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HYBRID STEEPEST DESCENT METHOD FOR SPLIT VARIATIONAL INCLUSION AND FINITE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. In the present paper, we use hybrid steepest descent method for finding a common point of the solution set of split variational inclusion and the fixed points set of a finite family of nonexpansive mappings in Hilbert spaces. Further, we prove that the sequences generated by our iterative algorithm converge strongly to the common point, which is the unique solution of a variational inequality. Our result improves and extends the corresponding results announced by many others. At the end of the paper, we extend our result to the more broad family of λ-strictly pseudo-contractive mappings.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Recall that a set-valued mapping $B : H_1 \to H_1$ is said to be monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \text{whenever} \quad u \in B(x), v \in B(y).$$

It is said to be maximal monotone if its graph: $graphB := \{(x, y) \in H_1 \times H_1 : y \in B(x)\}$ is not properly contained in the graph of any other monotone

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operators. As we all know that, when B is maximal monotone, then for each $x \in H_1$ and $\lambda > 0$ there is a unique $z \in H_1$ such that $x \in (I + \lambda B)z$.

In 2011, Moudafi [12] proposed the following Split Monotone Variational Inclusion Problem (SMVIP): find $x^* \in H_1$ such that

$$0 \in f(x^*) + B_1(x^*) \tag{1.1}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*), \tag{1.2}$$

where $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given single-valued operators, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two set-valued maximal monotone mappings. Moudafi proposed the following iterative method for solving (1.1)-(1.2): let $\gamma > 0$ and $x_0 \in H_1$ be arbitrary,

$$x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k), \quad k \in N, \tag{1.3}$$

where $\gamma \in (0, 1/L)$ with L being the spectral radius of the operator A^*A, the operator $U := J_\lambda^{B_1}(I - \lambda f)$ and $T := J_\lambda^{B_2}(I - \lambda g)$. He showed that the sequence generated by (1.3) weakly converges to a solution of SMVIP(1.1)-(1.2).

If $f \equiv 0$ and $g \equiv 0$ then SMVIP (1.1)-(1.2) reduces to Split Variational Inclusion Problem (SVIP): find $x^* \in H_1$ such that

$$0 \in B_1(x^*) \tag{1.4}$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in B_2(y^*). \tag{1.5}$$

We denote the solution set of SVIP(1.4) and SVIP(1.5) by SOLVIP(B_1) and SOLVIP(B_2), respectively. The set of solution of SVIP(1.4)-(1.5) is denoted by $\Gamma = \{x^* \in H_1 : x^* \in \text{SOLVIP}(B_1) \text{ and } Ax^* \in \text{SOLVIP}(B_2)\}$. The SVIP has extensively been investigated in recent years, for example [3, 6, 8, 13] and the references therein.

In 2012, Byrne et al. [2] introduced the following iterative method for SVIP(1.4)-(1.5) and the sequence $\{x_n\}$ is generated by

$$x_{n+1} = J_\lambda^{B_1}(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n).$$

Motivated by Moudafi and Byrne, Kazmi and Rizvi proposed a viscous iteration method as follows:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

where $f : H_1 \to H_1$ be a contraction mapping, $S : H_1 \to H_1$ be a non-expansive mapping, $\lambda > 0$, $\gamma \in (0, \frac{1}{\|A\|^2})$ and $\alpha_n \in (0, 1)$. Under certain conditions, they proved that the sequence generated by the proposed iterative method converges strongly to a common solution of the split variational inclusion problem and fixed point problem for a nonexpansive mapping which is the unique solution of the variational inequality problem.

On the other hand, let us recall some iterative methods for solving the fixed point problems of nonexpansive mappings. In 2011, Yamada [15] considered the variational inequality problem over the set of fixed point of nonexpansive mapping and proved strong convergence of the sequence generated by the hybrid steepest-decent method as follows:

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n), \tag{1.7}$$

where $x_1 \in H_1$, $\{\lambda_n\} \subset (0,1)$, $F : H_1 \to H_1$ is a strongly monotone and Lipschitz continuous mapping and μ is a positive real number.

Motivated by the Krasnoselskij-Mann type algorithm and the steepest descent method, a new explicit iterative algorithm is introduced by Buong and Duong [1] as follows:

$$x_{k+1} = (1-\beta_k^0)x_k + \beta_k^0 T_0^k T_N^k \cdots T_1^k x_k, \tag{1.8}$$

where $T_i^k = (1-\beta_k^i)I + \beta_k^i T_i$ for $i = 1, 2, \ldots, N$, $T_0^k = I - \lambda_k \mu F$ and F is a L-Lipschitz continuous and η-strongly monotone mapping. Under certain conditions, they proved that the sequence $\{x_k\}$ converges strongly to the unique solution of the following variational inequality :

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall\, x \in \bigcap_{i=1}^{N} Fix(T_i). \tag{1.9}$$

In 2014, Zhou and Wang [17] introduced a new iterative algorithm and their iterative algorithm is simpler than (1.8) given by Buong and Duong. Let $\{x_n\}$ is generated by

$$x_{k+1} = (I - \lambda_k \mu F) T_N^k \cdots T_1^k x_k. \tag{1.10}$$

They proved that the sequence $\{x_k\}$ defined by (1.10) converges strongly to the unique solution of the variational inequality (1.9) in a faster rate of convergence.

Motivated and inspired by the results of Zhou, Wang and Yamada, in this paper, we consider a new iterative algorithm to solve the class of variational inequalities (1.9). The iterative algorithm improves and extends the results of Zhou, Wang and Yamada, and the corresponding results announced by many others. At the end of this paper, we extend our iterative algorithm to the more broad family of λ-strictly pseudo-contractive mappings.

2. Preliminaries

We need some facts and tools in a real Hilbert space H which are listed as definitions and lemmas below.

Definition 2.1. A mapping $T : H \to H$ is said to be

(i) *nonexpansive*, if $\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in H$;
(ii) *monotone*, if $\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall\, x, y \in H$;
(iii) $\eta(\eta > 0)$ *strongly monotone*, if $\langle x-y, Tx-Ty\rangle \geq \eta\|x-y\|^2, \quad \forall\, x, y \in H$.

Definition 2.2. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity mapping I and a nonexpansive mapping $S : H \to H$; that is,

$$T \equiv (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$. More precisely, when the last equality holds, we say that T is α-averaged. We know that, firmly nonexpansive mappings are $(1/2)$-averaged mappings.

It is well known that every nonexpansive mapping $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, the inequality

$$\langle (x - Tx) - (y - Ty), Ty - Tx\rangle \leq \frac{1}{2}\|(Tx - x) - (Ty - y)\|^2 \tag{2.1}$$

and therefore, we get, for all $(x, y) \in H \times Fix(T)$,

$$\langle x - Tx, y - Tx\rangle \leq \frac{1}{2}\|Tx - x\|^2 \tag{2.2}$$

see e.g., ([4], Theorem 3.1) and ([5], Theorem 2.1).

Lemma 2.3. ([10]) *Averaged mappings have the following properties:*

(i) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then both T_1T_2 and T_2T_1 are α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*
(ii) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} Fix(T_i) = Fix(T_1 \cdots T_N).$$

In particular, if $N = 2$, we have $Fix(T_1) \bigcap Fix(T_2) = Fix(T_1T_2) = Fix(T_2T_1)$.

Lemma 2.4. ([11]) *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if the following inequality holds:*

$$\langle x - z, y - z\rangle \leq 0,$$

for every $y \in C$.

Lemma 2.5. *In a real Hilbert space H, there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall\, x, y \in H.$$

Lemma 2.6. ([9]) *Let $B : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the resolvent mapping $J_\lambda^B : H \to H$ is defined by*

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x), \;\; \forall\, x \in H,$$

for some $\lambda > 0$. The resolvent operator J_λ^B is single-valued and firmly nonexpansive. It is easy to be deduced that J_λ^B is nonexpansive and $\frac{1}{2}$-averaged.

Lemma 2.7. ([7]) *Let H be a Hilbert space, C a closed convex subset of H and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and $\{(I - T)x_n\}$ converges strongly to $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.8. ([16]) *Let $F : H \to H$ be a k-Lipschitz continuous and η-strongly monotone mapping with $k > 0$ and $\eta > 0$. For each $\lambda \in (0, 1)$ and a fixed $\mu \in (0, \frac{2\eta}{k^2})$, write $T^\lambda := (I - \lambda\mu F)$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0, 1)$. Then we have*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|,$$

for all $x, y \in H$, i.e., $T^\lambda : H \to H$ is a contraction on H with contractive coefficient $(1 - \lambda\tau)$.

Lemma 2.9. ([14]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \;\; n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} \mid \delta_n \mid < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. ([18]) *Assume S is a λ-strictly pseudo-contractive mapping on a Hilbert space H. Define a mapping T by $Tx = \alpha x + (1-\alpha)Sx$ for all $x \in H$ and $\alpha \in [\lambda, 1)$. Then T is a nonexpansive mapping such that $Fix(T) = Fix(S)$.*

3. Main results

Now we state and prove our main results in this paper.

Theorem 3.1. *Let H_1 be a Hilbert space and $F : H_1 \to H_1$ be a k-Lipschitz continuous and η-strongly monotone mapping with $k > 0$ and $\eta > 0$. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H_1. Assume that $\Omega = \bigcap_{i=1}^N Fix(T_i)$ $\cap\Gamma \neq \emptyset$. For any point $x_0 \in H_1$, define a sequence $\{x_n\}$ as following manner:*

$$y_n = J_\lambda^{B_1}(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n),$$
$$x_{n+1} = (I - \mu\alpha_n F)T_N^n T_{N-1}^n \cdots T_1^n y_n,$$

where $T_i^n = (1-\gamma_n^i)I + \gamma_n^i T_i$ for $i = 1, 2, \cdots, N$. Suppose that $\beta \in (0, \frac{1}{\|A\|^2})$, $\alpha_n \in (0, 1]$, $0 < \mu < \frac{2\eta}{k^2}$ and $\gamma_n^i \in (a, b)$ for some $a, b \in (0, 1)$. If the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$;
(iii) $\sum_{n=1}^\infty |\gamma_{n+1}^i - \gamma_n^i| < \infty$, *for* $i = 1, \cdots, N$ *and* $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality:

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall\, x \in \Omega. \tag{3.1}$$

Equivalently, we have $P_\Omega(I - \mu F)x^* = x^*$.

Proof. Since our methods easily deduce the general case, we prove Theorem 3.1 for $N = 2$.

Step 1. We show that $\{x_n\}$ is bounded. Take $p \in \Omega$, then we have $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2}(Ap)$ and $T_i^n p = p$, for all $i = 1, 2$. By Lemma 2.8,

$$\begin{aligned} \|x_{n+1} - p\| &= \|(I - \mu\alpha_n F)T_2^n T_1^n y_n - p\| \\ &\leq (1 - \alpha_n\tau)\|y_n - p\| + \mu\alpha_n\|Fp\|. \end{aligned} \tag{3.2}$$

On the other hand, since $J_\lambda^{B_1}$ is firmly-nonexpansive, so we have

$$\begin{aligned}\|y_n - p\|^2 &= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - p\|^2 \\ &\le \|x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &= \|x_n - p\|^2 + \beta^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\quad + 2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &\le \|x_n - p\|^2 + \beta^2\|A\|^2\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\quad + 2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle\end{aligned} \tag{3.3}$$

and using (2.2) we estimate

$$\begin{aligned}&2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\beta\langle A(x_n - p), (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\beta\{\langle J_\lambda^{B_2}Ax_n - Ap, (J_\lambda^{B_2} - I)Ax_n\rangle - \|(J_\lambda^{B_2} - I)Ax_n\|^2\} \\ &\le 2\beta\left\{\frac{1}{2}\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \|(J_\lambda^{B_2} - I)Ax_n\|^2\right\} \\ &\le -\beta\|(J_\lambda^{B_2} - I)Ax_n\|^2.\end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we get

$$\|y_n - p\|^2 \le \|x_n - p\|^2 + \beta(\beta\|A\|^2 - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2. \tag{3.5}$$

Since $\beta \in (0, \frac{1}{\|A\|^2})$, we have

$$\|y_n - p\|^2 \le \|x_n - p\|^2. \tag{3.6}$$

Now, by using (3.6), the inequality (3.2) becomes

$$\begin{aligned}\|x_{n+1} - p\| &\le (1 - \alpha_n\tau)\|x_n - p\| + \mu\alpha_n\|Fp\| \\ &\le \max\left\{\|x_n - p\|, \frac{\mu}{\tau}\|Fp\|\right\} \\ &\;\;\vdots \\ &\le \max\left\{\|x_1 - p\|, \frac{\mu}{\tau}\|Fp\|\right\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Subsequently, we deduce that $\{y_n\}$, $\{T_i^n y_n\}$ and $\{FT_2^nT_1^n y_n\}$ are bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Noting that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(I - \mu\alpha_n F)T_2^nT_1^n y_n - (I - \mu\alpha_{n-1}F)T_2^{n-1}T_1^{n-1}y_{n-1}\| \\ &\le (1 - \alpha_n\tau)\|T_2^nT_1^n y_n - T_2^{n-1}T_1^{n-1}y_{n-1}\| \\ &\quad + \mu|\alpha_n - \alpha_{n-1}|\|FT_2^{n-1}T_1^{n-1}y_{n-1}\|.\end{aligned} \tag{3.7}$$

Noting that T_1^n and T_2^n are γ_n^1-averaged and γ_n^2-averaged respectively, by Lemma 2.3, we see that $T_2^nT_1^n$ is t_n-averaged for every n, where $t_n=\gamma_n^1+\gamma_n^2-\gamma_n^1\gamma_n^2$. Let $a^*=2a-a^2$ and $b^*=2b-b^2$. Since $t_n=1-(1-\gamma_n^1)(1-\gamma_n^2)$, we can see that $0<a^*\le t_n\le b^*<1$ for all n, $\lim_{n\to\infty}|t_{n+1}-t_n|=0$ and $\sum_{n=1}^{\infty}|t_{n+1}-t_n|<\infty$. So we can find $\{S_n\}$ a family of nonexpansive mappings such that

$$T_2^nT_1^n=(1-t_n)I+t_nS_n,\ n\ge 0. \tag{3.8}$$

So we have

$$\begin{aligned}&\|T_2^nT_1^ny_n-T_2^{n-1}T_1^{n-1}y_{n-1}\|\\&=\|(1-t_n)y_n+t_nS_ny_n-(1-t_{n-1})y_{n-1}-t_{n-1}S_{n-1}y_{n-1}\|\\&\le(1-t_n)\|y_n-y_{n-1}\|+|t_n-t_{n-1}|\|y_{n-1}\|+t_n\|y_n-y_{n-1}\|\\&\quad+|t_n-t_{n-1}|\|S_ny_{n-1}\|+t_{n-1}\|S_ny_{n-1}-S_{n-1}y_{n-1}\|\\&=\|y_n-y_{n-1}\|+|t_n-t_{n-1}|(\|y_{n-1}\|+\|S_ny_{n-1}\|)\\&\quad+t_{n-1}\|S_ny_{n-1}-S_{n-1}y_{n-1}\|.\end{aligned} \tag{3.9}$$

Since $J_\lambda^{B_1}\big(I+\beta A^*(J_\lambda^{B_2}-I)A\big)$ is a nonexpansive mapping, we have

$$\begin{aligned}&\|y_n-y_{n-1}\|\\&=\|J_\lambda^{B_1}\big(I+\beta A^*(J_\lambda^{B_2}-I)A\big)x_n-J_\lambda^{B_1}\big(I+\beta A^*(J_\lambda^{B_2}-I)A\big)x_{n-1}\|\\&\le\|x_n-x_{n-1}\|.\end{aligned} \tag{3.10}$$

By (3.8), we can deduce that $S_n=\frac{T_2^nT_1^n-(1-t_n)I}{t_n}$, it follows that

$$\begin{aligned}&\|S_ny_{n-1}-S_{n-1}y_{n-1}\|\\&=\|\frac{T_2^nT_1^ny_{n-1}-(1-t_n)y_{n-1}}{t_n}-\frac{T_2^{n-1}T_1^{n-1}y_{n-1}-(1-t_{n-1})y_{n-1}}{t_{n-1}}\|\\&=\|\frac{1}{t_n}T_2^nT_1^ny_{n-1}-\frac{1}{t_{n-1}}T_2^nT_1^ny_{n-1}+\frac{1}{t_{n-1}}T_2^nT_1^ny_{n-1}\\&\quad-\frac{1}{t_{n-1}}T_2^{n-1}T_1^{n-1}y_{n-1}-\frac{1}{t_n}y_{n-1}+\frac{1}{t_{n-1}}y_{n-1}\|\\&\le\frac{|t_n-t_{n-1}|}{t_nt_{n-1}}(\|T_2^nT_1^ny_{n-1}\|+\|y_{n-1}\|)\\&\quad+\frac{1}{t_{n-1}}(\|T_2^nT_1^ny_{n-1}-T_2^nT_1^{n-1}y_{n-1}\|\\&\quad+\|T_2^nT_1^{n-1}y_{n-1}-T_2^{n-1}T_1^{n-1}y_{n-1}\|)\\&\le\frac{|t_n-t_{n-1}|}{t_nt_{n-1}}(\|T_2^nT_1^ny_{n-1}\|+\|y_{n-1}\|)+\frac{1}{a^*}(\|T_1^ny_{n-1}-T_1^{n-1}y_{n-1}\|\\&\quad+\|T_2^nT_1^{n-1}y_{n-1}-T_2^{n-1}T_1^{n-1}y_{n-1}\|).\end{aligned} \tag{3.11}$$

Noting that

$$\begin{aligned}&\|T_1^n y_{n-1} - T_1^{n-1} y_{n-1}\| \\ &= \|(1-\gamma_n^1) y_{n-1} + \gamma_n^1 T_1 y_{n-1} - (1-\gamma_{n-1}^1) y_{n-1} - \gamma_{n-1}^1 T_1 y_{n-1}\| \\ &\le |\gamma_n^1 - \gamma_{n-1}^1| (\|y_{n-1}\| + \|T_1 y_{n-1}\|).\end{aligned}$$

So, we can easily deduce that

$$\begin{aligned}&\|T_2^n T_1^{n-1} y_{n-1} - T_2^{n-1} T_1^{n-1} y_{n-1}\| \\ &\le |\gamma_n^2 - \gamma_{n-1}^2| (\|T_1^{n-1} y_{n-1}\| + \|T_2 T_1^{n-1} y_{n-1}\|).\end{aligned}$$

Combining (3.7), (3.9), (3.10) and (3.11), by Lemma 2.9, we can derive that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty} \|T_2^n T_1^n x_n - x_n\| = 0$. Note that

$$\begin{aligned}\|T_2^n T_1^n y_n - x_n\| &\le \|T_2^n T_1^n y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &= \mu\alpha_n \|F T_2^n T_1^n y_n\| + \|x_{n+1} - x_n\|.\end{aligned}$$

Taking n approaches to infinity, we get

$$\lim_{n\to\infty} \|T_2^n T_1^n y_n - x_n\| = 0. \tag{3.12}$$

Next, we claim that $\|x_n - y_n\| \to 0$ as $n \to \infty$. By Lemma 2.5 and (3.5), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(I - \mu\alpha_n F) T_2^n T_1^n y_n - p\|^2 \\ &= \|(I - \mu\alpha_n F) T_2^n T_1^n y_n - (I - \mu\alpha_n F) T_2^n T_1^n p - \mu\alpha_n F p\|^2 \\ &\le (1 - \alpha_n \tau)^2 \|y_n - p\|^2 - 2\mu\alpha_n \langle Fp, x_{n+1} - p\rangle \\ &\le (1 - \alpha_n \tau)^2 (\|x_n - p\|^2 + \beta(\beta\|A\|^2 - 1)\|(J_\lambda^{B_2} - I) A x_n\|^2) \\ &\quad - 2\mu\alpha_n \langle Fp, x_{n+1} - p\rangle \\ &\le \|x_n - p\|^2 + \beta(\beta\|A\|^2 - 1)\|(J_\lambda^{B_2} - I) A x_n\|^2 \\ &\quad + 2\mu\alpha_n \|Fp\| \|x_{n+1} - p\|.\end{aligned} \tag{3.13}$$

which is equivalent to

$$\begin{aligned}&\beta(1 - \beta\|A\|^2)\|(J_\lambda^{B_2} - I) A x_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - 2\mu\alpha_n \langle Fp, x_{n+1} - p\rangle \\ &\le \|x_{n+1} - x_n\| (\|x_n - p\| + \|x_{n+1} - p\|) - 2\mu\alpha_n \langle Fp, x_{n+1} - p\rangle.\end{aligned}$$

Since $1 - \beta\|A\|^2 > 0$, $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} \|(J_\lambda^{B_2} - I) A x_n\| = 0. \tag{3.14}$$

Furthermore, using (3.3), (3.5) and $\beta \in (0, \frac{1}{\|A\|^2})$, we observe that

$$\begin{aligned}\|y_n - p\|^2 &= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - p\|^2 \\ &= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - J_\lambda^{B_1}p\|^2 \\ &\leq \langle y_n - p, x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\rangle \\ &= \frac{1}{2}\{\|y_n - p\|^2 + \|x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\quad - \|(y_n - p) - \big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\big)\|^2\} \\ &\leq \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 + \beta(\|A\|^2\beta - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\quad - \|y_n - x_n - \beta A^*(J_\lambda^{B_2} - I)Ax_n\|^2\} \\ &\leq \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \big(\|y_n - x_n\|^2 \\ &\quad + \beta^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 - 2\beta\langle y_n - x_n, A^*(J_\lambda^{B_2} - I)Ax_n\rangle\big)\} \\ &\leq \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \|y_n - x_n\|^2 \\ &\quad + 2\beta\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|\}.\end{aligned}$$

Hence, we obtain

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2\beta\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|.$$

By (3.13), we observe that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n\tau)^2\|y_n - p\|^2 - 2\mu\alpha_n\langle Fp, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n\tau)^2(\|x_n - p\|^2 - \|y_n - x_n\|^2 \\ &\quad + 2\beta\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|) - 2\mu\alpha_n\langle Fp, x_{n+1} - p\rangle \\ &\leq \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2\beta\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\ &\quad - 2\mu\alpha_n\langle Fp, x_{n+1} - p\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|y_n - x_n\|^2 &\leq \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) \\ &\quad + 2\beta\|A(y_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\mu\alpha_n\|Fp\|\|x_{n+1} - p\|.\end{aligned}$$

Since $\alpha_n \to 0$ as $n \to \infty$ and by (3.14), we can easily deduce that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.15}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\|T_2^nT_1^nx_n - x_n\| &= \lim_{n\to\infty}(\|T_2^nT_1^nx_n - T_2^nT_1^ny_n\| + \|T_2^nT_1^ny_n - x_n\|)\\ &= \lim_{n\to\infty}(\|x_n - y_n\| + \|T_2^nT_1^ny_n - x_n\|)\\ &= 0.\end{aligned} \tag{3.16}$$

Step 4. We show that $\limsup_{n\to\infty}\langle Fx^*, x^* - x_n\rangle \geq 0$.

From [8, Theorem 3.2], we know that the solution of the variational inequality (3.1) is unique. We use x^* to denote the unique solution of (3.1). Since $\{x_n\}_{n\geq 0}$ is bounded, there exists a subsequence $\{x_{n_j}\}_{j\geq 1}$ of $\{x_n\}_{n\geq 0}$ such that $x_{n_j} \rightharpoonup \hat{x}$ as $j \to \infty$ and

$$\limsup_{n\to\infty}\langle Fx^*, x^* - x_n\rangle = \lim_{j\to\infty}\langle Fx^*, x^* - x_{n_j}\rangle.$$

Since $\{\gamma_n^i\}$ is bounded for $i = 1, 2$, we can assume that $\gamma_{n_j}^i \to \gamma_\infty^i$ as $j \to \infty$, where $0 < a \leq \gamma_\infty^i \leq b < 1$ for $i = 1, 2$. Define $T_i^\infty = (1 - \gamma_\infty^i)I + \gamma_\infty^i T_i$ $(i = 1, 2)$. Then we have $Fix(T_i^\infty) = Fix(T_i)$ for $i = 1, 2$. Note that

$$\|T_i^{n_j}x - T_i^\infty x\| \leq |\gamma_{n_j}^i - \gamma_\infty^i|(\|x\| + \|T_ix\|).$$

Hence, we deduce that

$$\lim_{j\to\infty}\sup_{x\in D}\|T_i^{n_j}x - T_i^\infty x\| = 0, \tag{3.17}$$

where D is an arbitrary bounded subset of H. Since $Fix(T_1^\infty)\bigcap Fix(T_2^\infty) = Fix(T_1)\bigcap Fix(T_2) \neq \emptyset$ and T_i^∞ is γ_∞^i-averaged for $i = 1, 2$, by Lemma 2.3, we know that $Fix(T_2^\infty T_1^\infty) = Fix(T_2^\infty)\bigcap Fix(T_1^\infty)$. Combine (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned}\|x_{n_j} - T_2^\infty T_1^\infty x_{n_j}\| &\leq \|x_{n_j} - T_2^{n_j}T_1^{n_j}x_{n_j}\| + \|T_2^{n_j}T_1^{n_j}x_{n_j} - T_2^\infty T_1^{n_j}x_{n_j}\|\\ &\quad + \|T_2^\infty T_1^{n_j}x_{n_j} - T_2^\infty T_1^\infty x_{n_j}\|\\ &\leq \|x_{n_j} - T_2^{n_j}T_1^{n_j}x_{n_j}\| + \|T_2^{n_j}T_1^{n_j}x_{n_j} - T_2^\infty T_1^{n_j}x_{n_j}\|\\ &\quad + \|T_1^{n_j}x_{n_j} - T_1^\infty x_{n_j}\|\\ &\leq \|x_{n_j} - T_2^{n_j}T_1^{n_j}x_{n_j}\| + \sup_{x\in D'}\|T_2^{n_j}x - T_2^\infty x\|\\ &\quad + \sup_{x\in D''}\|T_1^{n_j}x - T_1^\infty x\|,\end{aligned}$$

where D' is a bounded subset including $\{T_1^{n_j}x_{n_j}\}$ and D'' is a bounded subset including $\{x_{n_j}\}$. Hence $\lim_{j\to\infty}\|x_{n_j} - T_2^\infty T_1^\infty x_{n_j}\| = 0$. From Lemma 2.7, we have $\hat{x} \in Fix(T_2^\infty T_1^\infty) = Fix(T_2^\infty)\bigcap Fix(T_1^\infty)$.

On the other hand, $y_{n_j} = J_\lambda^{B_1}\big(x_{n_j} + \beta A^*(J_\lambda^{B_2} - I)Ax_{n_j}\big)$ can be written as

$$\frac{x_{n_j} - y_{n_j} + \beta A^*(J_\lambda^{B_2} - I)Ax_{n_j}}{\lambda} \in B_1 y_{n_j}. \tag{3.18}$$

By passing to limit $j \to \infty$ in (3.18) and by taking into account (3.14), (3.15) and the fact that the graph of maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(\hat{x})$, *i.e.*, $\hat{x} \in \text{SOLVIP}(B_1)$. Since $\{Ax_{n_j}\}$ weakly converges to $A\hat{x}$. Again, by (3.14) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive and Lemma 2.7, we obtain that $A\hat{x} = Fix(J_\lambda^{B_2})$, *i.e.*, $A\hat{x} \in \text{SOLVIP}(B_2)$. Thus $\hat{x} \in \Omega$. Using Lemma 2.4,

$$\begin{aligned}\limsup_{n\to\infty}\langle Fx^*, x^* - x_n\rangle &= \lim_{j\to\infty}\langle Fx^*, x^* - x_{n_j}\rangle \\ &= \langle Fx^*, x^* - \hat{x}\rangle \le 0.\end{aligned} \tag{3.19}$$

Step 5. We show that $\lim_{n\to\infty}\|x_{n+1} - x^*\| = 0$.

Using (3.6), we estimate

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(I - \mu\alpha_n F)T_2^n T_1^n y_n - x^*\|^2 \\ &= \|(I - \mu\alpha_n F)T_2^n T_1^n y_n - (I - \mu\alpha_n F)T_2^n T_1^n x^* - \mu\alpha_n Fx^*\|^2 \\ &= \|(I - \mu\alpha_n F)T_2^n T_1^n y_n - (I - \mu\alpha_n F)T_2^n T_1^n x^*\|^2 + \mu^2\alpha_n^2\|Fx^*\|^2 \\ &\quad + 2\mu\alpha_n\langle Fx^*, (I - \mu\alpha_n F)T_2^n T_1^n x^* - (I - \mu\alpha_n F)T_2^n T_1^n y_n\rangle \\ &\le (1 - \tau\alpha_n)\|y_n - x^*\|^2 + 2\mu\alpha_n\langle Fx^*, x^* - x_n\rangle \\ &\quad + 2\mu\alpha_n\|Fx^*\|\|x_n - T_2^n T_1^n y_n\| + 2\mu^2\alpha_n^2\|Fx^*\|\|FT_2^n T_1^n y_n\| \\ &\le (1 - \tau\alpha_n)\|x_n - x^*\|^2 + 2\mu\alpha_n\langle Fx^*, x^* - x_n\rangle \\ &\quad + 2\mu\alpha_n\|Fx^*\|\|x_n - T_2^n T_1^n y_n\| + 2\mu^2\alpha_n^2\|Fx^*\|\|FT_2^n T_1^n y_n\| \\ &\le (1 - \tau\alpha_n)\|x_n - x^*\|^2 + \tau\alpha_n c_n,\end{aligned}$$

where

$$c_n = \frac{2\mu}{\tau}(\langle Fx^*, x^* - x_n\rangle + \|Fx^*\|\|x_n - T_2^n T_1^n y_n\|) + \frac{2\mu^2\|Fx^*\|\|FT_2^n T_1^n y_n\|}{\tau}\alpha_n.$$

From (3.12), (3.19), we know that $\limsup_{n\to\infty} c_n \le 0$. By Lemma 2.9, we conclude that $x_n \to x^*$ as $n \to \infty$, completing the proof. □

4. An extension of our result

In this section, we extend our result to the more broad family of λ-strictly pseudo-contractive mappings. Now let us recall that a mapping $S : H \to H$

is said to be λ-strictly pseudo-contractive if there exists a constant $\lambda \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \lambda\|(I - S)x - (I - S)y\|^2, \ \ \forall\, x, y \in H.$$

Let $\{S_i\}_{i=1}^N$ be a family of λ_i-strictly pseudo-contractive self-mappings of H with $0 \leq \lambda_i < 1$. For $i = 1, 2, \ldots, N$, define

$$\hat{T}_i = \omega_i I + (1 - \omega_i)S_i, \tag{4.1}$$

where $0 \leq \lambda_i \leq \omega_i < 1$. By virtue of Lemma 2.10, we know that $\{\hat{T}_i\}_{i=1}^N$ is a family of N nonexpansive mappings. Thus we extend Theorem 3.1 to the family of λ_i-strictly pseudo-contractions.

Theorem 4.1. *Let H_1 be a real Hilbert space, $F : H_1 \to H_1$ be a k-Lipschitizian continuous and η-strongly monotone operator on H_1 with $k > 0$ and $\eta > 0$. Let $\{S_i\}_{i=1}^N$ be N λ_i-strictly pseudo-contractive mappings on H such that $\Omega = \bigcap_{i=1}^N Fix(S_i) \bigcap \Gamma \neq \emptyset$. Suppose $0 < \mu < \frac{2\eta}{k^2}$, $\alpha_n \in (0,1)$, $\gamma_n^i \in (a,b)$ for some $a, b \in (0,1)$ and $0 \leq \lambda_i \leq \omega_i < 1$ for $i = 1, 2, \ldots, N$. If the condition* (i)-(iii) *of Theorem* 3.1 *are satisfied, the sequence $\{x_k\}_{k\geq 0}$ defined by Theorem* 3.1 *with T_i replaced by* (4.1)*, converges strongly to the unique solution x^* of the following variational inequality:*

$$\langle Fx^*, x - x^*\rangle \geq 0, \ \ \forall\, x \in \bigcap_{i=1}^N Fix(S_i) \cap \Gamma.$$

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