Nonlinear Functional Analysis and Applications
Vol. 16, No. 4 (2011), pp. 501-509

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

NONLINEAR MIXED VECTOR VARIATIONAL INEQUALITY PROBLEMS

Suhel Ahmad Khan[1], **Q. H. Khan**[2] **and Farhat Suhel**[3]

[1]Department of Mathematics
BITS, PILANI-DUBAI, Dubai, U.A.E.
e-mail: khan.math@gmail.com

[2]Department of Mathematics, Zakir Husain College
University of Delhi, India, INDIA
e-mail: qhkhan.ssitm@gmail.com

[3]Department of Mathematics
Aligarh Muslim University, INDIA
e-mail: farhat.math@gmail.com

Abstract. In this paper, we consider nonlinear mixed vector variational inequality problems in the setting of topological vector spaces. We extend the concept of upper sign continuity for vector-valued functions and establish some existence results for solutions of vector variational inequalities. By utilizing a new version of Ky Fan lemma we investigated the nonemptyness and compactness of solution sets of these problems under some suitable assumptions.

1. Introduction

It has been shown that a wide class of linear and nonlinear problems arising in various branches of mathematical and engineering sciences can be studied in the unified and general framework of variational inequalities. Variational inequalities were introduced and considered by Stampacchia [9] in early sixties. Variational inequalities have been generalized and extended in several directions using new techniques. As a useful and important branch of variational inequality theory, vector variational inequalities were initially introduced and considered by Giannessi [3] in a finite-dimensional Euclidean space in 1980. Ever since the theory of vector variational inequalities has been extensively

[0]Received March 5, 2011. Revised August 27, 2011.
[0]2000 Mathematics Subject Classification: 49J40, 47H10, 47H17.
[0]Keywords: KKM mapping; P_x-η-pseudomonotonicity; P_x-η-upper sign continuity.

studied in the last two decades because of its applications to vector optimization problems, vector complementarity problems, game theory, economics, etc; See, for example, [3, 4, 6, 7, 8] and references therein.

Inspired and motivated by the recent research activities going on in this direction, we introduce the concept of P_x-η-upper sign continuity which extend the previous concept of upper sign continuity introduced by Hadjisavvas [5] and considered two classes of nonlinear vector variational inequality problems. Further, we obtain an existence result for their solutions in real Hausdorff topological vector spaces setting for a moving cone by relaxing continuity and compactness by using a new version of famous Ky Fan lemma which is due to Ben-El-Mechaiekh *et al.* [1]. The results of this paper are generalizations and refinement of several results recently appeared in the literature.

2. Preliminaries

Throughout the paper unless otherwise specified, let X and Y be two topological vector spaces, $K \subset X$ be a nonempty convex subset of X. A nonempty set $P \subset Y$ be a convex cone if $\lambda P \subseteq P$, for all $\lambda \geq 0$ and $P + P = P$ also P is said to be a pointed cone with its apex at the origin if $P \cap (-P) = \{0\}$.

The partial order $\leq_P$ in Y, induced by the pointed cone P, is defined by declaring

$$x \leq_P y \text{ if and only if } y - x \in P, \ \forall x, y \in Y.$$

An ordered topological vector space is a pair (Y, P) with the partial order induced by P. Let $L(X, Y)$ denote the space of all continuous linear mappings from X into Y. We will denote by 2^A, the family of all subsets of A and by $F(A)$ the family of all nonempty finite subsets of A. A set-valued mapping $P : K \to 2^Y$ be such that for each $x \in K$, $P(x)$ is closed and convex cone.

Now, we recall the following concepts and results which are needed in the sequel. Throughout the paper, unless otherwise specified, let $P_x = \bigcap\limits_{x \in K} P(x)$ is closed, convex and pointed cone in Y.

Definition 2.1. *A mapping $f : K \to Y$ is said to be*

(i) P_x*-convex, if* $tf(x) + (1-t)f(y) - f(tx + (1-t)y) \in P_x, \ \forall x, y \in K, \ t \in [0, 1]$;
(ii) P_x*-concave, if* $-f$ *is* P_x*-convex.*

Definition 2.2. *Let K_0 be a nonempty subset of K. A set-valued mapping $\Gamma : K_0 \to 2^K$ is said to be a KKM mapping if, $coA \subseteq \bigcup_{x\in A} \Gamma(x),\ \forall A \in F(K_0)$, where co denotes the convex hull.*

Definition 2.3 ([1]). *Consider a subset A of a topological vector space and a topological space Y. A family $\{C_i, K_i\}_{i\in I}$ of pairs of sets is said to be coercing for a map $G : A \to 2^Y$ if and only if*

(i) *for each $i \in I$, C_i is contained in a compact convex subset of A, and K_i is a compact subset of Y;*
(ii) *for each i, j, there exists $k \in I$ such that $C_i \cup C_j \subseteq C_k$;*
(iii) *for each $i \in I$, there exists $k \in I$ with $\bigcap_{x\in C_k} G(x) \subseteq K_i$.*

Theorem 2.4 ([1]). *Let $F : K \to 2^Y$ be a KKM mapping with compactly closed (in K) values. If F admits a coercing family, then $\bigcap_{x\in K} F(x) \neq \emptyset$.*

3. Main Results

Throughout this section we let X and Y be two topological vector spaces, K a nonempty convex subset of X, $P : K \to 2^Y$ with convex cone values, and let $T : K \to L(X,Y)$ be a mapping and $\eta : K \times K \to X$, $f : K \times K \to Y$ are two nonlinear mappings. Now, we consider following *nonlinear mixed vector variational inequality* (for short, NMVVI) problem that consists of finding $x \in K$ such that

$$\langle Tx, \eta(y,x)\rangle + f(y,x) \in P(x),\ \forall y \in K. \tag{3.1}$$

Also, we consider *dual nonlinear mixed vector variational inequality* (for short, DNMVVI) problem with respect to f that consists in finding $x \in K$ such that

$$\langle Ty, \eta(x,y)\rangle + f(y,x) \in -P(y),\ \forall y \in K. \tag{3.2}$$

We denote the solution set of NMVVI and DNMVVI with S_1 and S_2, respectively.

In order to establish existence results for the solutions of NMVVI and DNMVVI, we define the following concepts.

Definition 3.1. *A mapping $T : K \in L(X,Y)$ is said to be P_x-η-pseudomonotone with respect to f if, for all $x, y \in K$,*

$$\langle Tx, \eta(y,x)\rangle + f(y,x) \in P(x) \Rightarrow \langle Ty, \eta(x,y)\rangle + f(y,x) \in -P(y).$$

Example 3.2. *Let X=$\mathbb{R}$, K=$\mathbb{R}_+$, Y=$\mathbb{R}^2$, $P(x)$ and $P(y)$=$\mathbb{R}^2_+, \forall x, y \in K$, $\eta(y,x) = y - x$, for all $x, y \in K$ and*

$$T(x) = \begin{pmatrix} 0 \\ 2.5 + \cos\, x \end{pmatrix} \text{ and } f(y,x) = \begin{pmatrix} y-x \\ y-x \end{pmatrix} \text{ for all } x, y \in K.$$

If

$$\langle Tx, \eta(y,x)\rangle + f(y,x) = \begin{pmatrix} y-x \\ (3.5 + \cos\, x)(y-x) \end{pmatrix} \in \mathbb{R}^2_+.$$

Then, we have $y \geq x$. It follows that

$$\langle Ty, \eta(x,y)\rangle + f(y,x) = \begin{pmatrix} y-x \\ (1.5 + \cos\, y)(x-y) \end{pmatrix} \in -\mathbb{R}^2_+.$$

So, T is P_x-η-pseudomonotone with respect to f.

Definition 3.3. *A mapping $T : K \to L(X,Y)$ is said to be P_x-η-upper sign continuous with respect to f if, for all $x, y \in K$,*

$$\langle Tu, \eta(y,u)\rangle + f(y,u) \in P(u),\ \forall u \in \,]x,y[\Rightarrow \langle Tx, \eta(y,x)\rangle + f(y,x) \in P(x).$$

It is remarked that, if X=Y=$\mathbb{R}$, K=$P(u)$=$[0,\infty)$ and $f \equiv 0$, $\eta(y,x)$=$y-x$, for all $x, y \in K$, then any positive mapping $T : K \to L(X,Y) = \mathbb{R}$ is P_x-η-upper sign continuous while it is not hemicontinuous. In this case, the concept of P_x-η-upper sign continuity reduces to upper sign continuity introduced by Hadjisavvas [5] in the framework of variational inequalities and later by Bianchi and Pini [2] for real bifunctions.

Theorem 3.4. *Let $K \subset X$ be a nonempty and convex subset of X. Let $f : K \times K \to Y$ and $\eta : K \times K \to X$ be two bi-mappings. Suppose following conditions hold:*

(i) *f is P_x-convex in first argument with the condition*

$$f(x,x) = 0,\ \forall x \in K;$$

(ii) *η is an affine mapping in first argument with the condition*

$$\eta(x,x) = 0,\ \forall x \in K;$$

(iii) *$T : K \to L(X,Y)$ is P_x-η-upper sign continuous and P_x-η-pseudomonotone mapping.*

Then, the solution sets of NMVVI and DNMVVI are equivalent.

Proof. By the concept of P_x-η-pseudomonotonicity of T with respect to f, we have

$$NMVVI \subseteq DNMVVI. \tag{3.3}$$

Conversely, let $x_0 \in K$ be the solution of DNMVVI. For any given $x \in K$, we know that

$$x_t = x_0 + t(x - x_0) \in K, \ \forall t \in (0,1),$$

as K is convex. Since $x_0 \in K$ is a solution of DNMVVI, so for each $x_0 \in K$, it follows that

$$\langle Tx_t, \eta(x_0, x_t)\rangle + f(x_t, x_0) \in -P(x_t). \tag{3.4}$$

If $\langle Tx_s, \eta(x, x_s)\rangle + f(x_s, x) \notin P(x_s)$, for some $s \in (0,1)$, then from inclusion (3.4) and conditions (i)-(ii), it follows that

$$\begin{aligned} 0 &= \langle Tx_s, \eta(x_s, x_s)\rangle + f(x_s, x_s) \\ &= \langle Tx_s, \eta(sx + (1-s)x_0, x_s)\rangle + f(x_s, sx + (1-s)x_0) \\ &\leq_P s[\langle Tx_s, \eta(x, x_s)\rangle + f(x_s, x)] \\ &\quad + (1-s)[\langle Tx_s, \eta(x_0, x_s)\rangle + f(x_s, x_0)] \notin P(x_s). \end{aligned}$$

Which leads to a contradiction, since $P(x_s)$ is a pointed convex cone. Hence, we have

$$\langle Tx_s, \eta(x, x_s)\rangle + f(x_s, x) \in P(x_s), \text{ for some } s \in (0,1).$$

From P_x-η-upper sign continuity of T with respect to f, we get

$$\langle Tx_0, \eta(x, x_0)\rangle \ + f(x, x_0) \in P(x), \ \forall x \in K.$$

Therefore, $x_0 \in K$ is solution of NMVVI. This completes the proof. □

We now establish existence results for NMVVI and DNMVVI under P_x-η-upper sign continuity.

Theorem 3.5. *Let $K \subset X$ be a nonempty and convex subset of X. Let $f : K \times K \to Y$ and $\eta : K \times K \to X$ be two bi-mappings. Suppose following conditions hold:*

(i) *f is P_x-convex in first argument with the condition*

$$f(x, x) = 0, \ \forall x \in K;$$

(ii) *η is an affine mapping in first argument with the condition*

$$\eta(x, x) = 0, \ \forall x \in K;$$

(iii) *any compact E of K, the set*

$$\{y \in E : \langle Ty, \eta(x, y)\rangle + f(y, x) \in P(y)\},$$

is closed in E;

(iv) *for each finite subset A of K and any $y \in \mathrm{co}A \backslash A$, there exists $x \in A$ such that*

$$\langle Ty, \eta(x, y)\rangle + f(y, x) \in P(y);$$

(v) *there exist compact subset $B \subseteq K$ and compact convex subset $D \subseteq K$ such that $\forall x \in K \backslash B, \exists y \in D$ such that*

$$\langle Tx, \eta(y, x)\rangle + f(y, x) \notin P(x).$$

Then, the solution set S_1 of NMVVI is nonempty and compact.

Proof. Define a set-valued mapping $\Gamma : K \to 2^K$ as follows:

$$\Gamma(y) = \{x \in K : \langle Tx, \eta(y, x)\rangle + f(y, x) \in P(x)\}.$$

From conditions (i)-(iii), Γ has compactly closed values. We claim that Γ is a KKM mapping. If this is not true, then there exist a finite set $\{y_1, ..., y_n\} \subset K$ and $z \in \mathrm{co}(\{y_1, ..., y_n\})$ such that

$$z \notin \bigcup_{i=1}^{n} \Gamma(y_i).$$

Thus by definition of Γ, we have

$$\langle Tz, \eta(y_i, z)\rangle + f(y_i, z) \notin P(z),\ i = 1, ..., n. \tag{3.5}$$

which is a contradiction (by (iv)). It is clear that $\{D, B\}$ is a coercive family for Γ. Now by Theorem 2.4,

$$\mathrm{NMVVI} = \bigcap_{x \in K} \Gamma(x) \neq \emptyset.$$

Using (v) we obtain

$$\mathrm{NMVVI} = \bigcap_{x \in K} \Gamma(x) \subseteq B, \tag{3.6}$$

and hence

$$\mathrm{NMVVI} = \bigcap_{x \in K} \Gamma(x) = \bigcap_{x \in K} (\Gamma(x) \cap B), \tag{3.7}$$

which is closed in B (by (iii)) and so a compact set of B. $\square$

Example 3.6. *Let* $X = \mathbb{R}$, $K = E = [0,1]$, $Y = \mathbb{R}^2$ *and* $P(x), P(y) = P = \{(u,v) \in \mathbb{R}^2 : u \geq 0, v \geq 0\}$ *for all* $x, y \in K$, *be a fixed closed convex cone in* Y. *Let us define*

$$T(x)(t) = \langle T(x), t\rangle = t(x, x^2),$$

$$\eta(y,x) = y - x, \ \forall x, y \in K$$

and $f \equiv 0$, *for all* $x \in K$ *and* $t \in X$. *Then,* f *is* P_x*-convex and* T *is* P_x-η-*pseudomonotone and* P_x-η-*upper sign continuous with respect to* f *and*

$$\langle T(x), \eta(y,x)\rangle + f(y,x) = (y-x)(x,x^2) = ((y-x)x, (y-x)x^2).$$

It is easy to see that the set

$$\{x \in E : \langle T(y), \eta(x,y) \in -P(x)\} = [0, y]$$

is closed. Since K *is compact, condition (v) of Theorem 3.5 trivially holds. Therefore,* T *satisfies all the assumptions of Theorem 3.5 and so the solution set of NMVVI is nonempty and compact. It is clear that only* $x = 0$ *satisfies the following relation*

$$\langle T(x), \eta(y,x)\rangle \in P(x), \ \forall y \in K.$$

Similarly, only $x = 0$ *satisfies the following relation*

$$\langle T(y), \eta(x,y)\rangle \in -P(y), \ \forall y \in K.$$

Hence the solution sets of NMVVI and DNMVVI are equal to the singleton set $\{0\}$.

Remark 3.7. (a) *If* X *is a real reflexive Banach space and* K *is a nonempty, bounded, closed and convex subset of* X, *then* K *is weakly compact. In this case, condition (v) of Theorem 3.5 can be removed.*

(b) *It is obvious that if* $f(y,.)$ *is continuous and the set-valued mapping* $P(y))$ *for all* $y \in K$, *is closed, then condition (iii) of Theorem 3.5 trivially holds.*

Theorem 3.8. *Let* $K \subset X$ *be a nonempty and convex subset of* X. *Let* $f : K \times K \to Y$ *and* $\eta : K \times K \to X$ *be two bi-mappings. Suppose following conditions hold:*

(i) f *is* P_x*-convex in first argument with the condition*

$$f(x,x) = 0, \ \forall x \in K;$$

(ii) η *is an affine mapping in first argument with the condition*

$$\eta(x,x) = 0, \ \forall x \in K;$$

(iii) *Any compact E of K, the set*

$$\{y \in E : \langle Ty, \eta(y,x)\rangle + f(y,x) \in -P(y)\}$$

is closed in E;

(iv) *for each finite subset A of K and any $y \in \mathrm{co}A\backslash A$, there exists $x \in A$ such that*

$$\langle Ty, \eta(y,x)\rangle + f(y,x) \in -P(x);$$

(v) *there exist compact subset $B \subseteq K$ and compact convex subset $D \subseteq K$ such that $\forall x \in K\backslash B; \exists y \in D$ such that*

$$\langle Ty, \eta(x,y)\rangle + f(y,x) \notin -P(y).$$

Then, the solution set S_2 of DNMVVI is nonempty and compact.

Proof. Define a set-valued mapping $\Gamma : K \to 2^K$ as follows:

$$\Gamma(y) = \{x \in K : \langle Ty, \eta(x,y)\rangle + f(y,x) \in -P(y)\}.$$

From conditions (ii)-(iii), Γ has compactly closed values. From (iv), Γ is a KKM mapping. It is obvious that $\{D, B\}$ is a coercive family for Γ. Now, by Theorem 2.4, DNMVVI $= \bigcap_{x\in K} \Gamma(x) \neq \emptyset$. Moreover, using (v) we obtain

$$\text{DNMVVI} = \bigcap_{x\in K} \Gamma(x) \subseteq B, \tag{3.8}$$

and hence

$$\text{DNMVVI} = \bigcap_{x\in K} \Gamma(x) = \bigcap_{x\in K} (\Gamma(x) \cap B), \tag{3.9}$$

which is closed in B (by (iii)) and so a compact set of B. $\square$

Remark 3.9. Condition (iii) of Theorem 3.7 holds when $f(y, .)$ is continuous and the mapping $P(y)$ is closed.

References

[1] H Ben-El-Mechaiekh, S. Chebbi and M. Florenzano, *A generalized KKMF principle*, J. Math. Anal. Appl. **309**(2005), 583-590.

[2] M. Bianchi and R. Pini, *Coercivity conditions for equilibrium problems*, J. Optim. Theory Appl. **124**(2005), 79-92.

[3] F. Giannessi, *Theorems of alternative, quadratic programs and complementarity problems*, In: Cottle, R.W., Giannessi, F., Lions, J.L. (eds.)Variational inequalities and complementarity problems, pp. 151-186. John Wiley and Sons, New York(1980).

[4] F. Giannessi, *Vector Variational Inequalities and Vector Equilibrium*, Kluwer Academic Publishers, Dordrecht(1999.)

[5] N. Hadjisavvas, *Continuity and maximality properties of pseudomonotone operators*, J. Convex Anal. **10**(2003), 459-469.

[6] G. Isac, *Topological Methods in Complementarity Theory*, Kluwer Academic Publishers, Dordrecht(2000).

[7] G. Isac, V.A. Bulavsky and V.V. Kalashnikov, *Complementarity, Equilibrium, Efficient and Economics*, Kluwer Academic Publishers, Dordrecht(2002).
[8] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York(1980).
[9] G. Stampacchia, *Formes bilinèaires coercitives sür les ensembles convexes*, Comptes Rendus deĺAcadèmie des Sciences **258**, 4413-4416(1964).