text{(iii)} \left\| \dot{F}_m^{(k)}(t) \right\| \le \sum_{i=0}^{N-1} \tilde{b}_i \left(\sqrt{S_m^{(k)}(t)} \right)^i, \end{aligned} \tag{2.40}$$

where $\tilde{b}_i,\ i = 0, 1, .., N-1$ *are defined as follows*

$$\begin{aligned} &\tilde{b}_i = (M+N) K_M(f) \tilde{a}_i, \quad i = 0, 1, ..., N-1, \\ &\tilde{a}_i = \begin{cases} 1 + \sum_{i=1}^{N-1} \frac{2^{i-1}}{i!} M^i, & i = 0, \\ \frac{2^{i-1}}{i!}, & i = 1, 2, ..., N-1. \end{cases} \end{aligned} \tag{2.41}$$

Proof. (i) Use inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$, for all $a,\ b \ge 0,\ p \ge 1$, we have

$$\begin{aligned} \left| F_m^{(k)}(x,t) \right| &\le \sum_{i=0}^{N-1} \left| \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m^{(k)} - u_{m-1})^i \right| \\ &\le K_M(f) \left[1 + \sum_{i=1}^{N-1} \frac{1}{i!} \left(\left| u_m^{(k)} \right| + \left| u_{m-1} \right| \right)^i \right] \\ &\le K_M(f) \left[1 + \sum_{i=1}^{N-1} \frac{1}{i!} \left(\left\| u_{mx}^{(k)}(t) \right\| + M \right)^i \right] \\ &\le K_M(f) \left[1 + \sum_{i=1}^{N-1} \frac{1}{i!} \left(\sqrt{S_m^{(k)}(t)} + M \right)^i \right] \\ &\le K_M(f) \left[1 + \sum_{i=1}^{N-1} \frac{2^{i-1}}{i!} \left(\left(\sqrt{S_m^{(k)}(t)} \right)^i + M^i \right) \right] \end{aligned}$$

$$\begin{aligned} &= K_M(f)\left[1+\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}M^i+\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\left(\sqrt{S_m^{(k)}(t)}\right)^i\right] \\ &\leq \sum_{i=0}^{N-1}\tilde{b}_i\left(\sqrt{S_m^{(k)}(t)}\right)^i, \end{aligned} \tag{2.42}$$

where $\tilde{b}_i$, $i=0,1,...,N-1$ are defined as (2.41). Hence, (i) follows.
(ii) We use below notations:

$$f[u]=f(x,t,u),\quad D_jf[u]=D_jf(x,t,u),\quad j=1,2,3.$$

We have

$$\begin{aligned} \left|F_{mx}^{(k)}(x,t)\right| &\leq \left|D_1f[u_{m-1}]+D_3f[u_{m-1}]\nabla u_{m-1}\right| \\ &\quad+\sum_{i=1}^{N-1}\tfrac{1}{i!}\left|\left(D_1D_3^if[u_{m-1}]+D_3^{i+1}f[u_{m-1}]\nabla u_{m-1}\ \right)\left(u_m^{(k)}-u_{m-1}\right)^i\right| \\ &\quad+\sum_{i=1}^{N-1}\tfrac{i}{i!}\left|D_3^if[u_{m-1}]\left(u_m^{(k)}-u_{m-1}\right)^{i-1}\left(\nabla u_m^{(k)}-\nabla u_{m-1}\right)\right| \\ &\leq K_M(f)(1+M)+K_M(f)(1+M)\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(\sqrt{S_m^{(k)}(t)}+M\right)^i \\ &\quad+K_M(f)\sum_{i=1}^{N-1}\tfrac{i}{i!}\left(\sqrt{S_m^{(k)}(t)}+M\right)^{i-1}\left(\sqrt{S_m^{(k)}(t)}+M\right), \end{aligned}$$

so

$$\begin{aligned} &\left|F_{mx}^{(k)}(x,t)\right| \\ &\leq K_M(f)(1+M)+K_M(f)(1+M)\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\left(\left(\sqrt{S_m^{(k)}(t)}\right)^i+M^i\right) \\ &\quad+K_M(f)\sum_{i=1}^{N-1}i\tfrac{2^{i-1}}{i!}\left(\left(\sqrt{S_m^{(k)}(t)}\right)^i+M^i\right), \\ &\leq K_M(f)(1+M)\left[1+\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\left(\left(\sqrt{S_m^{(k)}(t)}\right)^i+M^i\right)\right] \\ &\quad+(N-1)K_M(f)\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\left(\left(\sqrt{S_m^{(k)}(t)}\right)^i+M^i\right) \\ &\leq \left[K_M(f)(1+M)+(N-1)K_M(f)\right]\left[1+\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\left(\left(\sqrt{S_m^{(k)}(t)}\right)^i+M^i\right)\right] \end{aligned}$$

$$\begin{aligned}&= (M+N)K_M(f)\left[1+\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}M^i+\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\left(\sqrt{S_m^{(k)}(t)}\right)^i\right]\\&=\sum_{i=0}^{N-1}\tilde{b}_i\left(\sqrt{S_m^{(k)}(t)}\right)^i.\end{aligned}\tag{2.43}$$

It implies that

$$\left\|F_{mx}^{(k)}(t)\right\|\leq\sum_{i=0}^{N-1}\tilde{b}_i\left(\sqrt{S_m^{(k)}(t)}\right)^i.\tag{2.44}$$

(iii) By $\dot{F}_m^{(k)}(t)$ is computed as follows

$$\begin{aligned}&\dot{F}_m^{(k)}(t)\\&=D_2f(x,t,u_{m-1})+D_3f(x,t,u_{m-1})\dot{u}_{m-1}\\&\quad+\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(D_2D_3^if(x,t,u_{m-1})+D_3^{i+1}f(x,t,u_{m-1})\dot{u}_{m-1}\right)(u_m^{(k)}-u_{m-1})^i\\&\quad+\sum_{i=1}^{N-1}\tfrac{i}{i!}D_3^if(x,t,u_{m-1})(u_m^{(k)}-u_m)^{i-1}(\dot{u}_m^{(k)}-\dot{u}_{m-1}),\end{aligned}\tag{2.45}$$

so

$$\begin{aligned}&\left|\dot{F}_m^{(k)}(t)\right|\\&\leq K_M(f)(1+M)+K_M(f)(1+M)\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(\sqrt{S_m^{(k)}(t)}+M\right)^i\\&\quad+K_M(f)\sum_{i=1}^{N-1}\tfrac{i}{i!}\left(\sqrt{S_m^{(k)}(t)}+M\right)^i\\&\leq(M+N)K_M(f)\left[1+\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(\sqrt{S_m^{(k)}(t)}+M\right)^i\right]\\&\leq(M+N)K_M(f)\left[1+\sum_{i=1}^{N-1}\tfrac{1}{i!}2^{i-1}\left(\left(\sqrt{S_m^{(k)}(t)}\right)^i+M^i\right)\right]\\&\leq(M+N)K_M(f)\left[1+\sum_{i=1}^{N-1}\tfrac{1}{i!}2^{i-1}M^i+\sum_{i=1}^{N-1}\tfrac{1}{i!}2^{i-1}\left(\sqrt{S_m^{(k)}(t)}\right)^i\right]\\&=\sum_{i=0}^{N-1}\tilde{b}_i\left(\sqrt{S_m^{(k)}(t)}\right)^i.\end{aligned}\tag{2.46}$$

Hence

$$\left\|\dot{F}_m^{(k)}(t)\right\|\leq\sum_{i=0}^{N-1}\tilde{b}_i\left(\sqrt{S_m^{(k)}(t)}\right)^i.\tag{2.47}$$

Lemma 2.4 is proved. □

Now, we estimate all intergals I_2, I_3, I_4.

Integral I_2. Using the inequality

$$x^q \leq 1 + x^N, \quad \forall\, x \geq 0, \quad \forall\, q \in [0, N], \tag{2.48}$$

we get from (2.40)−(i), that

$$\begin{aligned} I_2 &= 2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle ds \leq 2\int_0^t \left\| F_m^{(k)}(s) \right\| \left\| \dot{u}_m^{(k)}(s) \right\| ds \\ &\leq 2\sum_{i=0}^{N-1} \tilde{b}_i \int_0^t \left(\sqrt{S_m^{(k)}(s)} \right)^{i+1} ds \\ &\leq 2\sum_{i=0}^{N-1} \tilde{b}_i \int_0^t \left[1 + (S_m^{(k)}(s))^N \right] ds \\ &\leq 2\sum_{i=0}^{N-1} \tilde{b}_i \left[T + \int_0^t (S_m^{(k)}(s))^N ds \right]. \end{aligned} \tag{2.49}$$

Integral I_3. We again use inequality (2.48) and from (2.40)-(ii), we have

$$\begin{aligned} I_3 &= 2\int_0^t \langle F_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s)\rangle ds \leq 2\int_0^t \left\| F_{mx}^{(k)}(s) \right\| \left\| \dot{u}_{mx}^{(k)}(s) \right\| ds \\ &\leq 2\sum_{i=0}^{N-1} \tilde{b}_i \int_0^t \left(\sqrt{S_m^{(k)}(s)} \right)^{i} \sqrt{S_m^{(k)}(s)} ds \\ &\leq 2\sum_{i=0}^{N-1} \tilde{b}_i \int_0^t \left(1 + (S_m^{(k)}(s))^N \right) ds \\ &\leq 2\sum_{i=0}^{N-1} \tilde{b}_i \left[T + \int_0^t \left(S_m^k(s) \right)^N ds \right]. \end{aligned} \tag{2.50}$$

Integral I_4. Similarly, by (2.48) and (2.40)-(iii), we have

$$\begin{aligned} I_4 &= 2\int_0^t \langle \dot{F}_m^{(k)}(s), \ddot{u}_m^{(k)}(s)\rangle ds \leq 2\int_0^t \left\| \dot{F}_m^{(k)}(s) \right\| \left\| \ddot{u}_m^{(k)}(s) \right\| ds \\ &\leq 2\sum_{i=0}^{N-1} \tilde{b}_i \int_0^t \left(\sqrt{S_m^{(k)}(s)} \right)^{i+1} ds \\ &\leq 2\sum_{i=0}^{N-1} \tilde{b}_i \left[T + \int_0^t \left(S_m^k(s) \right)^N ds \right]. \end{aligned} \tag{2.51}$$

Combining (2.35), (2.38), (2.49)–(2.51), after arrangement and choose $M > 0$ such that

$$S_0 \leq \tfrac{M^2}{4}, \tag{2.52}$$

we have

$$S_m^{(k)}(t) \leq \tfrac{M^2}{4} + T\bar{C}_1(M) + \bar{C}_1(M)\int_0^t \left(S_m^{(k)}(s) \right)^N ds, \quad 0 \leq t \leq T, \tag{2.53}$$

where

$$\bar{C}_1(M) = 2\left(|\lambda| + 3\sum_{i=0}^{N-1}\tilde{b}_i\right). \tag{2.54}$$

Then we have the following lemma.

Lemma 2.5. *There exists constant $T > 0$ independent of k and m such that*

$$S_m^{(k)}(t) \le M^2, \quad \forall\, t \in [0,T], \quad \forall\, k,\, m \in \mathbb{N}. \tag{2.55}$$

Proof. Put

$$S(t) = \tfrac{M^2}{4} + T\bar{C}_1(M) + \bar{C}_1(M)\int_0^t \left(S_m^{(k)}(s)\right)^N ds, \quad 0 \le t \le T. \tag{2.56}$$

Clearly

$$\begin{cases} S(t) > 0,\ 0 \le S_m^{(k)}(t) \le S(t),\ 0 \le t \le T, \\ S^{'}(t) \le \bar{C}_1(M)S^N(t),\ 0 \le t \le T, \\ S(0) = M^2/4 + T\bar{C}_1(M). \end{cases} \tag{2.57}$$

Intergrating of (2.57), we have

$$\begin{aligned} S^{1-N}(t) &\ge \left[M^2/4 + T\bar{C}_1(M)\right]^{1-N} - (N-1)\,\bar{C}_1(M)t \\ &\ge \left[M^2/4 + T\bar{C}_1(M)\right]^{1-N} - (N-1)\,T\bar{C}_1(M),\ \ \forall\, t \in [0,T]. \end{aligned} \tag{2.58}$$

By

$$\begin{aligned} &\lim_{T\to 0^+}\left[\left(M^2/4 + T\bar{C}_1(M)\right)^{1-N} - (N-1)\,T\bar{C}_1(M)\right] \\ &= \left(M^2/4\right)^{1-N} > \left(M^2\right)^{1-N}, \end{aligned} \tag{2.59}$$

then, from (2.59), we always choose a constant $T > 0$ such that

$$\left(M^2/4 + T\bar{C}_1(M)\right)^{1-N} - (N-1)\,T\bar{C}_1(M) > \left(M^2\right)^{1-N}. \tag{2.60}$$

Finally, it follows from (2.57), (2.58) and (2.60), that

$$\begin{aligned} 0 &\le S_m^{(k)}(t) \le S(t) \\ &= \frac{1}{\sqrt[N-1]{\left[M^2/4+T\bar{C}_1(M)\right]^{1-N} - (N-1)\bar{C}_1(M)t}} \le M^2, \quad \forall\, t \in [0,T]. \end{aligned} \tag{2.61}$$

Lemma 2.5 is proved. □

Remark 2.6. The function

$$S(t) = \frac{1}{\sqrt[N-1]{\left[M^2/4 + T\bar{C}_1(M)\right]^{1-N} - (N-1)\,\bar{C}_1(M)t}}, \quad 0 \le t \le T,$$

is the maximal solution of the Volterra integral equation with non-decreasing kernel [8].

$$S(t)=\frac{M^2}{4}+T\bar{C}_1(M)+\bar{C}_1(M)\int_0^t S^N(s)ds,\quad 0\le t\le T. \tag{2.62}$$

By Lemma 2.5, we can take constant $T_m^{(k)}=T$ for all m and k. Therefore,

$$u_m^{(k)}\in W(M,T),\quad \text{for all}\quad m \text{ and } k. \tag{2.63}$$

From (2.63), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ still also so denoted, such that

$$\begin{cases} u_m^{(k)}\to u_m & \text{in}\quad L^\infty(0,T;V\cap H^2) \text{ weakly*},\\ \dot{u}_m^{(k)}\to u_m' & \text{in}\quad L^\infty(0,T;V\cap H^2) \text{ weakly*},\\ \ddot{u}_m^{(k)}\to u_m'' & \text{in}\quad L^\infty(0,T;V) \text{ weakly*},\\ u_m\in W(M,T). \end{cases} \tag{2.64}$$

By the compactness lemma of Lions ([7], p.57), from (2.64), there exists a subsequence of $\{u_m^{(k)}\}$, denoted by the same symbol, such that

$$\begin{cases} u_m^{(k)}\to u_m \quad \text{strongly in}\quad L^2(0,T;V) \quad \text{and a.e. in } Q_T,\\ \dot{u}_m^{(k)}\to u_m' \quad \text{strongly in}\quad L^2(0,T;V) \quad \text{and a.e. in } Q_T. \end{cases} \tag{2.65}$$

On the other hand, using the inequality

$$\left|x^j-y^j\right|\le jM^{j-1}\left|x-y\right|,\quad \forall\, x,y\in[-M,M],\ \forall\, M>0,\ \forall\, j\in\mathbb{N}, \tag{2.66}$$

we deduce from (2.63), $(2.64)_4$, that

$$\left|(u_m^{(k)})^i-(u_m)^i\right|\le iM^{i-1}\left|u_m^{(k)}-u_m\right|. \tag{2.67}$$

Therefore, (2.65) and (2.67) yield

$$(u_m^{(k)})^i\to(u_m)^i \quad \text{strongly in}\quad L^2(Q_T). \tag{2.68}$$

Hence, we deduce from (2.10), (2.14) and (2.68) that

$$F_m^{(k)}\to F_m \quad \text{strongly in}\quad L^2(Q_T). \tag{2.69}$$

Passing to limit in (2.11), (2.12), we have u_m satisfying (2.9), (2.10) in $L^2(0,T)$.

On the other hand, it follows from $(2.9)_1$ and $(2.64)_4$ that

$$\frac{\partial^2}{\partial x^2}\left(u_m''(t)+u_m(t)\right)=u_m''(t)+\lambda u_m'(t)-F_m(t)\in L^\infty(0,T;V). \tag{2.70}$$

Consequently

$$u_m''(t)+u_m(t)=\Phi\in L^\infty(0,T;V\cap H^2), \tag{2.71}$$

so

$$u_m''(t)=\Phi-u_m(t)\in L^\infty(0,T;V\cap H^2). \tag{2.72}$$

Hence $u_m \in W_1(M,T)$ and the proof of Theorem 2.2 is complete. □

We note that

$$W_T = \{u \in L^\infty(0,T;V) : u' \in L^\infty(0,T;V)\}$$

is a Banach space with respect to the norm

$$\|v\|_{W_T} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;V)}.$$

Then we have the following theorem.

Theorem 2.7. *Let* (A_1)-(A_2) *hold. Then*
(i) *Prob.*(1.1)–(1.3) *has a unique weak solution* $u \in W_1(M,T)$, *where the constants* $M > 0$ *and* $T > 0$ *are chosen as in Theorem* 2.2.
Furthermore,
(ii) *The recurrent sequence* $\{u_m\}$, *defined by* (2.9) *and* (2.10), *converges at a rate of order* N *to the solution* u *strongly in the space* W_T *in the sense*

$$\|u_m - u\|_{W_T} \le C \|u_{m-1} - u\|_{W_T}^N, \tag{2.73}$$

for all $m \ge 1$, *where* C *is a suitable constant. On the other hand, the estimate is fulfilled*

$$\|u_m - u\|_{W_T} \le C_T (k_T)^{N^m}, \quad \text{for all} \quad m \in \mathbb{N}, \tag{2.74}$$

where C_T *and* $k_T < 1$ *are the constants depending only on* T.

Proof. (a) *Existence.*

We shall prove that $\{u_m\}$ is a Cauchy sequence in W_T. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + \langle w_{mx}(t) + w_{mx}''(t), w_x\rangle = \langle F_{m+1}(t) - F_m(t), w\rangle, \forall w \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{2.75}$$

Taking $w = w_m'$ in (2.75), after integrating in t, we get

$$Z_m(t) = 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle\, ds, \tag{2.76}$$

where $Z_m(t) = \|w_m'(t)\|^2 + \|w_{mx}(t)\|^2 + \|w_{mx}'(t)\|^2$.

Using Taylor's expansion of the function $f(x,t,u_m)$ around the point u_{m-1} up to order N, we obtain

$$\begin{aligned} &f(x,t,u_m) - f(x,t,u_{m-1}) \\ &= \sum_{i=1}^{N-1} \tfrac{1}{i!} D_3^i f(x,t,u_{m-1}) w_{m-1}^i + \tfrac{1}{N!} D_3^N f(x,t,\lambda_m) w_{m-1}^N, \end{aligned} \tag{2.77}$$

where $\lambda_m = \lambda_m(x,t) = u_{m-1} + \theta_1 (u_m - u_{m-1})$, $0 < \theta_1 < 1$. Hence, it follows from (2.10) and (2.77) that

$$\begin{aligned} &F_{m+1}(x,t) - F_m(x,t) \\ &= \sum_{i=1}^{N-1} \tfrac{1}{i!} D_3^i f(x,t,u_m) w_m^i + \tfrac{1}{N!} D_3^N f(x,t,\lambda_m) w_{m-1}^N. \end{aligned} \tag{2.78}$$

So, we have

$$\|F_{m+1}(t) - F_m(t)\| \le \eta_T^{(1)} \sqrt{Z_m(t)} + \eta_T^{(2)} \left(\sqrt{Z_{m-1}(t)}\right)^N, \tag{2.79}$$

where $\eta_T^{(1)} = K_M(f) \sum_{i=1}^{N-1} \frac{M^{i-1}}{i!}$, $\eta_T^{(2)} = \frac{1}{N!} K_M(f)$. Then we deduce from (2.76) and (2.79) that

$$Z_m(t) \le T\eta_T^{(2)} \|w_{m-1}\|_{W_T}^{2N} + \left(2\eta_T^{(1)} + \eta_T^{(2)}\right) \int_0^t Z_m(s)ds. \tag{2.80}$$

By using Gronwall's Lemma, (2.80) leads to

$$\|w_m\|_{W_T} \le \mu_T \|w_{m-1}\|_{W_T}^N, \tag{2.81}$$

where $\mu_T = 2\sqrt{T\eta_T^{(2)} \exp\left(T(2\eta_T^{(1)} + \eta_T^{(2)})\right)}$. Then, it follows from (2.81) that

$$\|u_m - u_{m+p}\|_{W_T} \le (1 - k_T)^{-1} (\mu_T)^{\frac{-1}{N-1}} (k_T)^{N^m}. \tag{2.82}$$

Choosing T small enough such that $k_T = M\mu_T^{\frac{1}{N-1}} < 1$. It follows that $\{u_m\}$ is a Cauchy sequence in W_T. Then there exists $u \in W_T$ such that

$$u_m \to u \quad \text{strongly in} \quad W_T. \tag{2.83}$$

Note that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;V\cap H^2) \quad \text{weakly*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;V\cap H^2) \quad \text{weakly*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^\infty(0,T;V) \quad \text{weakly*}, \\ u \in W(M,T). \end{cases} \tag{2.84}$$

We note that

$$\|F_m(x,t) - f(\cdot,t,u(t))\| \le K_M(f) \sum_{i=1}^{N-1} \tfrac{1}{i!} \|u_m - u_{m-1}\|_{W_T}^i. \tag{2.85}$$

Hence, from (2.83) and (2.85), we obtain

$$F_m(t) \to f(\cdot,t,u(t)) \quad \text{strongly in} \quad L^\infty(0,T;L^2). \tag{2.86}$$

Finally, passing to limit in (2.9), (2.10) as $m = m_j \to \infty$, there exists $u \in W(M,T)$ satisfying the problem (2.4), (2.5).

On the other hand, by applying a similar argument used in the proof of Theorem 2.2, $u \in W_1(M,T)$ is the local unique weak solution of problem (1.1)–(1.3). Passing to the limit as $p \to +\infty$ for fixed m, we obtain the estimate (2.74) from (2.82). Theorem 2.7 is proved. □

Remark 2.8. In order to construct a N–order iterative scheme, we need the condition $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$. Then, we get a convergent sequence at a rate of order N to a local unique weak solution of problem and the existence follows. However, this condition of f can be relaxed if we only consider the existence of solution, see [6], [15], [16].

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