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EXISTENCE RESULTS FOR ABSTRACT IMPULSIVE DAMPED SECOND ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We establish the existence of mild solutions for a class of impulsive damped second-order partial neutral functional differential equations with infinite delay in a Banach space. The results are obtained using the cosine function theory and fixed point criterions.

1. Introduction

In this paper, we study the existence of mild solutions for a class of impulsive damped second-order abstract neutral functional differential equations with infinite delay of the form

$$\frac{d^2}{dt^2}\Big[x(t) - g(t, x_t)\Big] = Ax(t) + Bx'(t) + f(t, x_t, x'(t)), \quad t \in I = [0, a], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = \xi \in X, \tag{1.2}$$

$$\triangle x(t_i) = I_i^1(x_{t_i}), \quad i = 1, ..., n, \tag{1.3}$$

$$\triangle x'(t_i) = I_i^2(x_{t_i}), \quad i = 1, ..., n. \tag{1.4}$$

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Here, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on a Banach space X, B is a bounded linear operator on X; the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; g, f, I_i^j, are appropriate functions; $0 < t_1 < \cdots < t_n < a$ are prefixed numbers and the symbol $\triangle u(t)$ represents the jump of the function u at t, which is defined by $\triangle u(t) = u(t^+) - u(t^-)$.

Neutral functional differential equations (abbreviated, NFDE) arise in many areas of applied mathematics. For this reason, this type of equations have received much attention in recent years. The literature concerning first and second-order ordinary neutral functional differential equations is very extensive. We only mention the works [10, 21, 23], which are directly related to this work. First-order partial neutral functional differential equations have been studied by different authors. The reader can consult Adimy [1], Hale [9], and Wu [34] for systems with finite delay and Hernández & Henríquez [13, 14] and Hernández [11] for the unbounded delay case.

On the other hand, the theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often especially in phenomena studied in physics, chemical technology , population dynamics, biotechnology and economics. The theory of impulsive differential equations has become an important area of investigation in recent years and is much richer than the corresponding theory of classical differential equations. Several authors [3, 4, 22, 25, 27, 28, 29] have investigated the impulsive differential equations. The literature concerning second order functional differential equations is very extensive. We refer the reader to Hernandez et al to [12, 16, 17, 18] for second order impulsive differential equations. The damped first and second order differential equations have been studied by many authors [5, 6, 24, 30, 33].

The study of existence and qualitative properties of solutions of impulsive damped abstract partial neutral functional differential equations described by the form (1.1)-(1.4) is an untreated topic in the literature and this fact is the main motivation of this paper.

Next, we review some basic concepts, notations and properties needed to establish our results. Throughout this paper, A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on the Banach space $(X, \|\cdot\|)$ and $(S(t))_{t\in\mathbb{R}}$ is the associated sine function defined by $S(t)x = \int_0^t C(s)x ds$, for $x \in X$ and $t \in \mathbb{R}$. In this paper, $[D(A)]$ is the domain of A endowed with the norm $\| x \|_A = \| x \| + \| Ax \|$, $x \in D(A)$. The notation E represents the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is a function of class C^1 on $\mathbb{R}$. We know from Kisińsky [20] that E

endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0\le t\le 1} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function $\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space $E\times X$ with generator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from this property that $AS(t) : E \to X$ is a bounded linear operator and that $\| AS(t)x \| \to 0$ as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0,\infty) \to X$ is locally integrable, then $y(t) = \int_0^t S(t-s)x(s)ds$ is an E-valued continuous function. This is a consequence of the fact that

$$\int_0^t \mathcal{H}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds, & \int_0^t C(t-s)x(s)\,ds \end{bmatrix}^T,$$

defines an $E \times X$-valued continuous function.

From the definition of the norm in E, it follows that $u \in C(I;E)$ if, and only if, $u \in C(I;X)$ and the set of functions $\{AS(t)u(\cdot) : 0 \le t \le 1\}$ is an equicontinuous subset of $C(I;X)$.

The existence of solutions of the second-order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad t \in I, \tag{1.5}$$

$$x(0) = w, \quad x'(0) = z, \tag{1.6}$$

where $h \in L^1(I,X)$, is studied in [31]. On the other hand, the semilinear case has been treated in [32]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)w + S(t)z + \int_0^t S(t-s)h(s)\,ds, \quad t \in I, \tag{1.7}$$

is called a mild solution of (1.5)-(1.6), and that when $w \in E$ the function $x(\cdot)$ is of class C^1 and

$$x'(t) = AS(t)w + C(t)z + \int_0^t C(t-s)h(s)\,ds, \quad t \in I.$$

A function $u : [\sigma,\tau] \to X$ is said to be a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $\mathcal{PC}([\sigma,\tau],X)$ the space of normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all normalized piecewise continuous functions $u : [0,a] \to X$ such that u is continuous at $t \ne t_i$, $i = 1,\ldots,n$. It is clear that $\mathcal{PC}$ endowed with the norm of uniform convergence is a Banach space.

In what follows, we set $t_0 = 0, t_{n+1} = a$, and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_i$, for $i = 0, 1, ..., n$, the function $\tilde{u}_i \in C([t_i, t_{i+1}]; X)$ given by $\widetilde{u}_i(t) = u(t)$ for $t \in (t_i, t_{i+1}]$ and $\widetilde{u}_i(t_i) = \lim_{t \to t_i^+} u(t)$. Moreover, for a set $F \subseteq \mathcal{PC}$, we denote by $\widetilde{F}_i$, for $i = 0, 1, ..., n$, the set $\widetilde{F}_i = \{\tilde{u}_i : u \in F\}$.

Lemma 1.1. *A set $F \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, each set $\widetilde{F}_i$, $i = 0, 1, \cdots, n$, is relatively compact in $C([t_i, t_{i+1}], X)$.*

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$, similar to the one used in [19] and suitably modified to treat retarded impulsive differential equations. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$ and we will assume that $\mathcal{B}$ satisfies the following axioms:

- **(A)** If $x : (-\infty, \sigma + b] \to X$, $b > 0$, is such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma, \sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b], X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:
  - (i) x_t is in $\mathcal{B}$,
  - (ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$,
  - (iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,

  where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.
- **(B)** The space $\mathcal{B}$ is complete.

Remark 1.2. In impulsive functional differential systems, the map $[\sigma, \sigma+b] \to \mathcal{B}$, $t \to x_t$, is in general discontinuous. For this reason, this property has been omitted from our description of the phase space $\mathcal{B}$.

Next, we consider some examples of phase spaces.

Example 1.3. The phase space $\mathcal{PC}_\rho(X)$

We say that a function $\varphi : (-\infty, 0] \to X$ is normalized piecewise continuous if φ is left continuous and the restriction of φ to any interval $[-r, 0]$ is piecewise continuous.

Let $\rho : (-\infty, 0] \to [1, \infty)$ be a continuous function which satisfies the conditions (g-1), (g-2) in the terminology of [19]. Next, we slightly modify the definition of spaces C_ρ, C_ρ^0 in [19]. We denote by $\mathcal{PC}_\rho(X)$ the space formed by the normalized piecewise continuous functions φ such that $\frac{\varphi}{\rho}$ is bounded on $(-\infty, 0]$, and by $\mathcal{PC}_\rho^0(X)$ the subspace of $\mathcal{PC}_\rho(X)$ consisting of functions φ such that $\frac{\varphi(\theta)}{\rho(\theta)} \to 0$ as $\theta \to -\infty$. It is easy to see that $\mathcal{B} = \mathcal{PC}_\rho(X)$ and $\mathcal{B} = \mathcal{PC}_\rho^0(X)$ endowed with the norm $\| \varphi \|_{\mathcal{B}} := \sup_{\theta \in (-\infty, 0]} \frac{\|\varphi(\theta)\|}{\rho(\theta)}$ are phase spaces in the sense defined above.

Example 1.4. The phase space $\mathcal{PC}_r \times L^p(\rho, X)$

Let $r \geq 0$, $1 \leq p < \infty$ and let $\rho : (-\infty, -r] \to \mathbb{R}$ be a non-negative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [19]. Briefly, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $\mathcal{B} = \mathcal{PC}_r \times L^p(\rho, X)$ consists of all classes of Lebesgue-measurable functions $\varphi : (-\infty, 0] \to X$ such that $\varphi |_{[-r,0]} \in \mathcal{PC}([-r, 0], X)$ and $\rho \parallel \varphi \parallel^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in this space is defined by

$$\parallel \varphi \parallel_{\mathcal{B}} = \sup\{\parallel \varphi(\theta) \parallel : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \rho(\theta) \parallel \varphi(\theta) \parallel^p \, d\theta \right)^{1/p}.$$

Proceeding as in the proof of [19, Theorem 1.3.8], it follows that $\mathcal{B}$ is a space which satisfies the axioms **(A)** and **(B)**. Moreover, when $r = 0$ this space coincides with $C_0 \times L^p(\rho, X)$ and if, in addition, $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^{0} \rho(\theta) d\theta \right)^{1/2}$ for $t \geq 0$.

Let $(Z, \| \cdot \|_Z)$ and $(W, \| \cdot \|_W)$ be Banach spaces. In this paper, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W endowed with the operator norm and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. On the other hand, $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z, and for a bounded function $\gamma : [0, a] \to Z$ and $0 \leq t \leq a$, we will employ the notation $\gamma_{Z,t}$ to mean

$$\gamma_{Z,t} = \sup\{\parallel \gamma(s) \parallel_Z : s \in [0, t]\}. \tag{1.8}$$

We simplify this notation to γ_t when no confusion about the space Z arises. Additionally, for a function $\zeta : I \times Z \to W$ and $h \in \mathbb{R}$, we use the notation $\partial_h \zeta(t, z) = \frac{\zeta(t+h,z)-\zeta(t,z)}{h}$. If ζ is differentiable, we employ the decomposition

$$\begin{aligned} \zeta(s, w) - \zeta(t, z) \quad &= \quad D_1\zeta(t, z)(s - t) + D_2\zeta(t, z)(w - z) \\ &\quad + \|(s - t, w - z)\| \; R(\zeta(t, z), s - t, w - z), \end{aligned}$$

where $\|R(\zeta(t, z), \tilde{s}, \tilde{w})\|_W \to 0$ as $\|(\tilde{s}, \tilde{w})\| = |\tilde{s}| + \|\tilde{w}\|_Z \to 0$.

This paper is divided into two additional sections. In Section 2, we discuss the existence of mild solutions for impulsive second order neutral systems. Then, some applications of the theory are considered in Section 3.

2. Existence results

In this section, we discuss the existence of mild solutions for the system (1.1)-(1.4). We begin by studying the following impulsive neutral system

$$\frac{d^2}{dt^2}\Big[x(t)-g(t,x_t)\Big] = Ax(t)+Bx'(t)+f(t,x_t), \quad t\in I=[0,a], \tag{2.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2.2}$$

$$\frac{d}{dt}\Big[x(t)-g(t,x_t)\Big]\Big|_{t=0} = z\in X, \tag{2.3}$$

$$\triangle x(t_i) = I_i^1(x_{t_i}), \quad i=1,...,n. \tag{2.4}$$

In this section, M,N are constants such that $\|C(t)\|\le M$ and $\|S(t)\|\le N$ for all $t\in I$. The notation $\mathcal{F}(a)$ stands for the space

$$\mathcal{F}(a)=\{u:(-\infty,a]\to X : u|_I\in\mathcal{PC},\, u_0=0\},$$

endowed with the sup norm. In addition, $y:(-\infty,a]\to X$ is the function defined by $y_0=\varphi$ and $y(t)=C(t)\varphi(0)$ for $t\in I$.

In the following definition we introduce the concept of mild solution for system (2.1)-(2.4).

Definition 2.1. A function $x:(-\infty,a]\to X$ is called a mild solution of (2.1)-(2.4) if $x_0=\varphi$; $x\,|_I\in\mathcal{PC}$ and

$$\begin{aligned} x(t) &= C(t)(\varphi(0)-g(0,\varphi))+S(t)z+g(t,x_t) \\ &\quad +\sum_{i=0}^{j-1}[S(t-t_{i+1})Bx(t_{i+1}^-)-S(t-t_i)Bx(t_i^+)] \\ &\quad -S(t-t_j)Bx(t_j^+)+\int_0^t C(t-s)Bx(s)ds \\ &\quad +\int_0^t AS(t-s)g(s,x_s)ds+\int_0^t S(t-s)f(s,x_s)ds \\ &\quad +\sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i}), \end{aligned}$$

for all $t\in[t_j,t_{j+1}]$ and every $j=0,...,n$.

Remark 2.2. The above equation can also be written as

$$\begin{aligned} x(t) &= C(t)(\varphi(0)-g(0,\varphi))+S(t)z+g(t,x_t)+\int_0^t S(t-s)Bx'(s)ds \\ &\quad +\int_0^t AS(t-s)g(s,x_s)ds+\int_0^t S(t-s)f(s,x_s)ds \\ &\quad +\sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i}), \quad t\in I. \end{aligned}$$

Now, a integration by parts permit us to infer that $x(\cdot)$ is a mild solution of (2.1)-(2.4).

Remark 2.3. Clearly, a mild solution of (2.1)-(2.4) satisfies (2.2), (2.4). Nevertheless, a mild solution may be not differentiable at zero.

Motivated by this definition, we introduce the following assumptions.

- **(H1)** There exists a Banach space $(Y, \|\cdot\|_Y)$ continuously included in X such that $AS(t) \in \mathcal{L}(Y, X)$, for all $t \in I$, and $AS(\cdot)x \in C(I; X)$ for every $x \in Y$. There exist constants N_Y, N_1 such that $\|y\| \leq N_Y \|y\|_Y$, for all $y \in Y$, and $\|AS(t)\|_{\mathcal{L}(Y,X)} \leq N_1$, for all $t \in I$.
- **(H2)** $\mathcal{R}(C(t) - I)$ is closed and $dim\, Ker(C(t) - I) < \infty$, for every $0 < t \leq a$.

Remark 2.4. The condition **(H1)** is motivated by the fact that, in general, the function $AS(\cdot)$ defined from $[0, a]$ into $\mathcal{L}(X)$ is not integrable. In fact, if $\| \; AS(\cdot) \; \|_{\mathcal{L}(X)} \in \mathrm{L}^1([0, a])$, then by using the relation $C(t)x - C(s)x = A \int_s^t S(s)x ds$, $x \in X$, and that A is closed, we obtain

$$\| \; C(t)x - C(s)x \; \| = \left\| A \int_s^t S(\theta) x d\theta \right\| \leq \int_s^t \| \; AS(\theta) \; \|_{\mathcal{L}(X)} \, d\theta \, \| \; x \; \|,$$

proving that $C(\cdot) \in C(I; \mathcal{L}(X))$, and hence that A is bounded (see [31, Proposition 4.1]).

Remark 2.5. If condition **(H1)** holds, then Y is continuously included in E. In fact, for $y \in Y$

$$C(t)y - y = A \int_0^t S(s) y \, ds = \int_0^t AS(s) y \, ds, \tag{2.5}$$

which implies that $C(\cdot)y$ is of class C^1 and, therefore $Y \subseteq E$. Moreover, the inequality

$$\|y\|_E = \|y\| + \sup_{0 \leq t \leq 1} \|AS(t)y\| \leq (1 + N_1)\|y\| \leq (1 + N_1) N_Y \|y\|_Y,$$

shows that the inclusion $\iota : Y \to E$ is continuous. We observe that $[D(A)]$ and E satisfy **(H1)**.

Remark 2.6. The assumption **(H2)** is satisfied by a long list of differential operators; see in particular, the example in Section 3.

The following properties of cosine functions will be used to establish our results [17].

Lemma 2.7. *Let condition* **(H2)** *be satisfied and* $F \subseteq Y$. *If* F *is bounded in* X *and the set* $\{AS(t)y : t \in [0, b],\ y \in F\}$ *is relatively compact in* X, *then* F *is relatively compact in* X.

Lemma 2.8. *Let* $F \subseteq E$ *and assume that* $V(b) = \{AS(t)x : t \in [0, b], x \in F\}$, $0 \le b \le a$, *is relatively compact in* X. *Then* $AS(h)x \to 0$, $h \to 0$, *uniformly for* $x \in F$.

We can consider a mild solution $x(\cdot)$ as a function which has jumps at the points t_i, for $i = 1, ..., n$, so that the tangent vector of the curve $x(t)$ does not change when t goes from t_i^- to t_i^+. Note however that, according to the definition, a mild solution $x(\cdot)$ need not be differentiable, so that this explanation is only a rough translation of the concept.

To establish existence of solutions we consider the following assumptions:

(H3) The function $g : I \times \mathcal{B} \to Y$ satisfies the following conditions:
- (i) For each $t \in I$, $g(t, \cdot) : \mathcal{B} \to Y$ is continuous.
- (ii) Let $x : (-\infty, a] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. Then the function $t \mapsto g(t, x_t)$ belongs to $\mathcal{PC}$ and is strongly measurable from I into Y.
- (iii) There exist a function $m_g \in L^1(I, [0, \infty))$ and a continuous nondecreasing function $W_g : [0, \infty) \to (0, \infty)$ such that $\|g(t, \psi)\|_Y \le m_g(t) W_g(\|\psi\|_{\mathcal{B}})$, for all $(t, \psi) \in I \times \mathcal{B}$.

(H4) The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:
- (i) For each $t \in I$, $f(t, \cdot) : \mathcal{B} \to X$ is continuous.
- (ii) Let $x : (-\infty, a] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. Then the function $[0, a] \to X$, $t \mapsto f(t, x_t)$, is strongly measurable.
- (iii) There exist a function $m_f \in L^1(I, [0, \infty))$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that $\|f(t, \psi)\| \le m_f(t) W_f(\| \psi \|_{\mathcal{B}})$, for all $(t, \psi) \in I \times \mathcal{B}$.

Remark 2.9. In what follows we set $W = \max\{W_g, W_f\}$.

In connection with these conditions it is worth mentioning the following remarks.

Remark 2.10. If $(\mathbf{H_1}), (\mathbf{H_3})$ are satisfied and $x : (-\infty, a] \to X$ is a function such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$, then from Bochner's criterion for integrable functions and the estimate

$$\| AS(t-s) g(s, x_s) \| \le N_1 m_g(s) W(K_a \| x \|_a + M_a \| \varphi \|_{\mathcal{B}}),$$

we infer that $s \to AS(t-s) g(s, x_s) \in L^1([0, t]; X)$ for every $t \in I$.

Remark 2.11. Concerning $(\mathbf{H_3})$(ii) and $(\mathbf{H_4})$(ii) we only remark that, in general, the function $t \to x_t$ is not continuous.

We have conditions now to establish our first existence result.

Theorem 2.12. *Assume that* **(H1)-(H4)** *are satisfied and that the following conditions hold:*

(a) *The set $U(r,t) = \{S(t)f(s,\psi) : s \in I, \psi \in B_r(0,\mathcal{B})\}$ is relatively compact in X, for each $t \in I$ and all $r > 0$.*

(b) *Let $r > 0$ and $V(r,g)$ be the set of functions $V(r,g) = \{t \to g(t, u_t + y_t); u \in B_r(0, \mathcal{F}(a))\}$. The set $V(r) = \{AS(\theta)g(s,\psi) : \theta, s \in I, \psi \in B_r(0,\mathcal{B})\}$ is pre-compact in X; for all $i = 1, \ldots, n$, the set $\{\widetilde{v}_i : v \in V(r,g)\}$ is an equicontinuous subset of $C([t_i, t_i+1], X)$ and there are positive constants c_1, c_2 such that $\|g(t,\psi)\|_Y \leq c_1\|\psi\|_{\mathcal{B}} + c_2$, for every $(t,\psi) \in I \times \mathcal{B}$.*

(c) *The maps B, $I_i^1 : \mathcal{B} \to X$ are completely continuous and there are positive constants d_i^j, $j = 1,2$, such that $\| I_i^1(\psi) \| \leq d_i^1 \| \psi \|_{\mathcal{B}} + d_i^2$, for every $i = 1, ..., n$, and all $\psi \in \mathcal{B}$.*

(d) *The constant $\mu = 1 - K_a[N_Y c_1 + 3N \| B \| H + M \sum_{i=1}^n d_i^1] > 0$, and*

$$\frac{K_a}{\mu}\Big[aM \| B \| H + \int_0^a [N_1 m_g(s) + N m_f(s)]\, ds\Big] < \int_C^\infty \frac{ds}{W(s)}, \qquad (2.6)$$

where

$$C = \frac{1}{\mu}\left[K_a\left(N_Y c_2 + M(\| g(0,\varphi)\| + \sum_{i=1}^n d_i^2) + N \| z \|\right) + \| y \|_{\mathcal{B},a}\right].$$

Then there exists a mild solution of (2.1)-(2.4).

Proof. We define the map Γ on the space $\mathcal{F}(a)$ by $(\Gamma u)_0 = 0$ and

$$\begin{aligned}\Gamma u(t) =& - C(t)g(0,\varphi) + S(t)z + g(t, u_t + y_t)\\ &+ \sum_{i=0}^{j-1}\Big[S(t-t_{i+1})B(u(t_{i+1}^-) + y(t_{i+1}^-))\\ &- S(t-t_i)B(u(t_i^+) + y(t_i^+))\Big] - S(t-t_j)B(u(t_j^+) + y(t_j^+))\\ &+ \int_0^t C(t-s)B(u(s)+y(s))ds + \int_0^t AS(t-s)g(s, u_s + y_s)ds\\ &+ \int_0^t S(t-s)f(s, u_s + y_s)ds + \sum_{0<t_i<t} C(t-t_i)I_i^1(u_{t_i} + y_{t_i}),\end{aligned}$$

if $t \in [t_j, t_{j+1}]$ for some $j = 0, \cdots, n$. From **(H3), (H4)** and Remark 2.10 we infer that $\Gamma u \in \mathcal{F}(a)$. In order to apply Theorem [8, Theorem 6.5.4], we need to

obtain an *a priori* bound for the solutions of the integral equation $u = \lambda\Gamma(u)$, $\lambda \in (0,1)$. To this end, let u^λ be a solution of $u = \lambda\Gamma(u)$, $\lambda \in (0,1)$.
Using the notation $\beta_\lambda(t) = \|u_t^\lambda + y_t\|_{\mathcal{B}} \leq K_a \| u^\lambda \|_t + \|y_s\|_{\mathcal{B},a}$, we observe that

$$\begin{aligned}
\| u^\lambda(t) \| &\leq M \| g(0,\varphi)\| + N \| z \| + N_Y c_1 \beta_\lambda(t) + N_Y c_2 + 3N \| B \| H\beta_\lambda(t) \\
&\quad + M \| B \| H \int_0^t \beta_\lambda(s)ds + M \sum_{0<t_i<t} d_i^1 \beta_\lambda(t_i) + M \sum_{0<t_i<t} d_i^2 \\
&\quad + \int_0^t \| AS(t-s) \|_{\mathcal{L}(Y,X)} \| g(s, x_s + y_s) \|_Y \, ds \\
&\quad + N \int_0^t m_f(s) W(\beta_\lambda(s))ds \\
&\leq \left(N_Y c_1 + 3N \| B \| H + M \sum_{i=1}^n d_i^1 \right) \beta_\lambda(t) + N_Y c_2 + M \| g(0,\varphi)\| \\
&\quad + N \| z \| + M \sum_{0<t_i<t} d_i^2 + M \| B \| H \int_0^t \beta_\lambda(s)ds \\
&\quad + \int_0^t [N_1 m_g(s) + N m_f(s)] W(\beta_\lambda(s))ds
\end{aligned}$$

which yields

$$\begin{aligned}
\beta_\lambda(t) &\leq \frac{K_a}{\mu} \left(N_Y c_2 + M \| g(0,\varphi)\| + N \| z \| + M \sum_{0<t_i<t} d_i^2 \right) + \frac{1}{\mu} \|y_s\|_{\mathcal{B},a} \\
&\quad + \frac{K_a}{\mu} \left(M \| B \| H \int_0^t \beta_\lambda(s)ds + \int_0^t [N_1 m_g(s) + N m_f(s)] W(\beta_\lambda(s))ds \right),
\end{aligned}$$

for $t \in I$. Denoting by $\alpha_\lambda(t)$ the right-hand side of the previous inequality, we see that

$$\alpha_\lambda'(t) \leq \frac{K_a}{\mu} \Big[M \| B \| H \alpha_\lambda(t) + (N_1 m_g(t) + N m_f(t)) W(\alpha_\lambda(t)) \Big],$$

and subsequently, upon integrating over $[0,t]$, we obtain

$$\begin{aligned}
\int_C^{\alpha_\lambda(t)} \frac{ds}{s + W(s)} &\leq \frac{K_a}{\mu} \Big[aM \| B \| H + \int_0^t [N_1 m_g(s) + N m_f(s)] ds \Big] \\
&< \int_C^\infty \frac{ds}{s + W(s)},
\end{aligned}$$

since $\alpha_\lambda(0) = C$. Now, (2.6) enables us to conclude that the set of functions $\{\alpha_\lambda : \lambda \in (0,1)\}$ is uniformly bounded on $[0,a]$ and, in turn, that $\{u^\lambda : \lambda \in (0,1)\}$ is uniformly bounded on $[0,a]$.

To prove that Γ is completely continuous, we introduce the decomposition $\Gamma = \sum_{i=1}^{4} \Gamma_i$, where $(\Gamma_i u)_0 = 0$ and

$$\begin{aligned}\Gamma_1 u(t) = &- C(t)g(0,\varphi) + S(t)z + g(t, u_t + y_t)\\ &+ \sum_{i=0}^{j-1} \Big[S(t-t_{i+1})B(u(t_{i+1}^-) + y(t_{i+1}^-)) - S(t-t_i)B(u(t_i^+) + y(t_i^+)) \Big]\\ &- S(t-t_j)B(u(t_j^+) + y(t_j^+)) + \sum_{0<t_i<t} C(t-t_i) I_i^1(u_{t_i} + y_{t_i}),\end{aligned}$$

$$\Gamma_2 u(t) = \int_0^t AS(t-s)g(s, u_s + y_s)ds,$$

$$\Gamma_3 u(t) = \int_0^t S(t-s)f(s, u_s + y_s)ds + \int_0^t C(t-s)B(u(s) + y(s))ds, \quad \text{for } t \in I.$$

From the assumptions and Lemma 1.1, it is easy to see that the map Γ_1 is completely continuous. From [17, Theorem 2.1], Γ_2 is completely continuous. Moreover, from [15, Lemma 3.1] we infer that Γ_3 is also completely continuous. Finally, from [8, Theorem 6.5.4] we infer the existence of a fixed point u of Γ. It is clear that the function $x = u + y$ is a mild solution of (2.5). This completes the proof. □

In applications the operator $S(t)$ is often compact. This motivates the next result.

Corollary 2.13. *Let assumptions* **(H1)-(H4)** *hold. If $S(t)$ is compact, for every $t \geq 0$, and conditions* (b), (c) *and* (d) *of Theorem* 2.12 *hold, then there exists a mild solution of* (2.1)-(2.4).

In the next results we establish conditions that guarantee a mild solution satisfies (2.3).

Proposition 2.14. *Assume that the hypotheses of Theorem* 2.12 *are fulfilled and that $\varphi(0) \in E$. If $x(\cdot)$ is a mild solution of* (2.1)-(2.4)*, then condition* (2.3) *holds.*

Proof. Clearly, $\frac{1}{t}\int_0^t S(t-s)f(s,x_s)ds \to 0$, $\frac{1}{t}\int_0^t S(t-s)Bx'(s)ds \to 0$ as $t \to 0^+$. In addition, for $\delta > 0$ we can write

$$\begin{aligned}\int_0^t AS(t-s)g(s,x_s)ds = &\Big(I - \frac{1}{\delta}S(\delta)\Big)\int_0^t AS(t-s)g(s,x_s)\,ds\\ &+ \frac{1}{\delta}\int_0^t S(t-s)AS(\delta)g(s,x_s)ds.\end{aligned}$$

Let r, r^* be positive constants such that $\|x_s\|_{\mathcal{B}} \leq r$, for all $s \in I$, and $V(r) \subset B_{r^*}(0, X)$. Since $AS(t-s)g(s, x_s) \in V(r)$ for every $s \in I$, it follows from the mean value theorem that

$$\Big(I - \frac{1}{\delta}S(\delta)\Big)\frac{1}{t}\int_0^t AS(t-s)g(s,x_s)\,ds \in \Big(I - \frac{1}{\delta}S(\delta)\Big)\overline{c(V(r))}.$$

Since $(I - \frac{1}{\delta}S(\delta))x \to 0, \delta \to 0$, for each $x \in X$ and $\overline{c(V(r))}$ is compact in X, we have that $(I - \frac{1}{\delta}S(\delta))x \to 0, \delta \to 0$, uniformly for $x \in \overline{c(V(r))}$. This, together with the estimate

$$\Big\|\frac{1}{\delta}\int_0^t S(t-s)AS(\delta)g(s,x_s)\,ds\Big\| \leq \frac{Nr^*}{\delta}\int_0^t (t-s)\,ds \leq \frac{Nr^*}{2\delta}t^2,$$

enables us to conclude that $\frac{1}{t}\int_0^t AS(t-s)g(s,x_s)ds \to 0,\ t \to 0^+$. From these remarks we see that

$$\begin{aligned}
&\lim_{t\to 0^+}\frac{x(t) - g(t,x_t) - \varphi(0) + g(0,\varphi)}{t}\\
&= \lim_{t\to 0^+}\frac{1}{t}\,(\,S(t)z - (C(t)-I)(g(0,\varphi)+\varphi(0))\,)\\
&\quad + \lim_{t\to 0^+}\frac{1}{t}\Big(\int_0^t S(t-s)f(s,x_s)ds + \int_0^t AS(t-s)g(s,x_s)ds\Big)\\
&= z,
\end{aligned}$$

which shows the assertion. □

Remark 2.15. In the sequel of this paper, $\mathcal{T}_t\varphi : (-\infty, 0] \to X$ is the function given by $\mathcal{T}_t\varphi(\theta) = \varphi(0)$, for $\theta \in [-t, 0]$ and $\mathcal{T}_t\varphi(\theta) = \varphi(t+\theta)$, for $\theta \in (-\infty, -t]$.

Theorem 2.16. *Assume that* **(H1), (H3)** *and* **(H4)** *hold and that f satisfies condition (a) of Theorem 2.12. Suppose that the function $g(\cdot, \varphi)$ is bounded on I and that there exist positive constants L_g, L_i^1, $i = 1, ..., n$, such that*

$$\begin{aligned}
\| g(t,\psi_1) - g(t,\psi_2) \|_Y &\leq L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad (t,\psi_i) \in I \times \mathcal{B},\\
\| I_i^1(\psi_1) - I_i^1(\psi_2) \| &\leq L_i^1 \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad \psi_i \in \mathcal{B},
\end{aligned}$$

and

$$\begin{aligned}
K_a\Big[&L_g(N_Y + aN_1) + \frac{1}{K_a}(3N + aM) \| B \| + M\sum_{i=1}^n L_i^1\\
&+ N\liminf_{\xi\to\infty}\frac{W_f(\xi)}{\xi}\int_0^a m_f(s)\,ds\Big] < 1.
\end{aligned}$$

Then there exists a mild solution of (2.1)-(2.4). *Moreover, if* $AS(h)g(t,\mathcal{T}_t\varphi)\to 0$, *as* $h\to 0$, *uniformly for* $t\in I$ *and* $\varphi(0)\in E$, *then each mild solution of* (2.1)-(2.4) *satisfies* (2.3).

Proof. Let Γ be defined as in the proof of Theorem 2.12. We claim that there exists an $r>0$ such that $\Gamma(B_r(0,\mathcal{F}(a)))\subseteq B_r(0,\mathcal{F}(a))$. If we assume that this assertion is false, then for each $r>0$ we can choose $u^r\in B_r(0,\mathcal{F}(a))$, $j\in\{0,...,n\}$ and $t^r\in[t_j,t_{j+1}]$ such that $r<\parallel \Gamma u^r(t^r)\parallel$. Consequently,

$$\begin{aligned}
r <\ & \parallel g(t^r,\varphi)-C(t^r)g(0,\varphi)\parallel + N_Y \parallel g(t^r,u_{t^r}+y_{t^r})-g(t^r,\varphi)\parallel_Y + N\parallel z\parallel \\
&+(3N+aM)\parallel B\parallel (r+\parallel y\parallel_a)+N_1\int_0^a \parallel g(s,u_s^r+y_s)-g(s,y_s)\parallel_Y ds \\
&+N_1\int_0^a \parallel g(s,y_s)\parallel_Y\ ds + N\int_0^a m_f(s)W_f(\parallel u_s^r+y_s\parallel_{\mathcal{B}})\,ds \\
&+M\sum_{i=1}^n \left[\parallel I_i^1(u_{t_i}^r+y_{t_i})-I_i^1(y_{t_i})\parallel + \parallel I_i^1(y_{t_i})\parallel\right] \\
\le\ & \parallel g(t^r,\varphi)-C(t^r)g(0,\varphi)\parallel + K_aN_YL_gr + N_YL_g\parallel y_s-\varphi\parallel_{\mathcal{B},a} + N\parallel z\parallel \\
&+(3N+aM)\parallel B\parallel (r+\parallel y\parallel_a)+aK_aL_grN_1+N_1\int_0^a \parallel g(s,y_s)\parallel_Y ds \\
&+N\int_0^a m_f(s)W_f(K_ar+\parallel y_s\parallel_{\mathcal{B},a})\,ds + M\sum_{i=1}^n\left[L_i^1K_ar+\parallel I_i^1(y_{t_i})\parallel\right]
\end{aligned}$$

and hence,

$$\begin{aligned}
1\le K_a\Big[&L_g(N_Y+aN_1)+\frac{1}{K_a}(3N+aM)\parallel B\parallel + M\sum_{i=1}^n L_i^1 \\
&+N\liminf_{\xi\to\infty}\frac{W_f(\xi)}{\xi}\int_0^a m_f(s)\,ds\Big],
\end{aligned}$$

which is contrary to our assumptions.

Let $r>0$ with $\Gamma(B_r(0,\mathcal{F}(a)))\subseteq B_r(0,\mathcal{F}(a))$ and consider the decomposition $\Gamma=\Gamma_1+\Gamma_2$, where

$$\begin{aligned}
\Gamma_1u(t) =& -C(t)g(0,\varphi)+S(t)z+g(t,u_t+y_t) \\
&+\sum_{i=0}^{j-1}\Big[S(t-t_{i+1})B(u(t_{i+1}^-)+y(t_{i+1}^-))-S(t-t_i)B(u(t_i^+)+y(t_i^+))\Big] \\
&-S(t-t_j)B(u(t_j^+)+y(t_j^+))+\int_0^t C(t-s)B(u(s)+y(s))ds
\end{aligned}$$

$$+\int_0^t AS(t-s)g(s,u_s+y_s)\,ds + \sum_{0<t_i<t} C(t-t_i)I_i^1(u_{t_i}+y_{t_i}),$$

$$\Gamma_2 u(t) = \int_0^t S(t-s)f(s,u_s+y_s)\,ds, \quad t\in I.$$

From [15, Lemma 3.1] we have that Γ_2 is completely continuous. This fact and the estimate

$$\|\,\Gamma_1 u - \Gamma_1 v\,\| \le K_a\Big[\,L_g(N_Y + aN_1) + \frac{1}{K_a}(3N + aM)\,\|\,B\,\| + M\sum_{i=1}^n L_i^1\Big]\,\|\,u-v\,\|_a,$$

together imply that Γ is condensing on $B_r(0,\mathcal{F}(a))$. Now, from [26, Corollary 4.3.2] we obtain the existence of a fixed point u of Γ. Clearly, $x=u+y$ is a mild solution of (2.1)-(2.4).

We next show that if $\varphi(0)\in E$, then each mild solution x satisfies (2.3). At first, we estimate

$$\left\|\frac{1}{t}\int_0^t AS(t-s)g(s,x_s)ds\right\| \le \frac{N_1}{t}\int_0^t \|g(s,x_s)-g(s,\mathcal{T}_s\varphi)\|_Y ds + \frac{1}{t}\int_0^t \|AS(t-s)g(s,\mathcal{T}_s\varphi)\|\,ds.$$

Since $\|\,x_s - \mathcal{T}_s\varphi\,\|_{\mathcal{B}} \to 0$ as $s\to 0$, and $\|AS(h)g(s,\mathcal{T}_s\varphi)\|\to 0, h\to 0$, (by assumption), it follows that

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^t AS(t-s)g(s,x_s)\,ds = 0.$$

Now, the proof can be completed proceeding as in the proof of Proposition 2.14 □

Remark 2.17. The condition $AS(h)g(t,\mathcal{T}_t\varphi)\to 0$, $h\to 0$, uniformly for $t\in I$, can be dropped when the map $AS: I\to\mathcal{L}(Y,X)$ is Lipschitz continuous [2, Proposition 1.3.7].

In Theorem 2.19 below, we establish the existence of mild solutions using the classical contraction principle. To prove this result, we need the following lemma.

Lemma 2.18. *Let $0<c_1<1$ and $c_2>0$. Let $a_0\in C([a,b]:\mathbb{R})$ and $(a_n)_{n\in\mathbb{N}}$ be the sequence in a $C([a,b]:\mathbb{R})$ defined by the recurrence relation $a_{n+1}(t)=c_1a_n(t)+c_1\int_0^t a_n(s)ds$, $n\in\mathbb{N}$. Then $a_n(t)\to 0$, uniformly on I.*

Proof. On the space $C([a,b]:\mathbb{R})$ we define the map $P: C([a,b]:\mathbb{R}) \to C([a,b]:\mathbb{R})$ by

$$Px(t) = c_1 x(t) + c_2 \int_0^t x(s)ds.$$

Clearly, P is a bounded linear operator with $\sigma(P) = \{c_1\}$. Thus, $r(P) = \lim_{n\to\infty} \| P^n \| = c_1 < 1$ which implies that $a_{n+1} = P^n(a_0) \to 0$ as $n \to \infty$. □

Theorem 2.19. *Assume that* **(H1), (H3)** *and* **(H4)** *hold. Suppose that the function $g(\cdot,\varphi)$ is bounded on I and that there exist constants L_f, L_g, L_i^1, for $i = 1,...,n$, such that*

$$\begin{aligned} \| g(t,\psi_1) - g(t,\psi_2) \|_Y &\leq L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \\ \| f(t,\psi_1) - f(t,\psi_2) \| &\leq L_f \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \\ \| I_i^1(\psi_1) - I_i^1(\psi_2) \| &\leq L_i^1 \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \end{aligned}$$

for all $t \in I$ and each $\psi_1,\psi_2 \in \mathcal{B}$, $i = 1,...,n$, and $K_a[L_g(N_Y + aN_1) + \frac{1}{K_a}(3N + aM)\|B\| + M\sum_{i=1}^n L_i^1] < 1$. Then, there exists a unique mild solution x of system (2.1)-(2.4)*. Furthermore, if $\varphi(0) \in E$ and $AS(h)g(t,\mathcal{T}_t\varphi) \to 0$, $h \to 0$, uniformly for $t \in I$, then x satisfies* (2.3)*.*

Proof. Let Γ be defined as in the proof of Theorem 2.12. For $u,v \in \mathcal{F}(a)$, we have that

$$\begin{aligned} \| \Gamma u(t) - \Gamma v(t) \| &\leq N_Y L_g K_a \| u - v \|_t + 3N \| B \| \| u - v \|_t \\ &\quad + MK_a \sum_{i=1}^n L_i^1 \| u - v \|_{t_i} + M \| B \| t \| u - v \|_t \\ &\quad + N_1 L_g K_a t \| u - v \|_t + N L_f K_a t \| u - v \|_t \\ &\leq (c_1 + c_2 t) \| u - v \|_t, \end{aligned}$$

where $c_1 = K_a(N_Y L_g + M\sum_{i=1}^n L_i^1) + 3N \| B \|$ and $c_2 = K_a(N_1 L_g + N L_f) + M \| B \|$. Proceeding as above, we get

$$\begin{aligned} \| \Gamma^2 u(t) - \Gamma^2 v(t) \| &\leq c_1 \| \Gamma u(t) - \Gamma v(t) \| + M \| B \| \int_0^t \| \Gamma u(s) - \Gamma v(s) \| ds \\ &\quad + N_1 L_g \int_0^t \| (\Gamma u)_s - (\Gamma v)_s \| ds \\ &\quad + N L_f \int_0^t \| (\Gamma u)_s - (\Gamma v)_s \| ds \end{aligned}$$

$$
\begin{aligned}
&\leq c_1(c_1 + c_2 t) \parallel u - v \parallel_t \\
&\quad + M \parallel B \parallel \int_0^t (c_1 + c_2 s) \parallel u - v \parallel_s ds \\
&\quad + N_1 L_g K_a \int_0^t (c_1 + c_2 s) \parallel u - v \parallel_s ds \\
&\quad + N L_f K_a \int_0^t (c_1 + c_2 s) \parallel u - v \parallel_s ds \\
&\leq c_1(c_1 + c_2 t) \parallel u - v \parallel_t + \Big[M \parallel B \parallel + N_1 L_g K_a \\
&\quad + N L_f K_a \Big] \int_0^t (c_1 + c_2 s) ds \parallel u - v \parallel_t \\
&\leq \Big[c_1(c_1 + c_2 t) + c_2 \int_0^t (\, c_1 + c_2 s \,) ds \Big] \parallel u - v \parallel_t .
\end{aligned}
$$

Repeating the argument inductively, we obtain that

$$\parallel \Gamma^n u - \Gamma^n v \parallel_t \leq a_n(t) \parallel u - v \parallel_t,$$

where the functions a_n satisfy the recurrence relation

$$a_{n+1}(t) = c_1 a_n(t) + c_2 \int_0^t a_n(s) ds, \qquad t \in I, \ n \in \mathbb{N}.$$

Now, from Lemma 2.18, we conclude that $a_n(t) \to 0$ uniformly for $t \in I$ which implies that Γ^n is a contraction for n sufficiently large. This shows that Γ has a unique fixed point and, consequently, that there exists a unique mild solution of (2.1)-(2.4). The assertion related condition (2.3) is proved arguing as in the proof of Proposition 2.14. The proof is complete. □

In the next result, we show that under suitable conditions a mild solution of (2.1)-(2.4) satisfies condition (1.2). To establish this property, we select $z = \xi - \eta$ where $\eta \in X$ is an element related to the function g in a form to be specified in the statement of Theorem 2.20. We also introduce the following subspaces of $\mathcal{B}$

$$
\begin{aligned}
\Lambda_1 &= \{ \varphi \in \mathcal{B} : \ t \to \mathcal{T}_t \varphi \text{ is continuous on } [0, \infty) \}, \\
\Lambda_2 &= \{ \varphi \in \mathcal{B} : \parallel \mathcal{T}_t \varphi - \varphi \parallel_{\mathcal{B}} \leq L_\varphi t, \ 0 \leq t \leq t_\varphi \ \text{ and some } L_\varphi > 0, t_\varphi > 0 \}, \\
\Lambda_3 &= \{ \varphi \in \mathcal{B} : \ t \to \mathcal{T}_t \varphi \text{ is differentiable at } t = 0 \}.
\end{aligned}
$$

It is easy to see that if $\varphi \in \Lambda_3$ with $\frac{d}{dt}\mathcal{T}_t \varphi \mid_{t=0} = \psi$ and $\psi(0) \in E$, then $\frac{d}{dt} y_t \mid_{t=0} = \psi$. Similarly, if $\varphi \in \Lambda_2$ and $\varphi(0) \in E$, then the function $t \to y_t$ is Lipschitz continuous.

Now we consider the following general assumptions.

(H5) For each $r > 0$, $AS(t)g(s, \psi) \to 0$ as $t \to 0$ uniformly for $0 \leq s \leq t$ and $\psi \in B_r(0, \mathcal{B})$.

(H6) Let $x : (-\infty, \sigma + b] \to X$, $b > 0$, be the function in axiom **A** and assume that $x(\theta) = 0$ for $\theta < \sigma$ and $\| x(\theta) \| \leq n$, for every $\theta \leq \sigma + b$. Then $\| x_t \|_{\mathcal{B}} \leq K^n(t - \sigma) \| x(t) \| + M^n(t - \sigma)$, where $K^n, M^n : [0, \infty) \to [1, \infty)$ are continuous, $M^n(0) = 0, K^n(0) = 1$ and K^n, M^n are independent of $x(\cdot)$.

We observe that assumption (**H6**) is satisfied, for instance, by the spaces $\mathcal{PC}_0 \times L^p(\rho, X)$, $p > 1$. We also point out that condition (**H5**) holds in the following cases.

(i) $\| AS(t) \|_Y \to 0$ as $t \to 0^+$ and $m_g \in L^\infty(I)$.
(ii) The set $g(I \times B_r(0, \mathcal{B}))$ is relatively compact in E.
(iii) The function $s \to g(s, x_s)$, belongs to $\mathcal{PC}(E)$ for all the functions involved in assumption **H3**(ii). Here, $\mathcal{PC}(E)$ is the space formed by all normalized piecewise continuous functions $u : [0, a] \to E$ such that u is continuous at $t \neq t_i$, $i = 1, \ldots, n$, endowed with the norm of the uniform convergence topology.
(iv) The set $g(I \times B_r(0, \mathcal{B}))$ is $[D(A)]$-bounded.

In the next result, for $x \in X$, we use the notation χ_x for the function $\chi_x : (-\infty, 0] \to X$ given by $\chi_x(\theta) = 0$ for $\theta < 0$ and $\chi_x(0) = x$.

Theorem 2.20. *Assume that* (**H1**), (**H3**), (**H4**) *and* (**H5**) *hold,* $\varphi(0) \in E$ *and that* g *is differentiable at* $(0, \varphi)$*. Let* x *be a mild solution of* (2.1)-(2.4) *which satisfies the condition* (2.3) *with* $z = \xi - \eta$*. If either of the following conditions hold*

(i) $\varphi \in \Lambda_2$, $D_2 g(0, \varphi) \equiv 0$ *and* $\xi = \eta = D_1 g(0, \varphi)$;
(ii) *the assumption* (**H6**) *holds and* $\chi_\xi \in \mathcal{B}$*. In addition,* $\varphi \in \Lambda_3$ *with* $\frac{d}{dt}\mathcal{T}_t \varphi \mid_{t=0} = \psi$, $\| D_2 g(0, \varphi) \| K(0) < 1$ *and* $\eta = D_1 g(0, \varphi) + D_2 g(0, \varphi)(\psi + \chi_\xi)$;

then x *satisfies the initial condition* (1.2).

Proof. Since $\frac{d}{dt} C(t)\varphi(0) \mid_{t=0} = 0$, we only need to prove that the function $u(t) = x(t) - C(t)\varphi(0)$ has derivative ξ at $t = 0$. In the sequel, we use the abbreviations $\tilde{g}(s) = g(s, x_s)$ and $\tilde{f}(s) = f(s, x_s)$, the notation introduced in (1.9) and the relation (2.5). Proceeding as in the previous results, we write $x(t) = u(t) + y(t)$. Using these notations, for $0 < t < t_1$ we have that

$$\begin{aligned} u(t) &= -C(t)g(0,\varphi) + S(t)z + g(t,x_t) + \int_0^t AS(t-s)\tilde{g}(s)ds \\ &\quad + \int_0^t S(t-s)Bx'(s)ds + \int_0^t S(t-s)\tilde{f}(s)ds \\ &= S(t)z + g(t,x_t) - g(0,\varphi) + \int_0^t AS(t-s)[\tilde{g}(s)-\tilde{g}(0)]ds \\ &\quad + \int_0^t S(t-s)Bx'(s)ds + \int_0^t S(t-s)\tilde{f}(s)ds \end{aligned}$$

and hence

$$\begin{aligned} u(t) = {} & S(t)z + D_1 g(0,\varphi)t + D_2 g(0,\varphi)(x_t-\varphi) \\ & + \parallel (t, x_t-\varphi) \parallel R(g(0,\varphi), t, x_t-\varphi) \\ & + \int_0^t AS(t-s)[\tilde{g}(s)-\tilde{g}(0)]ds \\ & + \int_0^t S(t-s)Bx'(s)ds + \int_0^t S(t-s)\tilde{f}(s)ds. \end{aligned} \tag{2.7}$$

It is follows from (2.7) that $u(t) \to 0$ as $t \to 0$ and from **A**(iii) we infer that $u_t \to 0$, as $t \to 0$. Since $\parallel x_t - \varphi \parallel_{\mathcal{B}} \leq \parallel u_t \parallel_{\mathcal{B}} + \parallel y_t - \varphi \parallel_{\mathcal{B}} \to 0$, as $t \to 0$, we know that for $\epsilon > 0$ there exists $0 < \delta(\epsilon) < t_1$ such that $\parallel R(g(0,\varphi), t, x_t - \varphi) \parallel \leq \epsilon$ for all $t \in [0, \delta(\epsilon)]$. From (2.7) we see that

$$\begin{aligned} \parallel \frac{u(t)}{t} \parallel \leq {} & N \parallel z \parallel + \parallel D_1 g(0,\varphi) \parallel + \parallel D_2 g(0,\varphi) \parallel \left(\parallel \frac{u_t}{t} \parallel_{\mathcal{B}} + \parallel \frac{y_t-\varphi}{t} \parallel_{\mathcal{B}} \right) \\ & + \epsilon \left(1 + \parallel \frac{u_t}{t} \parallel_{\mathcal{B}} + \parallel \frac{y_t-\varphi}{t} \parallel_{\mathcal{B}} \right) \\ & + \frac{1}{t} \int_0^t \parallel AS(t-s)[\tilde{g}(s)-\tilde{g}(0)] \parallel ds \\ & + N \int_0^t \parallel Bx'(s) \parallel ds + N \int_0^t \parallel \tilde{f}(s) \parallel ds, \end{aligned}$$

for every $t \in [0, \delta(\epsilon)]$. Choosing $\epsilon > 0$ sufficiently small, applying (**H5**) and **(A)**(iii), we can conclude from the above estimate that in both cases, (i) and (ii), the set $\{\frac{u_t}{t} : t \in (0, t_1]\}$ is bounded in $\mathcal{B}$.

If condition (i) is valid, then from (2.7) we see that

$$\begin{aligned} u(t) =& D_1g(0,\varphi)t+ \parallel (t, x_t - \varphi) \parallel R(g(0,\varphi), t, x_t - \varphi) \\ &+ \int_0^t AS(t-s)[\tilde{g}(s) - \tilde{g}(0)]ds \\ &+ \int_0^t S(t-s)Bx'(s)ds + \int_0^t S(t-s)\tilde{f}(s)ds. \end{aligned} \tag{2.8}$$

Proceeding as before we obtain

$$\begin{aligned} \parallel \frac{u(t)}{t} - \xi \parallel \leq \epsilon \left[1 + \parallel \frac{u_t + y_t - \varphi}{t} \parallel\right] + \frac{1}{t}\int_0^t \parallel AS(t-s)[\tilde{g}(s) - \tilde{g}(0)] \parallel ds \\ + N\int_0^t \parallel Bx'(s) \parallel ds + N\int_0^t \parallel \tilde{f}(s) \parallel ds \end{aligned}$$

for all $t \in [0, \delta(\epsilon)]$, which enables us to conclude that $\parallel \frac{u(t)}{t} - \xi \parallel \to 0$ as $t \to 0$ since ϵ is arbitrary. Therefore, $u'(0) = \xi$ and $x'(0) = u'(0) + y'(0) = \xi$. This establishes the first assertion.

Assume that condition (ii) holds. In this case, for $t \in [0, \delta(\epsilon)]$ and n large enough, we get

$$\begin{aligned} \left\|\frac{u(t)}{t} - \xi\right\| \leq& \left\|\frac{S(t)}{t}z - z\right\| + \left\|D_2g(0,\varphi)\left[\frac{u_t + y_t - \varphi}{t} - \psi - \chi_\xi\right]\right\| \\ &+ \left[1 + \left\|\frac{u_t + y_t - \varphi}{t}\right\|_{\mathcal{B}}\right]\epsilon + \int_0^t \left\|AS(t-s)[\tilde{g}(s) - \tilde{g}(0)]\right\|_Y d\theta \\ &+ tN\int_0^t \left\|Bx'(s)\right\|ds + tN\int_0^t \left\|\tilde{f}(\theta)\right\|dt, \\ \leq& \zeta_1(t) + \|D_2g(0,\varphi)\|\left\|\frac{u_t}{t} - \mathcal{T}_t\chi_\xi\right\|_{\mathcal{B}} + \|D_2g(0,\varphi)\|\|\mathcal{T}_t\chi_\xi - \chi_\xi\|_{\mathcal{B}} \\ &+ \|D_2g(0,\varphi)\|\left\|\frac{y_t - \varphi}{t} - \psi\right\|_{\mathcal{B}} + \left(1 + \left\|\frac{u_t}{t}\right\|_{\mathcal{B}} + \left\|\frac{y_t - \varphi}{t}\right\|_{\mathcal{B}}\right)\epsilon \\ \leq& \zeta_2(t) + M^n(t)\|D_2g(0,\varphi)\| + K^n(t)\|D_2g(0,\varphi)\|\left\|\frac{u(t)}{t} - \xi\right\| \\ &+ \left(1 + \left\|\frac{u_t}{t}\right\|_{\mathcal{B}} + \left\|\frac{y_t - \varphi}{t}\right\|_{\mathcal{B}}\right)\epsilon, \end{aligned}$$

where $\zeta_i(t) \to 0$ as $t \to 0$ for $i = 1, 2$. Since K^n is continuous and $K^n(0) = 1$, we can assume that $\mu = K^n_{\delta(\epsilon)} \parallel D_2g(0,\varphi) \parallel < 1$, which implies that

$$\left\|\frac{u(t)}{t} - \xi\right\| \leq \zeta_3(t) + (1-\mu)^{-1}\left(1 + \left\|\frac{u_t}{t}\right\|_{\mathcal{B}} + \left\|\frac{y_t - \varphi}{t}\right\|_{\mathcal{B}}\right)\epsilon, \qquad t \in [0, \delta(\epsilon)],$$

where $\zeta_3(t) \to 0$ as $t \to 0$. Thus, $\frac{d^+}{dt}u(t) \mid_{t=0} = \xi$ and $\frac{d^+}{dt}x(t) \mid_{t=0} = \frac{d^+}{dt}(u(t) + y(t)) \mid_{t=0} = \xi$. This completes the proof. □

We conclude this section with a discussion about the existence of mild solutions for (1.1)-(1.4). We begin by introducing some appropriate terminology. A normalized piecewise continuous function $x : [\sigma, \tau] \to X$ is said to be normalized piecewise smooth on $[\sigma, \tau]$ if x is continuously differentiable except on a finite set G, the left derivative exists on $(\sigma, \tau]$ and the right derivative exists on $[\sigma, \tau)$. In this case, we represent by $x'(t)$ the left derivative at $t \in (\sigma, \tau]$ and by $x'(\sigma)$ the right derivative at σ. We denote by $\mathcal{PC}^1([\sigma, \tau], X)$ the space of normalized piecewise smooth functions from $[\sigma, \tau]$ into X and by $\mathcal{PC}^1$ the space formed by all normalized piecewise smooth functions $x : [0, a] \to X$ such that $G = \{t_i : i = 1, ..., n\}$. It is clear that $\mathcal{PC}^1$ endowed with the norm $\|x\|_1 = \|x\|_\infty + \|x'\|_\infty$ is a Banach space.

Lemma 2.21. *A set $F \subseteq \mathcal{PC}^1$ is relatively compact in $\mathcal{PC}^1$ if, and only if, each set $\widetilde{F}_i$, for $i = 0, ..., n$, is relatively compact in $C^1([t_i, t_{i+1}], X)$.*

In this statement the derivative of a function $x \in \widetilde{F}_i$ is taken to be the right derivative $x'_R(t_i)$ of $x(\cdot)$ at t_i and the left derivative $x'_L(t_{i+1})$ at t_{i+1}. It is easy to see that F is relatively compact in $\mathcal{PC}^1$ if, and only if, the following conditions hold:

(i) The set $\{x(t_i^+), x(t_i^-) : x \in F\},\ i = 0, ..., n$, is relatively compact in X.
(ii) The sets $\{x'(t) : x \in F\}$, for $t \neq t_i,\ i = 0, ..., n+1$, $\{x'_R(t_i) : x \in F\}$, for $i = 0, ..., n$, and $\{x'_L(t_i) : x \in F\}$, for $i = 1, ..., n+1$, are relatively compact in X.
(iii) The sets $\{\widetilde{(x')}_i : x \in F\}$ are equicontinuous on each interval $[t_i, t_{i+1}]$.

To establish our results we introduce the following conditions.

(H7) The function $f : I \times \mathcal{B} \times X \to X$ satisfies the following conditions:
  (i) The function $f(t, \cdot, \cdot) : \mathcal{B} \times X \to X$ is continuous a.e. $t \in I$.
  (ii) If $x : (-\infty, a] \to X$ is such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}^1$, then the function $I \to X,\ t \mapsto f(t, x_t, x'(t))$, is strongly measurable.
  (iii) There exist a function $m_f \in L^1(I, [0, \infty))$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that $\|f(t, \psi, x)\| \leq m_f(t) W_f\left(\|\psi\|_{\mathcal{B}} + \|x\|\right)$, for all $(t, \psi, x) \in I \times \mathcal{B} \times X$.

(H8) If $x : (-\infty, a] \to X$ is a function such that $x_0 = \varphi$, $x|_I \in \mathcal{PC}^1$ and $x'(0) = \xi$, then the function $t \mapsto g(t, x_t)$ is of class C^1 on I and $\frac{d}{dt} g(t, x_t)|_{t=0} = \eta$ is independent of x.

(H9) For every $t \in I$, $AC(t) \in \mathcal{L}(Y, X)$ and the operator function $AC(\cdot) : I \to \mathcal{L}(Y, X)$ is strongly continuous. We denote by

$$N_2 = \sup_{0 \leq t \leq a} \|AC(t)\|_{\mathcal{L}(Y,X)}.$$

Remark 2.22. If assumption (**H7**) holds, then $Y \subseteq D(A)$ and (**H1**) is also valid. In addition, if (**H7**) is satisfied and $h : I \to Y$ is integrable, then the function $t \mapsto \int_0^t AS(t-s)h(s)ds$ is of class C^1 and

$$\frac{d}{dt}\int_0^t AS(t-s)h(s)\,ds = \int_0^t AC(t-s)h(s)\,ds.$$

Remark 2.23. The reader can observe in the second example of Section 3, see (3.5), that assumption (**H8**) is satisfied by a large class of functions.

Definition 2.24. A function $x : (-\infty, a] \to X$ is called a mild solution of (1.1)-(1.4) if $x_0 = \varphi$, $x'(0) = \xi$, $x(\cdot)_{|I} \in \mathcal{PC}^1$ and

$$\begin{aligned} x(t) &= C(t)(\varphi(0) - g(0,\varphi)) + S(t)(\xi - \eta) + g(t, x_t) + \int_0^t S(t-s)Bx'(s)ds \\ &\quad + \int_0^t AS(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s,x'(s))ds \\ &\quad + \sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i}) + \sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i}), \qquad t \in I. \end{aligned}$$

Next we need modify some of our previous notations. The space $\mathcal{F}(a)$ is collection of the functions $u : (-\infty, a] \to X$ such that $u_0 = 0$, $u'(0) = 0$ and $u\ |_{[0,a]} \in \mathcal{PC}^1$. We consider $\mathcal{F}(a)$ endowed with the norm $\|u\|_1 = \|u\ \|_\infty + \|u'\ \|_\infty$, and we denote by $P : \mathcal{F}(a) \to C(I, X)$ the function given by $P(u)(t) = \frac{d}{dt}g(t, u_t + y_t)$.

Theorem 2.25. *Assume that* (**H3**), (**H7**), (**H8**) *and* (**H9**) *are satisfied,* $\varphi(0) \in E$, *and the following conditions hold:*

(a) *For each* $r > 0$, *the set* $U(r) = \{f(t, u_t + y_t, u'(t) + y'(t)) : t \in I, u \in B_r(0, \mathcal{F}(a))\}$ *is relatively compact in* X.

(b) *For each* $r > 0$, *the set* $g(I \times B_r(0, \mathcal{B}))$ *is relatively compact in* Y. *The map* P *is completely continuous and there are constants* c_P, d_P *such that* $\|P(u)\|_\infty \le c_P\|u\|_1 + d_P$, *for all* $u \in \mathcal{F}(a)$. *The map* B *is completely continuous.*

(c) *The maps* $I_i^1 : \mathcal{B} \to E$, *for* $= 1, ..., n$, *are completely continuous and there are constants* d_i^j, *for* $j = 1, 2$, *such that* $\|\ I_i^1(\psi)\ \|_E \le d_i^1\ \|\ \psi\ \|_{\mathcal{B}} + d_i^2, i = 1, 2, ..., n$, *for all* $\psi \in \mathcal{B}$.

(d) *The maps* $I_i^2 : \mathcal{B} \to X$, *for* $i = 1, ..., n$, *are completely continuous and there are constants* e_i^j, *for* $j = 1, 2$, *such that* $\|\ I_i^2(\psi)\ \| \le e_i^1\ \|\ \psi\ \|_{\mathcal{B}} + e_i^2, i = 1, ..., n$, *for every* $\psi \in \mathcal{B}$.

If

$$c_P(1+a)+(M+N)\parallel B\parallel a+K_a\sum_{i=1}^{n}((M+N_1)d_i^1+(M+N)e_i^1)$$
$$+K_a\liminf_{\xi\to\infty}\frac{W(\xi)}{\xi}\int_0^a((N_1+N_2)m_g(s)+(M+N)m_f(s))ds<1, \qquad (2.9)$$

then there exists a mild solution of (1.1)-(1.4).

Proof. For $u\in\mathcal{F}(a)$, we define Γu by $(\Gamma u)_0=0$ and

$$\Gamma u(t)=-C(t)g(0,\varphi)-S(t)\eta+g(t,u_t+y_t)+\int_0^t S(t-s)B(u'(s)+y'(s))ds$$
$$+\int_0^t AS(t-s)g(s,u_s+y_s)ds+\int_0^t S(t-s)f(s,u_s+y_s,u'(s)$$
$$+y'(s))ds+\sum_{0<t_i<t}C(t-t_i)I_i^1(u_{t_i}+y_{t_i})+\sum_{0<t_i<t}S(t-t_i)I_i^2(u_{t_i}+y_{t_i}),$$

for $t\in I$. It is easy to see that $\Gamma u\in\mathcal{PC}^1$ and from Remark 2.22 we see that

$$\frac{d}{dt}\Gamma u(t)=AS(t)g(0,\varphi)-C(t)\eta+P(u)(t)+\int_0^t C(t-s)B(u'(s)+y'(s))ds$$
$$+\int_0^t AC(t-s)g(s,u_s+y_s)ds$$
$$+\int_0^t C(t-s)f(s,u_s+y_s,u'(s)+y'(s))ds$$
$$+\sum_{0<t_i<t}AS(t-t_i)I_i^1(u_{t_i}+y_{t_i})$$
$$+\sum_{0<t_i<t}C(t-t_i)I_i^2(u_{t_i}+y_{t_i}). \qquad (2.10)$$

It follows from this expression that Γ is a map from $\mathcal{F}(a)$ into $\mathcal{F}(a)$. Moreover, the Lebesgue dominated convergence theorem and our assumptions on f and g imply that Γ is continuous.

On the other hand, for $u\in\mathcal{F}(a)$ with $\|u\|_1\le r$, we have

$$\|\Gamma u(t)\|\le c+c_Pa\|u\|_1+N\parallel B\parallel ra+N_1\int_0^t m_g(s)W_g(\|u_s+y_s\|_{\mathcal{B}})ds$$
$$+N\int_0^t m_f(s)W_f(\|u_s+y_s\|_{\mathcal{B}}+\|u'(s)+y'(s)\|)ds$$
$$+M\sum_{0<t_i<t}d_i^1\|u_{t_i}+y_{t_i}\|_{\mathcal{B}}+N\sum_{0<t_i<t}e_i^1\|u_{t_i}+y_{t_i}\|_{\mathcal{B}}$$

and hence

$$\|\Gamma u(t)\| \leq c + (c_P + N \parallel B \parallel) ar + rK_a \sum_{i=1}^{n} (Md_i^1 + Ne_i^1) + \int_0^t (N_1 m_g(s) + N m_f(s)) W(K_a r + c) ds, \tag{2.11}$$

where c denotes a generic constant. Similarly, it follows from (2.10) that

$$\left\| \frac{d}{dt} \Gamma u(t) \right\| \leq c + c_P r + M \parallel B \parallel ra + rK_a \sum_{0<t_i<t} (N_1 d_i^1 + Me_i^1) + \int_0^t (N_2 m_g(s) + M m_f(s)) W(K_a r + c) ds. \tag{2.12}$$

Therefore, adding (2.11) and (2.12) yields

$$\begin{aligned} \|\Gamma u(t)\|_1 \leq {} & c + (a+1) c_P r + (M+N) \parallel B \parallel ra \\ & + rK_a \sum_{i=1}^{n} \Big((M+N_1) d_i^1 + (M+N) e_i^1 \Big) \\ & + \int_0^a ((N_1 + N_2) m_g(s) + (M+N) m_f(s)) W(K_a r + c) \, ds. \end{aligned}$$

If we assume that for each $r > 0$, there is $u^r \in B_r(0, \mathcal{F}(a))$ such that $\|\Gamma u^r\|_1 > r$, then replacing u by u^r in the above estimate we obtain

$$\begin{aligned} 1 \leq & (a+1) c_P + (M+N) \parallel B \parallel a + K_a \sum_{i=1}^{n} ((M+N_1) d_i^1 + (M+N) e_i^1) \\ & + K_a \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^a ((N_1 + N_2) m_g(s) + (M+N) m_f(s)) \, ds, \end{aligned}$$

which is absurd. Thus, there exists $r > 0$ such that $\Gamma(B_r(0, \mathcal{F}(a)) \subseteq B_r(0, \mathcal{F}(a))$.

To prove that Γ is completely continuous, we consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where $(\Gamma_i u)_0 = 0$ and

$$\begin{aligned} \Gamma_1 u(t) = & - C(t) g(0, \varphi) - S(t) \eta + g(t, u_t + y_t) \\ & + \int_0^t S(t-s) B(u'(s) + y'(s)) ds + \int_0^t AS(t-s) g(s, u_s + y_s) ds \\ & + \int_0^t S(t-s) f(s, u_s + y_s, u'(s) + y'(s)) ds, \\ \Gamma_2 u(t) = & \sum_{0<t_i<t} C(t-t_i) I_i^1(u_{t_i} + y_{t_i}) + \sum_{0<t_i<t} S(t-t_i) I_i^2(u_{t_i} + y_{t_i}), \end{aligned}$$

for $t \in I$. From the assumptions and [15, Lemma 3.1] we can conclude that Γ_1 is completely continuous. Similarly, using Lemma 2.21 instead of Lemma 1.1, specifically the conditions (i), (ii) and (iii) following the statement of Lemma 2.21, and the conditions (c) and (d), we can show that Γ_2 is also completely continuous.

Schauder's theorem now implies that Γ has a fixed point $u \in \mathcal{F}(a)$, which establishes the existence of a mild solution. The proof is complete. □

We also obtain existence results by assuming that the involved functions are Lipschitz.

Theorem 2.26. *Assume that* (**H7**)*,* (**H8**)*,* (**H9**) *and condition* (a) *of Theorem* 2.25 *are satisfied. Further, suppose* $\varphi(0) \in E$ *and the following conditions:*

(a) *The function* $g : I \times \mathcal{B} \to Y$ *satisfies the Lipschitz condition*

$$\| g(t,\psi_1) - g(t,\psi_2) \|_Y \leq L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad (t,\psi_i) \in I \times \mathcal{B},$$

for some constant $L_g > 0$*, and for each function* $x : (-\infty, a] \to X$ *such that* $x_0 = \varphi$*,* $x|_I \in \mathcal{PC}^1$ *and* $x'(0) = \xi$*, the function* $t \mapsto g(t, x_t)$ *is strongly measurable from* I *into* Y*.*

(b) *There exists a continuous function* $L_P : I \times [0,\infty) \to (0,\infty)$ *such that*

$$\|P(u)(t) - P(v)(t)\| \leq L_P(t,r)\|u - v\|_1,$$

for all $u, v \in B_r(0,\mathcal{F}(a))$*,* $t \in I$ *and each* $r > 0$*.*

(c) *There exist positive constants* L_i^j *for* $i = 1,2,...,n,\ j = 1,2,$ *such that*

$$\|I_i^1(\psi_1) - I_i^1(\psi_2)\|_Y \leq L_i^1\|\psi_1 - \psi_2\|_{\mathcal{B}}, \qquad \psi_i \in \mathcal{B},$$
$$\|I_i^2(\psi_1) - I_i^2(\psi_2)\| \leq L_i^2\|\psi_1 - \psi_2\|_{\mathcal{B}}, \qquad \psi_i \in \mathcal{B}.$$

If

$$\liminf_{\xi\to\infty}\Big[\sup_{s\in I} L_P(s,\xi) + \int_0^a L_P(s,\xi)ds + \frac{K_a(M+N)W_f(\xi)}{\xi}\int_0^a m_f(s)ds\Big] + (M+N) \| B \| a + K_a\Big[a(N_1+N_2)L_g + \sum_{0<t_i\leq a}[(M+N_1)L_i^1 + (M+N)L_i^2]\Big] < 1, \qquad (2.13)$$

then there exists a mild solution of (1.1)-(1.4)*.*

Proof. Let Γ be defined as in the proof of Theorem 2.25. We claim that there exists an $r > 0$ such that $\Gamma(B_r(0,\mathcal{F}(a))) \subseteq B_r(0,\mathcal{F}(a))$. Initially we observe that for $\|u\|_1 \leq r$,

$$\begin{aligned}\|g(t, u_t + y_t) - g(0, \varphi)\| &= \| \int_0^t Pu(s)\, ds\| \\ &\leq \int_0^t L_P(s, r) ds \|u\|_1 + \int_0^t \|P(0)(s)\| ds\end{aligned}$$

and hence,

$$\begin{aligned}\|\Gamma u(t)\| &\leq c + \int_0^t L_P(s, r) ds \|u\|_1 + N \parallel B \parallel rt + N_1 L_g \int_0^t \|u_s\|_{\mathcal{B}} ds \\ &\quad + N \int_0^t m_f(s) W_f(\|u_s + y_s\|_{\mathcal{B}} + \|u'(s) + y'(s)\|) ds \\ &\quad + M \sum_{0<t_i<t} L_i^1 \|u_{t_i}\|_{\mathcal{B}} + N \sum_{0<t_i<t} L_i^2 \|u_{t_i}\|_{\mathcal{B}} \\ &\leq c + r \int_0^t L_P(s, r) ds + N \parallel B \parallel rt + t N_1 L_g K_a r \\ &\quad + N W_f(K_a r + c) \int_0^t m_f(s) ds + r K_a \sum_{0<t_i<t} (M L_i^1 + N L_i^2),\end{aligned}$$

where c denotes a generic constant. Similarly, it follows from (2.10) that

$$\begin{aligned}\left\| \frac{d}{dt} \Gamma u(t) \right\| &\leq c + \sup_{s \in I} L_P(s, r) r + M \parallel B \parallel rt + t N_2 L_g K_a r \\ &\quad + r K_a \sum_{0<t_i<t} (N_1 L_i^1 + M L_i^2) + M W_f(K_a r + c) \int_0^t m_f(s) ds.\end{aligned}$$

Therefore, for $t \in I$, we see that

$$\begin{aligned}\|\Gamma u(t)\|_1 &\leq c + \left[\sup_{s \in I} L_P(s, r) + \int_0^a L_P(s, r) ds \right] r + (M + N) \parallel B \parallel ra \\ &\quad + a(N_1 + N_2) L_g K_a r + (M + N) W_f(K_a r + c) \int_0^a m_f(s) ds \\ &\quad + r K_a \sum_{i=1}^n [(M + N_1) L_i^1 + (M + N) L_i^2].\end{aligned}$$

Now, from the last inequality and (2.13) we can prove that there exists r large enough such that $\Gamma(B_r(0, \mathcal{F}(a))) \subseteq B_r(0, \mathcal{F}(a))$. Moreover, we can choose r

so that

$$\Theta = \sup_{s\in I} L_P(s,r) + \int_0^a L_P(s,r)ds + (M+N) \parallel B \parallel a$$
$$+ K_a \left[a(N_1+N_2)L_g + \sum_{i=1}^{n} [(M+N_1)L_i^1 + (M+N)L_i^2] \right] < 1.$$

Consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 x(t) = -C(t)g(0,\varphi) - S(t)\eta + g(t, x_t + y_t) + \int_0^t AS(t-s)g(s, x_s + y_s)ds$$
$$+ \sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i} + y_{t_i}) + \sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i} + y_{t_i}),$$
$$\Gamma_2 x(t) = \int_0^t S(t-s)f(s, x_s + y_s)ds, \quad t \in I.$$

From [15, Lemma 3.1], the map Γ_2 is completely continuous. This property, and the estimate

$$\parallel \Gamma_1 u - \Gamma_1 v \parallel_1 \leq \Theta \|u - v\|_1,$$

implies that Γ is condensing from $B_r(0, \mathcal{F}(a))$ into $B_r(0, \mathcal{F}(a))$. Now, the assertion follows from an application of [26, Corollary 4.3.2]. The proof is complete. □

The proof of the next result is standard. Hence the proof is omitted.

Theorem 2.27. *Assume that* **(H8), (H9)** *and conditions* (a), (b) *and* (c) *of Theorem* 2.26 *are fulfilled. Further,* $\varphi(0) \in E$*, the function* $f : I \times \mathcal{B} \times X \to X$ *satisfies the Lipschitz condition*

$$\parallel f(t, \psi_1, x_1) - f(t, \psi_2, x_2) \parallel \leq L_f(\parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}} + \|x_1 - x_2\|),$$

$(t, \psi_i, x_i) \in I \times \mathcal{B} \times X$ *for some constant* $L_f \geq 0$*, and for each function* $x : (-\infty, a] \to X$ *such that* $x_0 = \varphi$*,* $x|_I \in \mathcal{PC}^1$ *and* $x'(0) = \xi$*, the function* $t \mapsto f(t, x_t, x'(t))$ *is integrable. If* $L_P(s,r) \leq L_P$ *for all* (s,r)*, and*

$$L_P(1+a) + (M+N) \parallel B \parallel a + K_a \Big[a(N_1+N_2)L_g$$
$$+ \Big(1 + \frac{1}{K_a}\Big)(M+N)L_f a + \sum_{i=1}^{n} (M+N_1)L_i^1 + (M+N)L_i^2 \Big] < 1,$$

then there exists a unique mild solution of (1.1)-(1.4).

3. Applications

The literature for neutral differential systems with $x(t) \in \mathbb{R}^k$ is extensive. In this case our results are easily applicable. In fact, the operator A is a matrix of order $n \times n$ which generates the uniformly continuous cosine function $C(t) = \cosh(tA^{1/2}) = \sum_{n=1}^{\infty} \frac{t^{2n}}{2n!} A^n$ with associated sine function $S(t) = A^{-\frac{1}{2}} \sinh(tA^{1/2}) = \sum_{n=1}^{\infty} \frac{t^{2n+2}}{(2n+1)!} A^n$. We note the expressions $\cosh(tA^{1/2})$ and $\sinh(t\|A\|^{1/2})$ are purely symbolic and do not assume the existence of the square roots of A. In this case, we can take $Y = X = \mathbb{R}^k$, from which it follows that assumptions (**H1**), (**H2**) and (**H9**) are automatically satisfied. Moreover, $\|C(t)\| \leq \cosh(t\|A\|^{1/2})$ and $\|S(t)\| \leq \|A\|^{1/2} \sinh(t\|A\|^{1/2})$. The next result is a consequence of Theorem 2.25.

Proposition 3.1. *Assume that* (**H3**), (**H7**), (**H8**) *hold, with* $m_f, m_g \in L^{\infty}(I)$. *Suppose further that the following conditions are satisfied:*

(a) *The map* P *is continuous and there exist constants* c_P, d_P *such that* $\|Pu(t)\| \leq c_p\|u\|_1 + d_P$, *for all* $(t, u) \in I \times \mathcal{F}(a)$.

(b) *The map* I_i^j *are continuous and there are constants* d_i^j, e_i^j *such that* $\| I_i^1(\psi) \| \leq d_i^1 \| \psi \|_{\mathcal{B}} + d_i^2$ *and* $\| I_i^2(\psi) \| \leq e_i^1 \| \psi \|_{\mathcal{B}} + e_i^2$ *for every* $i = 1, ..., n$, *and all* $\psi \in \mathcal{B}$.

If

$$c_P(1+a) + \alpha \| B \| a + \alpha K_a \Big[\sum_{i=1}^{n} ((1+ \| A \|) d_i^1 + e_i^1) + \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^a (\|A\| m_g(s) + m_f(s)) ds \Big] < 1,$$

where $\alpha = \cosh(a\|A\|^{\frac{1}{2}}) + \|A\|^{\frac{1}{2}} \sinh(a\|A\|^{\frac{1}{2}})$, *then there exists a mild solution of* (1.1)-(1.4).

We next consider an application of the theory developed in Section 2. In the sequel, $X = L^2([0, \pi])$; $\mathcal{B} = \mathcal{PC}_0 \times L^2(\rho, X)$ and $A : D(A) \subseteq X \to X$ is the map defined by $Af = f''$ with domain $D(A) = \{f \in X : f'' \in X,\ f(0) = f(\pi) = 0\}$. Clearly A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t \in \mathbb{R}}$ on X. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2, n \in I\!N$, with corresponding eigenvectors $z_n(\theta) = \left(\frac{2}{\pi}\right)^{1/2} \sin(n\theta)$; the set $\{z_n : n \in I\!N\}$ is an orthonormal basis of X and the following properties hold:

(a) For $\varphi \in D(A)$, $A\varphi = -\sum_{n=1}^{\infty} n^2 < \varphi, z_n > z_n$.

(c) For $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt) < \varphi, z_n > z_n$ and $S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < \varphi, z_n > z_n$. Consequently, $\|C(t)\| = \|S(t)\| \leq 1$ for all $t \in \mathbb{R}$ and $S(t)$ is compact for every $t \in \mathbb{R}$.

(d) If Φ is the group of translations on X defined by $\Phi(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t))$ and $A = B^2$, where B is the generator of Φ and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$ (see [7] for details). In particular, we observe that the inclusion $\iota : E \to X$ is compact.

Consider the impulsive partial neutral differential equation

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\Big[u(t,\tau) - \int_{-\infty}^{t}\int_0^{\pi} b(t-s,\vartheta,\tau)u(s,\vartheta)d\vartheta ds\Big] \\ &= \frac{\partial^2}{\partial \tau^2}u(t,\tau) + \alpha\frac{\partial}{\partial t}u(t,\xi) + \int_0^{\pi}\beta(\xi)\frac{\partial}{\partial t}u(t,\xi)d\xi \\ &\quad + \int_{-\infty}^{t} c(t-s)u(s,\tau)ds, \end{aligned} \tag{3.1}$$

for $t \in I = [0,a], \tau \in J = [0,\pi]$, subject to the conditions

$$u(t,0) = u(t,\pi) = 0, \quad t \in I, \tag{3.2}$$

$$u(s,\tau) = \varphi(s,\tau), \;\; \frac{\partial}{\partial t}u(0,\tau) = \xi(\tau), \;\; s \in (-\infty, 0], \;\; \tau \in J, \tag{3.3}$$

$$\triangle u(t_i,\tau) = \int_{-\infty}^{t_i}\gamma_i(t_i - s)u(s,\tau)ds, \tag{3.4}$$

where α is a prefixed number and we assume that $\varphi \in \mathcal{B}$, with the identification $\varphi(s)(\tau) = \varphi(s,\tau)$, $\varphi(0,\cdot) \in H^1([0,\pi])$, $\xi \in X$ and $0 < t_1 < ... < t_n < a$.

To treat this system, we assume that b, c, γ_i satisfy the following conditions:

(i) The functions $b(\cdot)$, $\frac{\partial}{\partial\zeta}b(\cdot)$, $\frac{\partial^2}{\partial\zeta^2}b(\cdot)$, $\frac{\partial}{\partial\theta}b(\cdot)$ are continuous, $b(\theta,\vartheta,\pi) = b(\theta,\vartheta,0) = 0$ for every $(\theta,\vartheta) \in (-\infty,0] \times I$ and

$$L_g = \max\left\{\left[\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{\rho(s)}\left(\frac{\partial^i b(s,\vartheta,\tau)}{\partial\tau^i}\right)^2 d\vartheta ds d\tau\right]^{\frac{1}{2}} : i = 0,1,2\right\} < \infty,$$

$$\tilde{L}_g = \int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{\rho(s)}\left(\frac{\partial}{\partial s}b(s,\vartheta,\tau)\right)^2 d\vartheta ds d\tau < \infty.$$

(ii) The functions $c(\cdot), \gamma_i^1$, are continuous, $L_f = \left(\int_{-\infty}^{0}\frac{c^2(-\theta)}{\rho(\theta)}d\theta\right)^{\frac{1}{2}} < \infty$ and $L_i^1 = \left(\int_{-\infty}^{0}\frac{\gamma_i^2(-\theta)}{\rho(\theta)}d\theta\right)^{\frac{1}{2}} < \infty$ for every $i = 1, ..., n$.

We now define the functions $B : X \to X$, and $f, g, I_i : \mathcal{B} \to X$ by

$$Bx(\tau) = \alpha u(t,\tau) + \int_0^\pi \beta(s)u(t,s)ds,$$
$$g(\psi)(\tau) = \int_{-\infty}^0 \int_0^\pi b(-s,\vartheta,\tau)\psi(s,\vartheta)\, d\vartheta ds,$$
$$f(\psi)(\tau) = \int_{-\infty}^0 c(-s)\psi(s,\tau)\, d\theta,$$
$$I_i(\psi)(\tau) = \int_{-\infty}^0 \gamma_i(-s)\psi(s,\tau)\, d\theta.$$

Under these conditions, the maps B, f, g, I_i^1 are bounded linear operators, $\parallel B \parallel_{\mathcal{L}(X)} \leq \mid \alpha \mid + \parallel \beta \parallel_{L^2(0,T)}$, $\|f\| \leq L_f$, $\|g\| \leq L_g$ and $\|I_i^1\| \leq L_i^1$. Moreover, using (i) we can prove that g is $D(A)$-valued and that $\parallel g \parallel_{\mathcal{L}(\mathcal{B},[D(A)])} \leq L_g$. For this reason, we take $Y = [D(A)]$. It is follows from the introduction that if $\iota : Y \hookrightarrow X$ is the inclusion, then $\|\iota(x)\| \leq \|x\|_A$, the function $t \mapsto AS(t)$ is uniformly continuous into $\mathcal{L}(Y, X)$, and $\parallel AS(t) \parallel_{\mathcal{L}(Y,X)} \leq 1$ for $t \in [0, a]$. On the other hand, if $x : (-\infty, a] \to X$ given by $x(t)(\tau) = u(t, \tau)$ is such that $x_0 = \varphi$ and x is continuous on $[0, t_1)$, then the right derivative

$$\frac{d}{dt}g(x_t)|_{t=0}(\tau) = -\int_{-\infty}^0 \int_0^\pi \frac{\partial b(-s,\vartheta,\tau)}{\partial s}\varphi(-s,\vartheta)d\vartheta ds + \int_0^\pi b(0,\vartheta,\tau)\varphi(0,\vartheta)d\vartheta = \eta(\tau), \tag{3.5}$$

exists and is independent of x. Consequently, the impulsive neutral system (3.1)-(3.4) can be written in the form (2.1)-(2.4) and the following result is obtained from Theorem 2.19 and Theorem 2.20.

Proposition 3.2. *Assume* $(3+a)[|\alpha| + \parallel \beta \parallel_{L^2(0,a)}] + [1 + (\int_{-a}^0 \rho(\theta)\, d\theta)^{\frac{1}{2}}][L_g(1 + a) + \sum_{i=1}^n L_i^1] < 1$. *Then there exists a unique mild solution of* (2.1)-(2.4). *Moreover, if the function* $t \to \mathcal{T}_t\varphi$ *is differentiable at zero with* $\frac{d}{dt}\mathcal{T}_t\varphi \mid_{t=0} = \psi$, x *is a mild solution of* (2.1)-(2.4) *with* $z = \xi - g(\psi + \chi_\xi)$, *then* $\frac{d}{dt}x(t)_{t=0} = \xi$.

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