Nonlinear Functional Analysis and Applications
Vol. 21, No. 2 (2016), pp. 345-366

http://nfaa.kyungnam.ac.kr/journal-nfaa

EXISTENCE AND EXPONENTIAL DECAY FOR A NONLINEAR LOVE EQUATION ASSOCIATED WITH MIXED HOMOGENEOUS CONDITIONS

Le Thi Phuong Ngoc[1], Nguyen Tuan Duy[2], Tran Minh Thuyet[3], Le Khanh Luan[4] and Nguyen Anh Triet[5]

[1]University of Khanh Hoa
01 Nguyen Chanh Str., Nha Trang City, Vietnam
e-mail: ngoc1966@gmail.com

[2]Department of Fundamental sciences, University of Finance and Marketing
306 Nguyen Trong Tuyen Str., Dist. Tan Binh, HoChiMinh City, Vietnam
and
Department of Mathematics and Computer Science, University of Natural Sciences
Vietnam National University Ho Chi Minh City
227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam
e-mail: tuanduy2312@gmail.com

[3]Department of Mathematics, University of Economics of HoChiMinh City
59C Nguyen Dinh Chieu Str., Dist. 3, HoChiMinh City, Vietnam
e-mail: tmthuyet@ueh.edu.vn

[4]Department of Mathematics, University of Economics of HoChiMinh City
59C Nguyen Dinh Chieu Str., Dist. 3, HoChiMinh City, Vietnam
e-mail: luanle@ueh.edu.vn

[5]Department of Mathematics, University of Architecture of HoChiMinh City
196 Pasteur Str., Dist.3, HoChiMinh City, Vietnam
e-mail: anhtriet1@gmail.com

Abstract. In this paper, a nonlinear Love equation with a mixed homogeneous condition is studied. The uniqueness and existence of a weak solution is proved with the help of an a priori estimate and the Galerkin method. Furthermore, a new result related exponential decay of a weak solution is also established.

[0]Received November 12, 2015. Revised January 31, 2016.
[0]2010 Mathematics Subject Classification: 35L20, 35L70, 35Q74, 37B25.
[0]Keywords: Faedo-Galerkin method, nonlinear Love's equation, local existence, global existence, exponential decay.

1. Introduction

In this paper, we consider the following Love equation with initial conditions and mixed homogeneous conditions

$$\begin{aligned} &u_{tt} - u_{xx} - u_{xxtt} - \lambda_1 u_{xxt} + \lambda u_t \\ &= a\left|u\right|^{p-2} u + f(x,t),\ x \in \Omega = (0,1),\ 0 < t < T, \end{aligned} \tag{1.1}$$

$$u_x(0,t) + \lambda_1 u_{xt}(0,t) + u_{xtt}(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $a = 1$, $p > 1$, $\lambda > 0$, $\lambda_1 > 0$ are constants and $\tilde{u}_0$, $\tilde{u}_1$, f are given functions satisfying conditions specified later.

When $f = 0$, $\lambda = \lambda_1 = a = 0$, $\Omega = (0, L)$, Eq. (1.1) is related to the Love equation

$$u_{tt} - \tfrac{E}{\rho} u_{xx} - 2\mu^2\omega^2 u_{xxtt} = 0, \tag{1.4}$$

presented by V. Radochová in 1978 (see [21]). This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy functional

$$\int_0^T dt \int_0^L \left[\tfrac{1}{2} F\rho \left(u_t^2 + \mu^2\omega^2 u_{tx}^2\right) - \tfrac{1}{2} F \left(E u_x^2 + \rho\mu^2\omega^2 u_x u_{xtt}\right)\right] dx. \tag{1.5}$$

The parameters in (1.5) have the following meaning: u is the displacement, L is the length of the rod, F is the area of cross-section, ω is the cross-section radius, E is the Young modulus of the material and ρ is the mass density. By using the Fourier method, Radochová [21] obtained a classical solution of Prob. (1.4) associated with initial condition (1.3) and boundary conditions

$$u(0,t) = u(L,t) = 0, \tag{1.6a}$$

or

$$\begin{cases} u(0,t) = 0, \\ \varepsilon u_{xtt}(L,t) + c^2 u_x(L,t) = 0, \end{cases} \tag{1.6b}$$

where $c^2 = \frac{E}{\rho}$, $\varepsilon = 2\mu^2\omega^2$. On the other hand, the asymptotic behaviour of the solution of Prob. (1.3), (1.4), (1.6a) or (1.6b) as $\varepsilon \to 0_+$ is also established by the method of small parameter.

Equations of Love waves or Love type waves have been studied by many authors, we refer to [5], [8], [15], [16], [20] and references therein.

In [16], by combining the linearization method for the nonlinear term, the Faedo-Galerkin method and the weak compactness method, the existence of a unique weak solution of a Dirichlet problem for the nonlinear Love equation $u_{tt} - u_{xx} - u_{xxtt} = f(x, t, u, u_x, u_t, u_{xt})$ is proved.

In [22], a symmetric version of the regularized long wave equation (SRLWE)

$$\begin{cases} u_{xxt} - u_t = \rho_x + uu_x, \\ \rho_t + u_x = 0, \end{cases} \tag{1.7}$$

has been proposed as a model for propagation of weakly nonlinear ion acoustic and space-charge waves. Obviously, eliminating ρ from (1.7), we get

$$u_{tt} - u_{xx} - u_{xxtt} = -uu_{xt} - u_x u_t. \tag{1.8}$$

The SRLWE (1.8) is explicitly symmetric in the x and t derivatives and it is very similar to the regularized long wave equation which describes shallow water waves and plasma drift waves [1], [2]. The SRLWE also arises in many other areas of mathematical physics [6], [13], [19]. We remark Eq. (1.1) and Eq. (1.8) are special forms of the equation discussed in [16].

The purpose of this paper is establishing the existence and exponential decay of weak solutions for Prob. (1.1)–(1.3). To our knowledge, there is not any decay result for equations of Love waves or Love type waves. However, the existence and exponential decay of weak solutions for the wave equation

$$u_{tt} - \Delta u = f(x, t, u, u_t), \tag{1.9}$$

with the different boundary conditions, have been extensively studied by many authors, for example, we refer to [4], [14], [17], [18] and references therein. In [4], the following problem was considered

$$\begin{cases} u_{tt} - \Delta u + g(u_t) + f(u) = 0, \ \ x \in \Omega, \ \ t > 0, \\ u = 0, \ \ x \in \partial\Omega, \ \ t \geq 0, \\ u(x,0) = \tilde{u}_0(x), \, u_t(x,0) = \tilde{u}_1(x), \ \ x \in \Omega, \end{cases} \tag{1.10}$$

where $f(u) = -b|u|^{p-2}u$, $g(u_t) = a(1 + |u_t|^{m-2})u_t$, a, $b > 0$, m, $p > 2$ and Ω is a bounded domain of $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$. Benaissa and Messaoudi showed that for suitably chosen initial data, (1.10) possesses a global weak solution, which decays exponentially even if $m > 2$. Nakao and Ono [14] extended previous results to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \lambda^2(x)u + g(u_t) + f(u) = 0, \ \ x \in \mathbb{R}^N, \ \ t > 0, \\ u(x,0) = \tilde{u}_0(x), \, u_t(x,0) = \tilde{u}_1(x), \ \ x \in \mathbb{R}^N, \end{cases} \tag{1.11}$$

where $g(u_t)$ behaves like $|u_t|^{m-2}u_t$, $f(u)$ behaves like $-|u|^{p-2}u$ and the initial data $(\tilde{u}_0, \tilde{u}_1)$ is small enough in $H^1(\Omega) \times L^2(\Omega)$. In [17], the existence and exponential decay for the following nonlinear wave equation

$$u_{tt} - u_{xx} + Ku + \lambda u_t = a\,|u|^{p-2}\,u + f(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1.12}$$

with a nonlocal boundary condition, in cases $a = 1$, $a = -1$, were also established.

Motivated by results for Love equations in [15], [16] and based on the use of Faedo – Galerkin method, Lyapunov's method as in [17], we show that a decay result can be obtained for Prob. (1.1)–(1.3). Our paper is organized as follows. In Section 2, we present preliminaries, where two existence results are proved via using Faedo–Galerkin method. In Section 3, the decay of solutions for (1.1)–(1.3) is investigated by the construction of a suitable Lyapunov functional, with respect to $\lambda > 0$, $\lambda_1 > 0$ and $p > 2$. We show that if $\|\tilde{u}_{0x}\|^2 - \|\tilde{u}_0\|_{L^p}^p > 0$ and if the initial energy $E(0)$, $\|f(t)\|$ are small enough, then the energy $E(t)$ of the solution decays exponentially as $t \to +\infty$.

Finally, we give a remark in case of $a = -1$, it means that we will consider (1.1) in the form

$$\begin{aligned} &u_{tt} - u_{xx} - u_{xxtt} - \lambda_1 u_{xxt} + \lambda u_t + |u|^{p-2}\, u \\ &= f(x,t), \quad 0 < x < 1, \quad t > 0. \end{aligned} \tag{1.13}$$

With suitable conditions for f, we obtain a unique global solution for (1.2), (1.3) and (1.13), with energy decaying exponentially as $t \to +\infty$, without any restrictions on the data size $(\tilde{u}_0, \tilde{u}_1)$.

2. Existence of solutions

First, we put $\Omega = (0,1)$; $Q_T = \Omega \times (0,T)$, $T > 0$ and we denote the usual function spaces used in this paper by the notations $C^m\left(\overline{\Omega}\right)$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, ...$ Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \quad \text{for} \quad p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1, we shall use the following norm

$$||v||_{H^1} = \left(||v||^2 + ||v_x||^2 \right)^{1/2}. \tag{2.1}$$

Then the following lemma is known.

Lemma 2.1. *The imbedding* $H^1 \hookrightarrow C^0(\overline{\Omega})$ *is compact and*

$$||v||_{C^0(\overline{\Omega})} \leq \sqrt{2}||v||_{H^1} \quad \textit{for all} \quad v \in H^1. \tag{2.2}$$

We put

$$V = \{v \in H^1 : v(1) = 0\}.$$

Then V is a closed subspace of H^1 and on V, $v \longmapsto ||v||_{H^1}$ and $v \longmapsto ||v_x||$ are equivalent norms. Furthermore,

$$||v||_{C^0(\overline{\Omega})} \leq ||v_x|| \quad \text{for all} \quad v \in V. \tag{2.3}$$

Now, we shall consider Prob. (1.1)–(1.3) in case $a = 1$, $\lambda > 0$, $p > 2$ and establish two local existence theorems. We note that the weak formulation of Prob. (1.1)–(1.3) can be given as follows.

Find $u \in L^\infty(0, T; V)$ with $u_t \in L^\infty(0, T; V)$, such that u satisfies the following equation

$$\begin{aligned} &\tfrac{d}{dt}\left[\langle u_t(t), v\rangle + \langle u_{xt}(t) + \lambda_1 u_x(t), v_x\rangle\right] + \langle u_x(t), v_x\rangle + \lambda \langle u_t(t), v\rangle \\ &= \left\langle |u(t)|^{p-2}u(t), v\right\rangle + \langle f(t), v\rangle, \end{aligned}$$

for all $v \in V$, a.e. $t \in (0, T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \; u_t(0) = \tilde{u}_1.$$

We need the following assumptions:

(A_1) $p > 2$, $\lambda > 0$, $\lambda_1 > 0$;

(A_2) $f \in L^\infty(0, T; L^2)$, $f' \in L^1(0, T; L^2)$;

(A_2') $f \in L^2(Q_T)$.

The first theorem about the existence of a "strong solution" as follows.

Theorem 2.2. *Suppose that* (A_1), (A_2) *hold. Let* $\tilde{u}_0,\ \tilde{u}_1 \in V \cap H^2$. *Then Prob.* (1.1)–(1.3) *has a unique local solution*

$$u,\ u_t,\ u_{tt} \in L^\infty\left(0, T_*; V \cap H^2\right), \tag{2.4}$$

for $T_* > 0$ *small enough.*

Remark 2.3. The regularity in (2.4) implies that Prob. (1.1)–(1.3) has a unique strong solution

$$u \in C^1\left([0, T_*]; V \cap H^2\right), \;\; u_{tt} \in L^\infty\left(0, T_*; V \cap H^2\right). \tag{2.5}$$

With less regular initial datas, we will obtain a local weak solution as follows.

Theorem 2.4. *Suppose that* (A_1), (A_2') *hold. Let* $\tilde{u}_0,\ \tilde{u}_1 \in V$. *Then Prob.* (1.1)–(1.3) *has a unique local solution*

$$u \in C^1\left([0,T_*];V\right), \tag{2.6}$$

for $T_* > 0$ *small enough.*

Proof. In the following, we prove Theorem 2.2. The proof is a combination of Galerkin method and compactness arguments, and consits of four steps.

Step 1. *The Faedo-Galerkin approximation*(introduced by Lions [12]).

Consider the basis in

$$V : w_j(x) = \sqrt{\frac{2}{1+\lambda_j^2}}\cos(\lambda_j x), \quad \lambda_j = (2j-1)\frac{\pi}{2}, \quad j \in \mathbb{N},$$

constructed by the eigenfunctions of the Laplace operator $-\Delta = -\frac{\partial^2}{\partial x^2}$. We find the approximate solution of Prob. (1.1)–(1.3) in the form

$$u_m(t) = \textstyle\sum_{j=1}^m c_{mj}(t) w_j, \tag{2.7}$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations

$$\begin{cases} \langle u_m''(t), w_j\rangle + \langle u_{mx}''(t) + \lambda_1 u_{mx}'(t) + u_{mx}(t), w_{jx}\rangle + \lambda\,\langle u_m'(t), w_j\rangle \\ = \langle |u_m(t)|^{p-2}\, u_m(t), w_j\rangle + \langle f(t), w_j\rangle\,, \quad 1 \le j \le m, \\ u_m(0) = \tilde{u}_0, \ \ u_m'(0) = \tilde{u}_1. \end{cases} \tag{2.8}$$

From the assumptions of Theorem 2.2, (2.8) has a solution u_m on an interval $[0,T_m] \subset [0,T]$.

Step 2. *The first estimate.*

Multiplying the j^{th} equation of (2.8) by $c_{mj}'(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, after some rearrangements, we get

$$\begin{aligned} &S_m(t) \\ &= S_m(0) + 2\textstyle\int_0^t \langle f(s), u_m'(s)\rangle\, ds + 2\int_0^t \left\langle |u_m(s)|^{p-2}\, u_m(s), u_m'(s)\right\rangle ds, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} S_m(t) = {} & \|u_m'(t)\|^2 + \|u_{mx}'(t)\|^2 + \|u_{mx}(t)\|^2 \\ & + 2\lambda_1 \textstyle\int_0^t \|u_{mx}'(s)\|^2\, ds + 2\lambda \int_0^t \|u_m'(s)\|^2\, ds, \end{aligned} \tag{2.10}$$

$$S_m(0) = \|\tilde{u}_1\|^2 + \|\tilde{u}_{1x}\|_1^2 + \|\tilde{u}_0\|_1^2 \equiv \bar{S}_0. \tag{2.11}$$

Use the following inequalities

$$\|u_m(t)\| \le \|u_m(t)\|_{C^0(\overline{\Omega})} \le \|u_{mx}(t)\| \le \sqrt{S_m(t)}, \tag{2.13}$$

we can estimate all terms in the right – hand side of (2.9) and obtain

$$\begin{aligned} &S_m(t) \\ &\leq \bar{S}_0 + \|f\|^2_{L^2(Q_T)} + \int_0^t S_m(s)\,ds + \int_0^t S_m^{p/2}(s)\,ds, \;\; 0 \leq t \leq T_m. \end{aligned} \tag{2.14}$$

Then by solving a nonlinear Volterra integral inequality (on the basis of the methods in [9]), we obtain the following lemma.

Lemma 2.5. *There exists a constant $T_* > 0$ depending on T (independent of m) such that*

$$S_m(t) \leq C_T, \quad \forall\, m \in \mathbb{N}, \quad \forall\, t \in [0, \;\; T_*], \tag{2.15}$$

where C_T is a constant depending only on T.

By Lemma 2.5, we can take constant $T_m = T_*$ for all m.

The second estimate.

Now, by differentiating $(2.8)_1$ with respect to t and substituting $w_j = u''_m(t)$, after integrating with respect to the time variable from 0 to t, we have

$$\begin{aligned} X_m(t) &= X_m(0) + 2\int_0^t \langle f'(s), u''_m(s)\rangle\,ds \\ &\quad +2(p-1)\int_0^t \left\langle |u_m(s)|^{p-2}\, u'_m(s), u''_m(s)\right\rangle ds, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} X_m(t) &= \|u''_m(t)\|^2 + \|u''_{mx}(t)\|^2 + \|u'_{mx}(t)\|^2 \\ &\quad +2\lambda_1 \int_0^t \|u''_{mx}(s)\|^2\,ds + 2\lambda\int_0^t \|u''_m(s)\|^2\,ds. \end{aligned} \tag{2.17}$$

First, we estimate $\eta_m = \|u''_m(0)\|^2 + \|u''_{mx}(0)\|^2$. Letting $t \to 0_+$ in Eq. $(2.8)_1$ and multiplying the result by $c''_{mj}(0)$ lead to

$$\begin{aligned} &\|u''_m(0)\|^2 + \langle u''_{mx}(0) + \lambda_1 \tilde{u}_{1x} + \tilde{u}_{0x}, u''_{mx}(0)\rangle + \lambda\,\langle \tilde{u}_1, u''_m(0)\rangle \\ &= \langle |\tilde{u}_0|^{p-2}\,\tilde{u}_0, u''_m(0)\rangle + \langle f(0), u''_m(0)\rangle\,. \end{aligned} \tag{2.18}$$

This implies that

$$\begin{aligned} \eta_m &= \|u''_m(0)\|^2 + \|u''_{mx}(0)\|^2 \\ &= -\langle \lambda_1 \tilde{u}_{1x} + \tilde{u}_{0x}, u''_{mx}(0)\rangle - \lambda\,\langle \tilde{u}_1, u''_m(0)\rangle \\ &\quad + \langle |\tilde{u}_0|^{p-2}\,\tilde{u}_0, u''_m(0)\rangle + \langle f(0), u''_m(0)\rangle \\ &\leq \|\lambda_1 \tilde{u}_{1x} + \tilde{u}_{0x}\|\,\|u''_{mx}(0)\| + \left[\lambda\,\|\tilde{u}_1\| + \left\||\tilde{u}_0|^{p-1}\right\| + \|f(0)\|\right]\|u''_m(0)\| \\ &\leq \left[\|\lambda_1 \tilde{u}_{1x} + \tilde{u}_{0x}\| + \lambda\,\|\tilde{u}_1\| + \left\||\tilde{u}_0|^{p-1}\right\| + \|f(0)\| \;\;\right]\sqrt{\eta_m} \\ &\leq \left[\|\lambda_1 \tilde{u}_{1x} + \tilde{u}_{0x}\| + \lambda\,\|\tilde{u}_1\| + \left\||\tilde{u}_0|^{p-1}\right\| + \|f(0)\| \;\;\right]^2 \\ &\equiv \tilde{X}_0 \quad \text{for all} \quad m, \end{aligned} \tag{2.19}$$

where $\tilde{X}_0$ is a constant depending only on p, λ, λ_1, $\tilde{u}_0$, $\tilde{u}_1$, f. By (2.17), (2.19), we get

$$\begin{aligned} X_m(0) &= \|u_m''(0)\|^2 + \|u_{mx}''(0)\|^2 + \|u_{mx}'(0)\|^2 \\ &= \eta_m + \|\tilde{u}_{1x}\|^2 \leq \tilde{X}_0 + \|\tilde{u}_{1x}\|^2 \equiv \overline{X}_0 \end{aligned} \tag{2.20}$$

Noting (2.15), (2.20) and the following inequalities

$$\left\|u_m'(t)\right\|_{C^0(\overline{\Omega})} \leq \left\|u_{mx}'(t)\right\| \leq \sqrt{X_m(t)},\ i = 1, 2, \tag{2.21}$$

we continue to estimate all terms in the right–hand side of (2.16), which gives

$$\begin{aligned} X_m(t) \leq \overline{X}_0 &+ \|f'\|_{L^1(0,T;L^2)} + (p-1)^2 C_T^{p-1} \\ &+ \int_0^t (1 + \|f'(s)\|)\, X_m(s)\, ds, \quad 0 \leq t \leq T_*. \end{aligned} \tag{2.22}$$

By Gronwall's lemma, we deduce from (2.22) that

$$\begin{aligned} &X_m(t) \\ &\leq \left(\overline{X}_0 + \|f'\|_{L^1(0,T;L^2)} + (p-1)^2 C_T^{p-1}\right) \exp\left[\int_0^T (1 + \|f'(s)\|)\, ds\right] \\ &\leq \bar{C}_T, \end{aligned} \tag{2.23}$$

for all $t \in [0, T_*]$, where $\bar{C}_T$ always indicates a bound depending on T.

Step 3. *Limiting process.*

From (2.10), (2.15), (2.17), (2.23), we deduce the existence of a subsequence of $\{u_m\}$, denoted by the same symbol, such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0, T_*; V) \quad \text{weakly*}, \\ u_m' \to u' & \text{in} \quad L^\infty(0, T_*; V) \quad \text{weakly*}, \\ u_m'' \to u'' & \text{in} \quad L^\infty(0, T_*; V) \quad \text{weakly*}. \end{cases} \tag{2.24}$$

By the compactness lemma of Lions ([12], p.57), we can deduce from (2.24) the existence of a subsequence still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to u & \text{strongly in} \quad L^2(Q_{T_*}) \quad \text{and a.e. in} \quad Q_{T_*}, \\ u_m' \to u' & \text{strongly in} \quad L^2(Q_{T_*}) \quad \text{and a.e. in} \quad Q_{T_*}. \end{cases} \tag{2.25}$$

Using the following inequality

$$\left|\, |x|^{p-2}x - |y|^{p-2}y \right| \leq (p-1) M^{p-2} \left| x - y \right|, \tag{2.26}$$

$\forall\, x, y \in [-M, M]$, $\forall\, M > 0$, $\forall\, p \geq 2$ with $M = \sqrt{2C_T}$, we deduce from (2.15) that

$$\left|\, |u_m|^{p-2}u_m - |u|^{p-2}u \right| \leq (p-1) M^{p-2} \left| u_m - u \right|, \tag{2.27}$$

for all m, $(x,t) \in Q_{T_*}$. Hence, by $(2.25)_1$, (2.27) implies

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{strongly in} \quad L^2(Q_{T_*}). \tag{2.28}$$

Passing to the limit in (2.8) by (2.24), (2.25) and (2.28), we have u satisfying the problem

$$\begin{cases} \langle u''(t), v\rangle + \langle u''_x(t) + \lambda_1 u'_x(t) + u_x(t), v_x\rangle + \lambda \langle u'(t), v\rangle \\ = \left\langle |u(t)|^{p-2}u(t), v\right\rangle + \langle f(t), v\rangle\,, \quad \text{for all } v \in V, \\ u(0) = \tilde{u}_0, \;\; u'(0) = \tilde{u}_1. \end{cases} \tag{2.29}$$

On the other hand, we have from (2.24), $(2.29)_1$ that

$$\tfrac{\partial^2}{\partial x^2}\left(u'' + \lambda_1 u' + u\right) = u'' + \lambda u' - |u|^{p-2}u - f \in L^\infty(0, T_*; L^2). \tag{2.30}$$

Therefore

$$u'' + \lambda_1 u' + u = \Phi \in L^\infty(0, T_*; V \cap H^2). \tag{2.31}$$

In order to continue the proof, now we deduce from (2.31) that, if

$$u \in L^\infty(0, T_*; V \cap H^2), \tag{2.32}$$

then

$$u',\, u'' \in L^\infty(0, T_*; V \cap H^2). \tag{2.33}$$

Indeed, let (2.31), (2.32) hold, we have

$$u'' + \lambda_1 u' = \Phi - u \equiv \Phi_1 \in L^\infty(0, T_*; V \cap H^2). \tag{2.34}$$

Integrating (2.34) leads to

$$u' + \lambda_1 u = \tilde{u}_1 + \lambda_1 \tilde{u}_0 + \textstyle\int_0^t \Phi_1(s) ds \equiv \Phi_2 \in L^\infty(0, T_*; V \cap H^2). \tag{2.35}$$

Hence

$$u' = \Phi_2 - \lambda_1 u \in L^\infty(0, T_*; V \cap H^2). \tag{2.36}$$

It follows from (2.31) that

$$u'' = -\lambda_1 u' - u + \Phi \in L^\infty(0, T_*; V \cap H^2). \tag{2.37}$$

We will prove that (2.32) holds as below. We consider three cases for λ_1.

Case 1. $\lambda_1 = 2$. By (2.31), we have

$$\begin{aligned} u(t) &= \tilde{u}_0 e^{-t} + \left(\tilde{u}_0 + \tilde{u}_1\right) t e^{-t} + \textstyle\int_0^t (t-s) e^{s-t} \Phi(s) ds \\ &\in L^\infty(0, T_*; V \cap H^2). \end{aligned} \tag{2.38}$$

Case 2. $\lambda_1 > 2$. Put $k_1 = \frac{-\lambda_1 + \sqrt{\lambda_1^2 - 4}}{2}$, $\;k_2 = \frac{-\lambda_1 - \sqrt{\lambda_1^2 - 4}}{2}$, we have from (2.31) that

$$\begin{aligned} u(t) = &\tfrac{1}{\sqrt{\lambda_1^2 - 4}} \left[\left(\tilde{u}_1 - k_2 \tilde{u}_0\right) e^{k_1 t} - \left(\tilde{u}_1 - k_1 \tilde{u}_0\right) e^{k_2 t}\right] \\ &+ \tfrac{1}{\sqrt{\lambda_1^2 - 4}} \textstyle\int_0^t \left(e^{k_1 (t-s)} - e^{k_2 (t-s)}\right) \Phi(s) ds \\ &\in L^\infty(0, T_*; V \cap H^2). \end{aligned} \tag{2.39}$$

Case 3. $0 < \lambda_1 < 2$. Put $\alpha = \frac{-\lambda_1}{2}$, $\beta = \frac{\sqrt{4-\lambda_1^2}}{2}$, it follows from (2.31) that

$$\begin{aligned} u(t) &= \tilde{u}_0 e^{\alpha t} \cos \beta t + \tfrac{1}{\beta} \left(\tilde{u}_1 - \alpha \tilde{u}_0 \right) e^{\alpha t} \sin \beta t \\ &\quad + \tfrac{1}{\beta} \int_0^t e^{\alpha (t-s)} \sin \left(\beta t (t-s) \right) \Phi(s) ds \\ &\in L^\infty(0, T_*; V \cap H^2). \end{aligned} \tag{2.40}$$

Thus u, u', $u'' \in L^\infty(0, T_*; V \cap H^2)$ and the existence of the solution is proved completely.

Step 4. *Uniqueness of the solution.*

Let u_1, u_2 be two weak solutions of Prob. (1.1)–(1.4), such that

$$u_i, \ u_i', \ u_i'' \in L^\infty(0, T_*; V \cap H^2), \quad i = 1, 2. \tag{2.41}$$

Then $w = u_1 - u_2$ verifies

$$\begin{cases} \langle w''(t), v \rangle + \langle w_x''(t) + \lambda_1 w_x'(t) + w_x(t), v_x \rangle + \lambda \langle w'(t), v \rangle \\ = \left\langle |u_1(t)|^{p-2} u_1(t) - |u_2(t)|^{p-2} u_2(t), v \right\rangle, \quad \text{for all} \quad v \in V, \\ w(0) = w'(0) = 0. \end{cases} \tag{2.42}$$

Taking $v = w = u_1 - u_2$ in (2.42) and integrating with respect to t, we obtain

$$\sigma(t) = 2 \int_0^t \left\langle |u_1(s)|^{p-2} u_1(s) - |u_2(s)|^{p-2} u_2(s), w'(s) \right\rangle ds, \tag{2.43}$$

where

$$\begin{aligned} \sigma(t) &= \left\| w'(t) \right\|^2 + \left\| w_x'(t) \right\|^2 + \left\| w_x(t) \right\|^2 \\ &\quad + 2\lambda_1 \int_0^t \left\| w_x'(s) \right\|^2 ds + 2\lambda \int_0^t \left\| w'(s) \right\|^2 ds. \end{aligned} \tag{2.44}$$

Using again the inequality (2.26), with $M = M_1 = \max_{i=1,2} \|u_i\|_{L^\infty(0,T_*;V)}$, we deduce that

$$\begin{aligned} \sigma(t) &= 2 \int_0^t \left\langle |u_1(s)|^{p-2} u_1(s) - |u_2(s)|^{p-2} u_2(s), w'(s) \right\rangle ds \\ &\leq 2(p-1) M_1^{p-2} \int_0^t \|w(s)\| \, \|w'(s)\| \, ds \\ &\leq (p-1) M_1^{p-2} \int_0^t \sigma(s) ds. \end{aligned} \tag{2.45}$$

By Gronwall's Lemma, it follows from (2.45) that $\sigma \equiv 0$, *i.e.*, $u_1 \equiv u_2$. Theorem 2.2 is proved completely. □

Next, we prove Theorem 2.4.

Proof. In order to obtain the existence of a weak solution, we use standard arguments of density.

Assume $(\tilde{u}_0, \tilde{u}_1, f) \in V \times V \times L^2(Q_T)$. Let sequences $\{(u_{0m}, u_{1m}, f_m)\} \subset \left(V \cap H^2\right) \times \left(V \cap H^2\right) \times C_0^\infty\left(\overline{Q}_T\right)$, such that

$$\begin{cases} u_{0m} \to \tilde{u}_0 & \text{strongly in} \quad V, \\ u_{1m} \to \tilde{u}_1 & \text{strongly in} \quad V, \\ f_m \to f & \text{strongly in} \quad L^2(Q_T). \end{cases} \tag{2.46}$$

Then, for each $m \in \mathbb{N}$, there exists a unique function u_m as in the Theorem 2.4. Thus, we can verify

$$\begin{cases} \langle u_m''(t), v\rangle + \langle u_{mx}''(t) + \lambda_1 u_{mx}'(t) + u_{mx}(t), v_x\rangle + \lambda \langle u_m'(t), v\rangle \\ = \left\langle |u_m(t)|^{p-2} u_m(t), v\right\rangle + \langle f_m(t), v\rangle, \quad \text{for all} \quad v \in V, \\ u_m(0) = u_{0m}, \;\; u_m'(0) = u_{1m}, \end{cases} \tag{2.47}$$

and

$$u_m \in C^1\left([0, T_*]; V \cap H^2\right), \;\; u_m'' \in L^\infty\left(0, T_*; V \cap H^2\right). \tag{2.48}$$

In the same way to obtain estimates as above, we get

$$\begin{aligned} &\|u_m'(t)\|^2 + \|u_{mx}'(t)\|^2 + \|u_{mx}(t)\|^2 \\ &+2\lambda_1 \int_0^t \|u_{mx}'(s)\|^2 ds + 2\lambda \int_0^t \|u_m'(s)\|^2 ds \le C_T, \quad \forall\, t \in [0, T_*], \end{aligned} \tag{2.49}$$

where C_T is a positive constant independent of m and t.

On the other hand, put $w_{m,l} = u_m - u_l$, $f_{m,l} = f_m - f_l$, it follows from (2.47) that

$$\begin{cases} \left\langle w_{m,l}''(t), v\right\rangle + \left\langle w_{m,lx}''(t) + \lambda_1 w_{m,lx}'(t) + w_{m,lx}(t), v_x\right\rangle + \lambda \left\langle w_{m,l}'(t), v\right\rangle \\ = \left\langle |u_m|^{p-2} u_m - |u_l|^{p-2} u_l, v\right\rangle + \langle f_{m,\,l}(t), v\rangle, \quad \text{for all} \quad v \in V, \\ w_{m,l}(0) = u_{0m} - u_{0l}, \quad w_{m,l}'(0) = u_{1m} - u_{1l}. \end{cases} \tag{2.50}$$

Taking $v = w_{m,l} = u_m - u_l$ in (2.50) and integrating with respect to t, we obtain

$$\begin{aligned} S_{m,l}(t) = S_{m,l}(0) &+ 2 \int_0^t \left\langle f_{m,l}(s), w_{m,l}'(s)\right\rangle ds \\ &+2 \int_0^t \left\langle |u_m(s)|^{p-2} u_m(s) - |u_l(s)|^{p-2} u_l(s), w_{m,l}'(s)\right\rangle ds, \end{aligned} \tag{2.51}$$

where

$$\begin{aligned} S_{m,l}(t) = &\left\|w_{m,l}'(t)\right\|^2 + \left\|w_{m,l\,x}'(t)\right\|^2 + \|w_{m,l\,x}(t)\|^2 \\ &+2\lambda_1 \int_0^t \left\|w_{m,lx}'(s)\right\|^2 ds + 2\lambda \int_0^t \left\|w_{m,l}'(s)\right\|^2 ds, \end{aligned} \tag{2.52}$$

$$\begin{aligned} S_{m,l}(0) = &\|u_{1m} - u_{1l}\|^2 + \|u_{1m\,x} - u_{1l\,x}\|^2 + \|u_{0m\,x} - u_{0l\,x}\|^2 \\ &\to 0, \quad \text{as } m,\, l \to \infty. \end{aligned} \tag{2.53}$$

Hence

$$S_{m,l}(t) \leq S_{m,l}(0) + \|f_{m,l}\|^2_{L^2(Q_T)} + \left(1 + (p-1)C_T^{p-2}\right) \int_0^t S_{m,l}(s)ds. \quad (2.54)$$

By Gronwall's Lemma, it follows from (2.54), that

$$\begin{aligned} S_{m,l}(t) &\leq \left[S_{m,l}(0) + \|f_{m,l}\|^2_{L^2(Q_T)}\right] \exp\left[\left(1 + (p-1)C_T^{p-2}\right) t\right] \\ &\leq \hat{C}_T \left[S_{m,l}(0) + \|f_{m,l}\|^2_{L^2(Q_T)}\right], \quad \forall\, t \in [0, T_*]. \end{aligned} \quad (2.55)$$

Note that the convergence of the sequence $\{(u_{0m}, u_{1m}, f_m)\}$ implies the convergence to zero (when $m,\, l \to \infty$) of terms on the right hand side of (2.55). Therefore, we get

$$u_m \to u \quad \text{strongly in} \quad C^1([0, T_*]; V). \quad (2.56)$$

On the other hand, from (2.49), we deduce the existence of a subsequence of $\{u_m\}$, still so denoted, such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0, T_*; V) \quad \text{weakly*}, \\ u'_m \to u' & \text{in} \quad L^\infty(0, T_*; V) \quad \text{weakly*}. \end{cases} \quad (2.57)$$

By the compactness lemma of Lions ([12], p.57) we can deduce from (2.57) the existence of a subsequence still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{strongly} \quad \text{in} \quad L^2(Q_{T*}) \text{ and a.e. in } Q_{T_*}. \quad (2.58)$$

Similarly, by (2.27), we deduce from (2.58), that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{strongly} \quad \text{in} \quad L^2(Q_{T_*}). \quad (2.59)$$

Passing to the limit in (2.47) by (2.56)–(2.59), we have u satisfying the problem

$$\begin{cases} \frac{d}{dt}\left[\langle u'(t), v\rangle + \langle u'_x(t) + \lambda_1 u_x(t), v_x\rangle\right] + \langle u_x(t), v_x\rangle + \lambda \langle u'(t), v\rangle \\ = \left\langle |u(t)|^{p-2}u(t), v\right\rangle + \langle f(t), v\rangle, \quad \text{for all} \quad v \in V, \\ u(0) = \tilde{u}_0, \;\; u'(0) = \tilde{u}_1. \end{cases} \quad (2.60)$$

Next, the uniqueness of a weak solution is obtained by using the well-known regularization procedure due to Lions [10]. See for example Ngoc *et al.* [15]. Theorem 2.4 is proved completely. □

Remark 2.6. In case $1 < p \leq 2$, $f \in L^2(Q_T)$ and $\tilde{u}_0,\, \tilde{u}_1 \in V$, the integral inequality (2.14) leads to the following global estimation

$$S_m(t) \leq C_T, \quad \forall\, m \in \mathbb{N}, \quad \forall\, t \in [0, T], \quad \forall\, T > 0. \quad (2.61)$$

Then, by applying a similar argument used in the proof of Theorem 2.4, we can obtain a global weak solution u of Prob. (1.1)–(1.4) satisfying

$$u,\, u' \in L^\infty(0, T; V). \quad (2.62)$$

However, in case $1 < p < 2$, we don't know a weak solution obtained here belonging to $C^1([0,T];V)$ or not. Furthermore, the uniqueness of a weak solution is also not asserted.

3. Exponential decay of solutions

This section investigates the decay of the solution of Prob. (1.1) – (1.4) corresponding to with $\lambda > 0$, $\lambda_1 > 0$ and $p > 2$.

We prove that if $\|\tilde{u}_{0x}\|^2 - \|\tilde{u}_0\|_{L^p}^p > 0$ and if the initial energy, $\|f(t)\|$ are small enough, then the energy of the solution decays exponentially as $t \to +\infty$. For this purpose, we make the following assumption

(A_2'') $f \in L^2((0,1) \times \mathbb{R}_+)$, and there exist two constants $C_0 > 0$, $\gamma_0 > 0$ such that $\|f(t)\| \le C_0 e^{-\gamma_0 t}$, for all $t \ge 0$.

First, we construct the following Lyapunov functional

$$L(t) = E(t) + \delta\psi(t), \tag{3.1}$$

where $\delta > 0$ is chosen later and

$$E(t) = \tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u_x'(t)\|^2 + \tfrac{1}{2}\|u_x(t)\|^2 - \tfrac{1}{p}\|u(t)\|_{L^p}^p, \tag{3.2}$$

$$\psi(t) = \langle u'(t), u(t)\rangle + \langle u_x'(t), u_x(t)\rangle + \tfrac{\lambda}{2}\|u(t)\|^2 + \tfrac{\lambda_1}{2}\|u_x(t)\|^2. \tag{3.3}$$

Put

$$I(t) = I(u(t)) = \|u_x(t)\|^2 - \|u(t)\|_{L^p}^p, \tag{3.4}$$

$$\begin{aligned} J(t) = J(u(t)) &= \tfrac{1}{2}\|u_x(t)\|^2 - \tfrac{1}{p}\|u(t)\|_{L^p}^p \\ &= \left(\tfrac{1}{2} - \tfrac{1}{p}\right)\|u_x(t)\|^2 + \tfrac{1}{p}I(t). \end{aligned} \tag{3.5}$$

Then we have the following theorem.

Theorem 3.1. *Assume that (A_2'') holds. Let $I(0) > 0$ and the initial energy $E(0)$ satisfy*

$$\eta_* = \left[\tfrac{2p}{p-2}\left(E(0) + \tfrac{1}{2\lambda}\int_0^\infty \|f(t)\|^2 dt\right)\right]^{\frac{p-2}{2}} < 1. \tag{3.6}$$

Then, there exist positive constants C, γ such that

$$E(t) \le C\exp(-\gamma t), \quad \textit{for all} \quad t \ge 0. \tag{3.8}$$

Proof. First, we need the following lemmas.

Lemma 3.2. *The energy functional $E(t)$ satisfies*

$$E'(t) \le -\tfrac{\lambda}{2}\|u'(t)\|^2 - \lambda_1\|u_x'(t)\|^2 + \tfrac{1}{2\lambda}\|f(t)\|^2. \tag{3.9}$$

Proof. Multiplying (1.1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$E'(t) = -\lambda \|u'(t)\|^2 - \lambda_1 \|u'_x(t)\|^2 + \langle f(t), u'(t)\rangle. \tag{3.10}$$

We have

$$\langle f(t), u'(t)\rangle \leq \frac{\lambda}{2} \|u'(t)\|^2 + \frac{1}{2\lambda} \|f(t)\|^2. \tag{3.11}$$

Combining (3.10)–(3.11), it is easy to see (3.9) holds. Lemma 3.2 is proved completely. □

Lemma 3.3. *Suppose that* (A''_2) *holds. Then, if we have* $I(0) > 0$ *and*

$$\eta_* = \left[\frac{2p}{p-2}\left(E(0) + \frac{1}{2\lambda}\int_0^\infty \|f(t)\|^2 dt\right)\right]^{\frac{p-2}{2}} < 1, \tag{3.14}$$

then $I(t) > 0, \ \ \forall\, t \geq 0.$

Proof. By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$I(t) = I(u(t)) \geq 0, \quad \forall\, t \in [0, T_1], \tag{3.15}$$

this implies

$$\begin{aligned} J(t) &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u_x(t)\|^2 + \frac{1}{p} I(t) \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_x(t)\|^2 \geq \frac{p-2}{2p} \|u_x(t)\|^2, \quad \forall\, t \in [0, T_1]. \end{aligned} \tag{3.16}$$

It follows from (3.15), (3.16) that

$$\|u_x(t)\|^2 \leq \frac{2p}{p-2} J(t) \leq \frac{2p}{p-2} E(t), \quad \forall\, t \in [0, T_1]. \tag{3.17}$$

From (3.9), (3.17), we get

$$\|u_x(t)\|^2 \leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2}\left(E(0) + \frac{1}{2\lambda}\int_0^\infty \|f(t)\|^2 dt\right), \ \forall\, t \in [0, T_1]. \tag{3.18}$$

Hence, (3.14) and (3.18) imply

$$\begin{aligned} \|u(t)\|_{L^p}^p &\leq \|u_x(t)\|^p = \|u_x(t)\|^{p-2} \|u_x(t)\|^2 \\ &\leq \left[\frac{2p}{p-2}\left(E(0) + \frac{1}{2\lambda}\int_0^\infty \|f(t)\|^2 dt\right)\right]^{\frac{p-2}{2}} \|u_x(t)\|^2 \\ &= \eta_* \|u_x(t)\|^2, \ \ \forall t \in [0, T_1]. \end{aligned} \tag{3.19}$$

Therefore

$$I(t) = \|u_x(t)\|^2 - \|u(t)\|_{L^p}^p \geq (1 - \eta_*) \|u_x(t)\|^2 > 0, \ \ \forall\, t \in [0, T_1].$$

Now, we put $T_\infty = \sup\{T > 0 : I(t) > 0, \forall t \in [0, T]\}$. If $T_\infty < +\infty$ then, by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$. By the same arguments as above, we can deduce that there exists $T'_\infty > T_\infty$ such that $I(t) > 0, \forall t \in [0, T'_\infty]$. Hence, we conclude that $I(t) > 0, \ \ \forall\, t \geq 0$. Lemma 3.3 is proved completely. □

Lemma 3.4. *Let* $I(0) > 0$ *and* (3.14) *hold. Put*

$$E_1(t) = \|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u_x(t)\|^2 + I(t). \tag{3.20}$$

Then there exist the positive constants β_1, β_2 *such that*

$$\beta_1 E_1(t) \le L(t) \le \beta_2 E_1(t), \quad \forall\, t \ge 0, \tag{3.21}$$

for δ *is small enough.*

Proof. It is easy to see that

$$\begin{aligned} L(t) = {} & \tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u_x'(t)\|^2 + \left(\tfrac{1}{2} - \tfrac{1}{p}\right)\|u_x(t)\|^2 + \tfrac{1}{p}I(t) \\ & + \delta\langle u'(t), u(t)\rangle + \delta\langle u_x'(t), u_x(t)\rangle + \tfrac{\delta\lambda}{2}\|u(t)\|^2 + \tfrac{\delta\lambda_1}{2}\|u_x(t)\|^2 . \end{aligned} \tag{3.22}$$

From the following inequalities

$$\begin{cases} \delta\langle u'(t), u(t)\rangle \le \delta\|u'(t)\|\,\|u_x(t)\| \le \tfrac{1}{2}\delta\|u'(t)\|^2 + \tfrac{1}{2}\delta\|u_x(t)\|^2 , \\ \delta\langle u_x'(t), u_x(t)\rangle \le \delta\|u_x'(t)\|\,\|u_x(t)\| \le \tfrac{1}{2}\delta\|u_x'(t)\|^2 + \tfrac{1}{2}\delta\|u_x(t)\|^2 , \\ \tfrac{\delta\lambda}{2}\|u(t)\|^2 \le \tfrac{\delta\lambda}{2}\|u_x(t)\|^2 , \end{cases} \tag{3.23}$$

we deduce that

$$\begin{aligned} L(t) \ge {} & \tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u_x'(t)\|^2 + \left(\tfrac{1}{2} - \tfrac{1}{p}\right)\|u_x(t)\|^2 + \tfrac{1}{p}I(t) \\ & + \delta\langle u'(t), u(t)\rangle + \delta\langle u_x'(t), u_x(t)\rangle \\ \ge {} & \tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u_x'(t)\|^2 + \left(\tfrac{1}{2} - \tfrac{1}{p}\right)\|u_x(t)\|^2 + \tfrac{1}{p}I(t) \\ & - \tfrac{1}{2}\delta\|u'(t)\|^2 - \tfrac{1}{2}\delta\|u_x(t)\|^2 - \tfrac{1}{2}\delta\|u_x'(t)\|^2 - \tfrac{1}{2}\delta\|u_x(t)\|^2 \\ = {} & \tfrac{1-\delta}{2}\|u'(t)\|^2 + \tfrac{1-\delta}{2}\|u_x'(t)\|^2 + \left(\tfrac{1}{2} - \tfrac{1}{p} - \delta\right)\|u_x(t)\|^2 + \tfrac{1}{p}I(t) \\ \ge {} & \beta_1 E_1(t), \end{aligned} \tag{3.24}$$

where we choose

$$\beta_1 = \min\left\{\tfrac{1-\delta}{2},\ \tfrac{1}{2} - \tfrac{1}{p} - \delta,\ \tfrac{1}{p}\right\}, \tag{3.25}$$

with δ is small enough, $0 < \delta < \frac{1}{2} - \frac{1}{p}$.

Similar, we can prove that

$$\begin{aligned} L(t) \le {} & \tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u_x'(t)\|^2 + \left(\tfrac{1}{2} - \tfrac{1}{p}\right)\|u_x(t)\|^2 + \tfrac{1}{p}I(t) \\ & + \tfrac{1}{2}\delta\|u'(t)\|^2 + \tfrac{1}{2}\delta\|u_x(t)\|^2 + \tfrac{1}{2}\delta\|u_x'(t)\|^2 + \tfrac{1}{2}\delta\|u_x(t)\|^2 \\ & + \tfrac{\delta\lambda}{2}\|u_x(t)\|^2 + \tfrac{\delta\lambda_1}{2}\|u_x(t)\|^2 \\ = {} & \tfrac{1+\delta}{2}\|u'(t)\|^2 + \tfrac{1+\delta}{2}\|u_x'(t)\|^2 \\ & + \left[\tfrac{1}{2} - \tfrac{1}{p} + \delta\left(1 + \tfrac{\lambda}{2} + \tfrac{\lambda_1}{2}\right)\right]\|u_x(t)\|^2 + \tfrac{1}{p}I(t) \\ \le {} & \beta_2 E_1(t), \end{aligned} \tag{3.26}$$

where

$$\beta_2 = \max\left\{\tfrac{1+\delta}{2},\ \tfrac{1}{2}-\tfrac{1}{p}+\delta(1+\tfrac{\lambda}{2}+\tfrac{\lambda_1}{2})\right\}. \tag{3.27}$$

Lemma 3.4 is proved completely. □

Lemma 3.5. *Let $I(0)>0$ and (3.14) hold. Then the functional $\psi(t)$ defined by (3.3) satisfies*

$$\begin{aligned}\psi'(t) \le{}& \|u'(t)\|^2+\|u_x'(t)\|^2-\tfrac{1}{2}I(t)\\ &-\tfrac{1}{2}(1-\eta_*-\varepsilon_1)\|u_x(t)\|^2+\tfrac{1}{2\varepsilon_1}\|f(t)\|^2,\end{aligned} \tag{3.28}$$

for all $\varepsilon_1>0$.

Proof. By multiplying (1.1) by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) &= \|u'(t)\|^2+\|u_x'(t)\|^2-\|u_x(t)\|^2+\|u(t)\|_{L^p}^p+\langle f(t),u(t)\rangle\\ &= \|u'(t)\|^2+\|u_x'(t)\|^2-\tfrac{1}{2}I(t)-\tfrac{1}{2}I(t)+\langle f(t),u(t)\rangle.\end{aligned} \tag{3.29}$$

Note that

$$I(t)\ge(1-\eta_*)\|u_x(t)\|^2, \tag{3.30}$$

and

$$\langle f(t),u(t)\rangle \le \tfrac{\varepsilon_1}{2}\|u_x(t)\|^2+\tfrac{1}{2\varepsilon_1}\|f(t)\|^2,\quad \forall\,\varepsilon_1>0, \tag{3.31}$$

we deduce that

$$\begin{aligned}\psi'(t) &\le \|u'(t)\|^2+\|u_x'(t)\|^2-\tfrac{1}{2}I(t)-\tfrac{1}{2}I(t)+\langle f(t),u(t)\rangle\\ &\le \|u'(t)\|^2+\|u_x'(t)\|^2-\tfrac{1}{2}I(t)-\tfrac{1}{2}(1-\eta_*)\|u_x(t)\|^2\\ &\quad+\tfrac{\varepsilon_1}{2}\|u_x(t)\|^2+\tfrac{1}{2\varepsilon_1}\|f(t)\|^2\\ &= \|u'(t)\|^2+\|u_x'(t)\|^2-\tfrac{1}{2}I(t)-\tfrac{1}{2}(1-\eta_*-\varepsilon_1)\|u_x(t)\|^2\\ &\quad+\tfrac{1}{2\varepsilon_1}\|f(t)\|^2.\end{aligned} \tag{3.32}$$

Hence, the lemma 3.5 is proved by using some simple estimates. □

Now we continue to prove Theorem 3.1. It follows from (3.1), (3.9) and (3.28), that

$$\begin{aligned}L'(t) \le{}& -\tfrac{\lambda}{2}\|u'(t)\|^2-\lambda_1\|u_x'(t)\|^2+\tfrac{1}{2\lambda}\|f(t)\|^2\\ &+\delta\|u'(t)\|^2+\delta\|u_x'(t)\|^2-\tfrac{\delta}{2}I(t)\\ &-\tfrac{\delta}{2}(1-\eta_*-\varepsilon_1)\|u_x(t)\|^2+\tfrac{\delta}{2\varepsilon_1}\|f(t)\|^2\\ ={}& -(\tfrac{\lambda}{2}-\delta)\|u'(t)\|^2-(\lambda_1-\delta)\|u_x'(t)\|^2\\ &-\tfrac{\delta}{2}I(t)-\tfrac{\delta}{2}(1-\eta_*-\varepsilon_1)\|u_x(t)\|^2+\rho(t),\end{aligned} \tag{3.33}$$

for all δ, $\varepsilon_1 > 0$, $0 < \delta < \frac{1}{2} - \frac{1}{p}$, where

$$\rho(t) = \frac{1}{2}\left(\frac{\delta}{\varepsilon_1} + \frac{1}{\lambda}\right)\|f(t)\|^2 \leq C_* e^{-2\gamma_0 t}. \tag{3.34}$$

Let δ, ε_1 satisfy

$$0 < \delta < \min\left\{\frac{\lambda}{2}, \lambda_1, \frac{1}{2} - \frac{1}{p}\right\}, \quad 0 < \varepsilon_1 < 1 - \eta_*. \tag{3.35}$$

Then, we deduce from (3.21), (3.33), (3.34) and (3.35) that there exists a constant $\gamma > 0$, such that

$$\begin{aligned} L'(t) &\leq -(\tfrac{\lambda}{2} - \delta)\|u'(t)\|^2 - (\lambda_1 - \delta)\|u_x'(t)\|^2 \\ &\quad -\tfrac{\delta}{2}I(t) - \tfrac{\delta}{2}(1 - \eta_* - \varepsilon_1)\|u_x(t)\|^2 + C_* e^{-2\gamma_0 t} \\ &\leq -\gamma_1 E_1(t) + C_* e^{-2\gamma_0 t} \leq -\tfrac{\gamma_1}{\beta_2} L(t) + C_* e^{-2\gamma_0 t} \\ &\leq -\gamma L(t) + C_* e^{-2\gamma_0 t}, \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} &\gamma_1 = \min\left\{\tfrac{\lambda}{2} - \delta,\ \lambda_1 - \delta,\ \tfrac{\delta}{2}(1 - \eta_* - \varepsilon_1)\right\} > 0, \\ &0 < \gamma < \min\left\{\gamma_1,\ \tfrac{\gamma_1}{\beta_2},\ 2\gamma_0\right\}. \end{aligned} \tag{3.37}$$

Combining (3.21) and (3.36), we get (3.8). Theorem 3.1 is proved completely. □

4. A remark

We consider the following problem

$$\begin{cases} u_{tt} - u_{xx} - u_{xxtt} - \lambda_1 u_{xxt} + \lambda u_t + |u|^{p-2} u = f(x,t), \\ u_x(0,t) + \lambda_1 u_{xt}(0,t) + u_{xtt}(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{4.1}$$

$0 < x < 1$, $t > 0$, where λ, λ_1, p are given constants and $\tilde{u}_0$, $\tilde{u}_1$, f are given functions. The global existence of a strong solution is as follows.

Theorem 4.1. *Let $T > 0$. Suppose that $p \geq 2$, $\lambda > 0$, $\lambda_1 > 0$ and (A_2) hold. Let $\tilde{u}_0$, $\tilde{u}_1 \in V \cap H^2$. Then Prob. (4.1) has a unique solution*

$$u \in C^1\left([0,T]; V \cap H^2\right), \quad u_{tt} \in L^\infty\left(0,T; V \cap H^2\right). \tag{4.2}$$

With less regular initial datas, we have the global existence of a weak solution as follows.

Theorem 4.2. *Let $T > 0$. Suppose that $\lambda > 0, \lambda_1 > 0$ and (A_2') hold. Let $\tilde{u}_0, \tilde{u}_1 \in V$.*

(i) *If* $p \geq 2$, *Prob.* (4.1) *has a unique solution*

$$u \in C^1\left([0,T];V\right). \tag{4.3}$$

(ii) *If* $1 < p < 2$, *Prob.* (4.1) *has a solution*

$$u \in L^\infty\left(0,T;V\right),\ u_t \in L^\infty\left(0,T;V\right). \tag{4.4}$$

The proofs of Theorems 4.1, 4.2 are similar to the ones in Theorems 2.2, 2.4. And in case $1 < p < 2$, we also note as above, see Remark 2.6.

In what follows, assume that $p > 2, \lambda > 0, \lambda_1 > 0$. With suitable conditions for f, we prove that Prob. (4.1) has a unique global solution $u(t)$ with energy decaying exponentially as $t \to +\infty$, without the initial data $(\tilde{u}_0, \tilde{u}_1)$ being small enough.

Theorem 4.3. *Assume that* (A_2'') *holds. Then, there exist positive constants* C, γ *such that*

$$\|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u_x(t)\|^2 + \|u(t)\|_{L^p}^p \leq C\exp(-\gamma t), \tag{4.5}$$

for all $t \geq 0$.

Proof. First, we construct the following Lyapunov functional

$$L_1(t) = \bar{E}(t) + \delta\psi(t), \tag{4.6}$$

where $\delta > 0$ is chosen later and

$$\bar{E}(t) = \tfrac{1}{2}\left\|u'(t)\right\|^2 + \tfrac{1}{2}\left\|u_x'(t)\right\|^2 + \tfrac{1}{2}\left\|u_x(t)\right\|^2 + \tfrac{1}{p}\left\|u(t)\right\|_{L^p}^p, \tag{4.7}$$

$$\psi(t) = \langle u'(t), u(t)\rangle + \langle u_x'(t), u_x(t)\rangle + \tfrac{\lambda}{2}\|u(t)\|^2 + \tfrac{\lambda_1}{2}\|u_x(t)\|^2. \tag{4.8}$$

Next, we need the following lemmas.

Lemma 4.4. *The energy functional* $\bar{E}(t)$ *satisfies*

$$\bar{E}'\left(t\right) \leq -\tfrac{\lambda}{2}\left\|u'\left(t\right)\right\|^2 - \lambda_1\left\|u_x'\left(t\right)\right\|^2 + \tfrac{1}{2\lambda}\left\|f\left(t\right)\right\|^2. \tag{4.9}$$

Proof. Multiplying $(4.1)_1$ by $u'(x,t)$ and integrating over $[0,1]$, we get

$$\bar{E}'\left(t\right) = -\lambda\left\|u'\left(t\right)\right\|^2 - \lambda_1\left\|u_x'\left(t\right)\right\|^2 + \langle f(t), u'(t)\rangle. \tag{4.10}$$

We have

$$\langle f(t), u'(t)\rangle \leq \tfrac{\lambda}{2}\left\|u'\left(t\right)\right\|^2 + \tfrac{1}{2\lambda}\left\|f\left(t\right)\right\|^2. \tag{4.11}$$

Combining (4.10), (4.11), it is easy to see (4.9) holds. Lemma 4.4 is proved completely. □

By (4.9), we obtain

$$\bar{E}'\left(t\right) \leq -\tfrac{\lambda}{2}\left\|u'\left(t\right)\right\|^2 - \lambda_1\left\|u_x'\left(t\right)\right\|^2 + \tfrac{1}{2\lambda}\left\|f\left(t\right)\right\|^2 \leq \tfrac{1}{2\lambda}\left\|f\left(t\right)\right\|^2. \tag{4.12}$$

Integrating with respect to t, we obtain

$$\bar{E}(t) \leq \bar{E}(0) + \tfrac{1}{2\lambda} \int_0^\infty \|f(t)\|^2 dt = E_*, \quad \text{for all } t \geq 0. \tag{4.13}$$

Put

$$\bar{E}_1(t) = \|u'(t)\|^2 + \|u'_x(t)\|^2 + \|u_x(t)\|^2 + \|u(t)\|_{L^p}^p, \tag{4.14}$$

we have the following lemma;

Lemma 4.5. *There exist the positive constants $\bar{\beta}_1$, $\bar{\beta}_2$ such that*

$$\bar{\beta}_1 \bar{E}_1(t) \leq L_1(t) \leq \bar{\beta}_2 \bar{E}_1(t), \quad \forall\, t \geq 0, \tag{4.15}$$

for δ is small enough.

Proof. It is clear that

$$\begin{aligned} L_1(t) = {} & \tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u'_x(t)\|^2 + \tfrac{1}{2}\|u_x(t)\|^2 + \tfrac{1}{p}\|u(t)\|_{L^p}^p \\ & + \delta\langle u'(t), u(t)\rangle + \delta\langle u'_x(t), u_x(t)\rangle + \tfrac{\delta\lambda}{2}\|u(t)\|^2 + \tfrac{\delta\lambda_1}{2}\|u_x(t)\|^2. \end{aligned} \tag{4.16}$$

From the following inequalities

$$\begin{cases} \delta\langle u'(t), u(t)\rangle \leq \delta\|u'(t)\|\,\|u_x(t)\| \leq \tfrac{1}{2}\delta\|u'(t)\|^2 + \tfrac{1}{2}\delta\|u_x(t)\|^2, \\ \delta\langle u'_x(t), u_x(t)\rangle \leq \delta\|u'_x(t)\|\,\|u_x(t)\| \leq \tfrac{1}{2}\delta\|u'_x(t)\|^2 + \tfrac{1}{2}\delta\|u_x(t)\|^2, \\ \tfrac{\delta\lambda}{2}\|u(t)\|^2 \leq \tfrac{\delta\lambda}{2}\|u_x(t)\|^2, \end{cases} \tag{4.17}$$

we deduce that

$$\begin{aligned} L_1(t) \geq {} & \tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u'_x(t)\|^2 + \tfrac{1}{2}\|u_x(t)\|^2 + \tfrac{1}{p}\|u(t)\|_{L^p}^p \\ & + \delta\langle u'(t), u(t)\rangle + \delta\langle u'_x(t), u_x(t)\rangle \\ = {} & \tfrac{1}{2}\|u'(t)\|^2 + \tfrac{1}{2}\|u'_x(t)\|^2 + \tfrac{1}{2}\|u_x(t)\|^2 + \tfrac{1}{p}\|u(t)\|_{L^p}^p \\ & - \tfrac{1}{2}\delta\|u'(t)\|^2 - \tfrac{1}{2}\delta\|u_x(t)\|^2 - \tfrac{1}{2}\delta\|u'_x(t)\|^2 - \tfrac{1}{2}\delta\|u_x(t)\|^2 \\ = {} & \tfrac{1-\delta}{2}\|u'(t)\|^2 + \tfrac{1-\delta}{2}\|u'_x(t)\|^2 + \tfrac{1-2\delta}{2}\|u_x(t)\|^2 + \tfrac{1}{p}\|u(t)\|_{L^p}^p \\ \geq {} & \bar{\beta}_1 \bar{E}_1(t), \end{aligned} \tag{4.18}$$

where we choose

$$\bar{\beta}_1 = \min\left\{\tfrac{1-2\delta}{2}, \tfrac{1}{p}\right\}, \tag{4.19}$$

δ is small enough, $0 < \delta < \frac{1}{2}$.

Similar, we can prove that

$$\begin{aligned} L_1(t) &\leq \tfrac{1}{2}\left\|u'(t)\right\|^2 + \tfrac{1}{2}\left\|u_x'(t)\right\|^2 + \tfrac{1}{2}\left\|u_x(t)\right\|^2 + \tfrac{1}{p}\left\|u(t)\right\|_{L^p}^p \\ &\quad + \tfrac{1}{2}\delta\left\|u'(t)\right\|^2 + \tfrac{1}{2}\delta\left\|u_x(t)\right\|^2 + \tfrac{1}{2}\delta\left\|u_x'(t)\right\|^2 + \tfrac{1}{2}\delta\left\|u_x(t)\right\|^2 \\ &\quad + \tfrac{\delta\lambda}{2}\|u_x(t)\|^2 + \tfrac{\delta\lambda_1}{2}\|u_x(t)\|^2 \\ &= \tfrac{1+\delta}{2}\left\|u'(t)\right\|^2 + \tfrac{1+\delta}{2}\left\|u_x'(t)\right\|^2 + \tfrac{1+\delta(2+\lambda+\lambda_1)}{2}\left\|u_x(t)\right\|^2 \\ &\quad + \tfrac{1}{p}\left\|u(t)\right\|_{L^p}^p \\ &\leq \tfrac{1+\delta(2+\lambda+\lambda_1)}{2}\bar{E}_1(t) = \bar{\beta}_2\bar{E}_1(t), \end{aligned} \tag{4.20}$$

where

$$\bar{\beta}_2 = \tfrac{1+\delta(2+\lambda+\lambda_1)}{2}. \tag{4.21}$$

Lemma 4.5 is proved completely. □

Lemma 4.6. *The functional* $\psi(t)$ *defined by* (4.8) *satisfies*

$$\psi'(t) \leq \left\|u'(t)\right\|^2 + \left\|u_x'(t)\right\|^2 - \tfrac{1}{2}\left\|u_x(t)\right\|^2 - \|u(t)\|_{L^p}^p + \tfrac{1}{2}\left\|f(t)\right\|^2. \tag{4.22}$$

Proof. By multiplying $(4.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\psi'(t) = \left\|u'(t)\right\|^2 + \left\|u_x'(t)\right\|^2 - \left\|u_x(t)\right\|^2 - \|u(t)\|_{L^p}^p + \langle f(t), u(t)\rangle. \tag{4.23}$$

Note that

$$\langle f(t), u(t)\rangle \leq \|f(t)\|\,\|u_x(t)\| \leq \tfrac{1}{2}\left\|u_x(t)\right\|^2 + \tfrac{1}{2}\left\|f(t)\right\|^2. \tag{4.24}$$

Combining (4.23), (4.24), it is easy to see (4.22) holds. Lemma 4.6 is proved completely. □

Now we continue to prove Theorem 4.4. It follows from (4.6), (4.9) and (4.22), that

$$\begin{aligned} L_1'(t) &\leq -\tfrac{\lambda}{2}\left\|u'(t)\right\|^2 - \lambda_1\left\|u_x'(t)\right\|^2 + \tfrac{1}{2\lambda}\left\|f(t)\right\|^2 \\ &\quad + \delta\left\|u'(t)\right\|^2 + \delta\left\|u_x'(t)\right\|^2 - \tfrac{\delta}{2}\left\|u_x(t)\right\|^2 - \delta\|u(t)\|_{L^p}^p + \tfrac{\delta}{2}\left\|f(t)\right\|^2 \\ &= -\left(\tfrac{\lambda}{2} - \delta\right)\left\|u'(t)\right\|^2 - \left(\lambda_1 - \tfrac{\delta}{2}\right)\left\|u_x'(t)\right\|^2 \\ &\quad - \tfrac{\delta}{2}\left\|u_x(t)\right\|^2 - \delta\|u(t)\|_{L^p}^p + \tfrac{1}{2}\left(\delta + \tfrac{1}{\lambda}\right)\left\|f(t)\right\|^2 \\ &= -\left(\tfrac{\lambda}{2} - \delta\right)\left\|u'(t)\right\|^2 - \left(\lambda_1 - \tfrac{\delta}{2}\right)\left\|u_x'(t)\right\|^2 \\ &\quad - \tfrac{\delta}{2}\left\|u_x(t)\right\|^2 - \delta\|u(t)\|_{L^p}^p + \rho_1(t), \end{aligned} \tag{4.25}$$

where

$$\rho_1(t) = \tfrac{1}{2}\left(\delta + \tfrac{1}{\lambda}\right)\left\|f(t)\right\|^2 \leq C_1 e^{-2\gamma_0 t}. \tag{4.26}$$

Choosing $0 < \delta < \min\{\frac{1}{2}, \frac{\lambda}{2}, \lambda_1\}$, we deduce from (4.25), (4.26) that

$$\begin{aligned} L_1'(t) &\leq -\beta_* \left[\|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u_x(t)\|^2 + \|u(t)\|_{L^p}^p \right] \\ &\quad + C_1 e^{-2\gamma_0 t} \\ &= -\beta_* \bar{E}_1(t) + C_1 e^{-2\gamma_0 t} \leq -\frac{\beta_*}{\beta_2} L_1(t) + C_1 e^{-2\gamma_0 t} \\ &\leq -\gamma L_1(t) + C_1 e^{-2\gamma_0 t}, \end{aligned} \tag{4.27}$$

where $\beta_* = \min\left\{\frac{\lambda}{2} - \delta, \lambda_1 - \frac{\delta}{2}, \frac{\delta}{2}\right\}$, $0 < \gamma < \min\left\{\frac{\beta_*}{\beta_2}, 2\gamma_0\right\}$. Combining (4.14), (4.15) and (4.27), we get (4.5). Theorem 4.4 is proved completely. □

Acknowledgments: The authors wish to express their sincere thanks to the referees for the suggestions and valuable comments. The authors are also extremely grateful for the support given by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under Project **101.01-2012.12.**

References

[1] J. Albert, *On the decay of solutions of the generalized Benjamin-Bona-Mahony equations*, J. Math. Anal. Appl., **141**(2) (1989), 527–537.

[2] C.J. Amick, J.L. Bona and M.E. Schonbek, *Decay of solutions of some nonlinear wave equations*, J. Differential Equations, **81**(1) (1989), 1–49.

[3] F.E. Browder, *On nonlinear wave equations*, Math. Z., **80** (1962), 249–264.

[4] A. Benaissa and S.A. Messaoudi, *Exponential decay of solutions of a nonlinearly damped wave equation*, Nonlinear Differ. Equ. Appl., **12** (2005) 391–399.

[5] A. Chattopadhyay, S. Gupta, A.K. Singh and S.A. Sahu, *Propagation of shear waves in an irregular magnetoelastic monoclinic layer sandwiched between two isotropic half-spaces*, International Journal of Engineering, Science and Technology, **1**(1) (2009), 228–244.

[6] P.A. Clarkson, *New similarity reductions and Painlevé analysis for the symmetric regularised long wave and modified Benjamin-Bona-Mahoney equations*, Journal of Physics A., **22**(18) (1989), 3821–3848.

[7] E.L.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955, p.43.

[8] Subhas Dutta, *On the propagation of Love type waves in an infinite cylinder with rigidity and density varying linearly with the radial distance*, Pure and Applied Geophysics, **98**(1) (1972), 35–39.

[9] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. **1**. Academic Press, NewYork, 1969.

[10] J.L. Lions and W.A. Strauss, *Some nonlinear evolution equations*, Bull. Soc. Math. France, **93** (1965), 43–96.

[11] J.L. Lions, *Equations differentielles operationelles et problèmes aux limites*, Springer-Verlag, Berlin, 1961, p.96, Section 7.

[12] J.L. Lions, *Quelques méthodes de ré solution des problèmes aux limites nonlinéaires*, Dunod, Gauthier–Villars, Paris, 1969.

[13] V.G. Makhankov, *Dynamics of classical solitons in nonintegrable systems*, Physics Reports C. **35**(1) (1978), 1–128.

[14] M. Nakao and K. Ono, *Global existence to the Cauchy problem of the semilinear wave equation with a nonlinear dissipation*, Funkcial Ekvacioj, **38** (1995), 417–431.

[15] L.T.P. Ngoc, N.T. Duy and N.T. Long, *Existence and properties of solutions of a boundary problem for a Love's equation*, Bulletin of the Malaysian Mathematical Sciences Society, **37**(4) (2014), 997–1016.

[16] L.T.P. Ngoc, N.T. Duy and N.T. Long, *A linear recursive scheme associated with the Love's equation*, Acta Mathematica Vietnamica, **38**(4) (2013), 551–562.

[17] L.T.P. Ngoc and N.T. Long, *Existence and exponential decay for a nonlinear wave equation with a nonlocal boundary condition*, Communications on Pure and Applied Analysis, **12**(5) (2013), 2001–2029.

[18] L.T.P. Ngoc and N.T. Long, *Existence, blow-up and exponential decay for a nonlinear Love equation associated with Dirichlet conditions*, Applications of Mathematics (accepted for publication).

[19] T. Ogino and S. Takeda, *Computer simulation and analysis for the spherical and cylindrical ion-acoustic solitons*, Journal of the Physical Society of Japan, **41**(1) (1976), 257–264.

[20] Mrinal K. Paul, *On propagation of love-type waves on a spherical model with rigidity and density both varying exponentially with the radial distance*, Pure and Applied Geophysics, **59**(1) (1964), 33–37.

[21] Věra Radochová, *Remark to the comparison of solution properties of Love's equation with those of wave equation*, Applications of Mathematics, **23**(3) (1978), 199–207.

[22] C.E. Seyler and D.L. Fenstermacher, *A symmetric regularized-long-wave equation*, Physics of Fluids, **27**(1) (1984), 4–7.