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# LOCAL CONVERGENCE OF SOME FIFTH AND SIXTH ORDER ITERATIVE METHODS

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**Abstract.** We present a local convergence analysis for some families of fifth and sixth order methods in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. Earlier studies [19] have used hypotheses on the fifth Fréchet derivative of the operator involved. We use hypotheses only on the first Fréchet derivative in our local convergence analysis. This way, the applicability of these methods is extended. Moreover the radius of convergence and computable error bounds on the distances involved are also given using Lipschitz constants. Numerical examples illustrating the theoretical results are also presented in this study.

## 1. Introduction

In this paper we are concerned with the problem of approximating a solution $x^*$ of the equation

$$F(x) = 0, \tag{1.1}$$

where $F$ is a Fréchet-differentiable operator defined on a convex subset $D$ of a Banach space $X$ with values in a Banach space $Y$.

Many problems in Computational Sciences and other disciplines can be brought in a form like (1.1) using mathematical modeling [5, 7, 10, 21]. It is known that, the solutions of these equations can rarely be found in closed

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form. So, most solution methods for these equations are iterative. Newton-like iterative methods [1]-[21] are famous for approximating a solution of the equation (1.1). These methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [1]-[21].

In this paper, we introduce the iterative method defined for each $n = 0, 1, 2, \cdots$, by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n + \frac{1}{2}F'(x_n)^{-1}F(x_n) - (3F'(y_n) - F'(x_n))^{-1}F(x_n), \\ x_{n+1} &= z_n - (bF'(x_n) + cF'(y_n))^{-1}(F'(x_n) + aF'(y_n))F'(x_n)^{-1}F(z_n), \end{aligned} \tag{1.2}$$

where $x_0$ is an initial point, $S$ is $\mathbb{R}$ or $\mathbb{C}$ and $a, b, c \in S$ are given parameters to generate a sequence $\{x_n\}$ approximating $x^*$. Method (1.2) has been studied in [19] in the special case when $X = Y = \mathbb{R}$ and $a, b, c \in \mathbb{R}$. In particular, it was shown that the convergence order of method (1.2) is sixth, if $a \neq -1, b = -\frac{1}{2}(3a+1), c = \frac{1}{2}(5a+3)$ and fifth, if $a = -1$ for functions that are four times differentiable in a neighborhood containing $x^*$. Notice however that method (1.2) uses only the first derivative. The requirement that $F^{(i)}, i = 1, 2, 3, 4$ exists limits the applicability of this method. As a motivational example, define function $f$ on $D = \bar{U}(1, \frac{3}{2})$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{1.3}$$

Choose $x^* = 1$. We also have that

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$
$$f''(x) = 6x \ln x^2 + 20x^3 + 12x^2 + 10x$$

and

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Notice that $f'''(x)$ is unbounded on $D$. Hence, the results in [19], cannot apply to show the convergence of method (1.2) or its special cases requiring hypotheses on the fifth derivative of function $F$ or higher. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations [1]-[21]. These results show that if the initial point $x_0$ is sufficiently close to the solution $x^*$, then the sequence $\{x_n\}$ converges to $x^*$. But how close to the solution $x^*$ the initial guess $x_0$ should be? These local results give no information on the radius of the convergence ball for the corresponding method.

Moreover, notice that the convergence ball of high convergence order methods is usually very small and in general decreases as the convergence order increases. Our approach establishes the local convergence result under hypotheses only on the first derivative. Our approach can give a larger convergence ball than the earlier studies, under weaker hypotheses. The same technique can be used to other methods.

The paper is organized as follows. In Section 2 we present the local convergence analysis of method (1.2). The numerical examples are given in the concluding Section 3.

## 2. Local convergence

We present the local convergence analysis of method (1.2) in this section. Let $L_0 > 0, L > 0, M > 0$, $a, b, c \in S$ be given parameters with $b + c \neq 0$. It is convenient for the local convergence analysis of method (1.2) that follows to define some functions and parameters. Define function $g_1$ on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)}$$

and parameter $r_1$ by

$$r_1 = \frac{2}{2L_0 + L}.$$

Then, we have that $g_1(r_1) = 1$ and $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$. Define functions $p$ and $h_p$ on the interval $[0, r_1)$ by

$$p(t) = \frac{L_0}{2}(1 + 3g_1(t))t$$

and set

$$h_p(t) = p(t) - 1.$$

We have that $h_p(0) = -1 < 0$ and $h_p(t) \longrightarrow +\infty$ as $t \longrightarrow \frac{1}{L_0}^-$. Then, it follows from the Intermediate Value Theorem that function $h_p$ has zeros in the interval $(0, \frac{1}{L_0})$. Denote by $r_p$ the smallest such zero.

- **Case** $2L_0 \leq L$

  Then, we have that $h_p(r_1) = 2L_0 r_1 - 1 \leq 0$, so

$$r_1 \leq r_p.$$

Define function on the interval $[0, r_1)$ by

$$g_2(t) = \frac{1}{2(1-L_0t)}[L + \frac{3L_0M(1+g_1(t))}{2(1-p(t))}]t$$

and set

$$h_2(t) = g_2(t) - 1.$$

We get for $r_1 < r_p$, $h_2(0) = -1$ and

$$\begin{aligned} h_2(r_1) &= \frac{Lr_1}{2(1-L_0r_1)} - 1 + \frac{3L_0M(1+g_1(r_1))r_1}{2(1-p(r_1))} \\ &= \frac{3L_0Mr_1}{1-p(r_1)} > 0, \quad \text{(by the definition of } r_1\text{)} \end{aligned}$$

and for $r_1 = r_p$ $h_2(t) \longrightarrow +\infty$ as $t \longrightarrow r_1^-$. In either case function $h_2$ has zeros in the interval $(0, r_1)$. Denote by $r_2$ the smallest such zero.

- **Case** $L < 2L_0$

  We have that $h_p(r_1) > 0$, so

$$r_p < r_1.$$

Define function $g_2$ as above but on the interval $[0, r_p)$. Then, we get that $h_2(0) = -1 < 0$ and $h_2(t) \longrightarrow +\infty$ as $t \longrightarrow r_p^-$. Hence, again function $h_2$ has a smallest zero in the interval $(0, r_p)$. Denote such zero also by $r_2$. Moreover, define functions $q$ and $h_q$ on the interval $[0, \frac{1}{L_0})$ by

$$q(t) = \frac{L_0}{|b+c|}(|b| + |c|g_1(t))t$$

and

$$h_q(t) = q(t) - 1.$$

We have that $h_q(0) = -1 < 0$ and $h_q(t) \longrightarrow +\infty$ as $t \longrightarrow \frac{1}{L_0}^-$. Then, function $h_q$ has a smallest zero in the interval $(0, \frac{1}{L_0})$ denoted by $r_q$.

- **Case** $\frac{L_0(|b|+|c|)r_1}{|b+c|} \leq 1$

  We have that $h_q(r_1) = \frac{L_0(|b|+|c|)r_1}{|b+c|} - 1 \leq 0$ so,

$$r_1 \leq r_q.$$

Define functions $g_3$ and $h_3$ on the interval $[0, r_2)$ by

$$g_3(t) = [1 + \frac{(L_0t + |a|L_0g_1(t)t + |1+a|)M}{|b+c|(1-L_0t)(1-q(t))}]g_2(t)$$

and set

$$h_3(t) = g_3(t) - 1.$$

Then, we get that $h_3(0) = -1 < 0$ and $h_3(t) > 0$. Hence, function $h_3$ has a smallest zero in the interval $(0, r_2)$ denoted by $r_3$.

- **Case** $\frac{L_0(|b|+|c|)r_1}{|b+c|} > 1$
  Then, $h_q(r_1) > 0$, so
  $$r_q < r_1.$$
  Define function $g_3$ on the interval $[0, r_q)$. Then, $h_3(0) = -1 < 0$ and $h_3(t) \longrightarrow +\infty$ as $t \longrightarrow r_q^-$. That is, in this case $h_3$ has a smallest zero denoted by $r_3$.

Hence, for each $t \in [0, r_3)$

$$0 \leq g_1(t) < 1, \tag{2.1}$$

$$0 \leq g_2(t) < 1,\ 0 \leq p(t) < 1,\ 0 \leq q(t) < 1 \tag{2.2}$$

and

$$0 \leq g_3(t) < 0. \tag{2.3}$$

Let $U(w, \rho)$, $\overline{U}(w, \rho)$ stand for the open and closed ball, respectively, with center $w \in X$ and of radius $\rho > 0$. Next, using the above notation we present the local convergence result for method (1.2).

**Theorem 2.1.** *Let $F : D \subset X \longrightarrow Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M \geq 1$, $a, b, c \in S$ with $b + c \neq 0$ such that for each $x, y \in D$,*

$$F(x^*) = 0,\ F'(x^*)^{-1} \in L(Y, X), \tag{2.4}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|, \tag{2.5}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \tag{2.6}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M \tag{2.7}$$

*and*

$$\bar{U}(x^*, r_3) \subseteq D, \tag{2.8}$$

*where $r_2$ is defined previously. Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r_3) - \{x^*\}$ is well defined, remains in $U(x^*, r_3)$ for each $n = 0, 1, 2, \cdots$, and converges to $x^*$. Moreover, the following estimates hold*

$$\|y_n - x^*\| \leq g_1(\|x_n - x_0\|)\|x_n - x^*\| < \|x_n - x^*\| < r_3, \tag{2.9}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x_0\|)\|x_n - x^*\| < \|x_n - x^*\| \tag{2.10}$$

*and*

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.11}$$

*where the $g$ functions are defined previously. Furthermore, for $T \in [r_3, \frac{2}{L_0})$ the limit point $x^*$ is the only solution of the equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.*

*Proof.* The proof uses induction to show estimates (2.9)–(2.11). By the definition of $r_3$, the hypothesis $x_0 \in U(x^*, r_3) - \{x^*\}$ and (2.5), we get that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le L_0\|x_0 - x^*\| \le L_0 r_3 < 1. \tag{2.12}$$

It follows from (2.12) and the Banach Lemma on invertible operators [5, 7, 15] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.13}$$

Hence, $y_0$ is well defined by the first sub-step of method (1.2) for $n = 0$. We can write by (2.4) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.14}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r_3$ for each $\theta \in [0, 1]$. That is $x^* + \theta(x_0 - x^*) \in U(x^*, r) \subset D$. Then, using (2.7) and (2.14) we get that

$$\begin{aligned} \|F'(x^*)^{-1}F(x_0)\| &= \|\int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta\| \\ &\le M\|x_0 - x^*\|. \end{aligned} \tag{2.15}$$

We also have (2.1), (2.6), (2.13) and the first sub step of method (1.2) for $n = 0$ that

$$\begin{aligned} \|y_0 - x^*\| &\le \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}\int_0^1 [F'(x^* + \theta(x_0 - x^*)) \\ &\quad -F'(x_0)]d\theta\|\|x_0 - x^*\| \\ &\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \le g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &< \|x_0 - x^*\| < r_3, \end{aligned} \tag{2.16}$$

which shows (2.9) for $n = 0$ and $y_0 \in U(x^*, r_3)$. Next, we show that $3F'(y_0) - F'(x_0)$ is invertible. Using the definition of function $p$, (2.2), (2.4) and (2.16), we have that

$$\begin{aligned} &\|(2F'(x^*))^{-1}(3F'(y_0) - F'(x_0) - 2F'(x^*))\| \\ &\le \frac{1}{2}[3\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| \\ &\quad +\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|] \\ &\le \frac{L_0}{2}[3\|y_0 - x^*\| + \|x_0 - x^*\|] \\ &\le \frac{L_0}{2}[1 + g_1(\|x_0 - x^*\|)]\|x_0 - x^*\|] \\ &= p(\|x_0 - x^*\|) < 1. \end{aligned} \tag{2.17}$$

It follows from (2.17) that $3F'(y_0) - F'(x_0)$ is invertible and

$$\|(3F'(y_0) - F'(x_0))^{-1}F'(x^*)\| \leq \frac{1}{2(1 - p(\|x_0 - x^*\|))}. \qquad (2.18)$$

Hence, $z_0$ is well defined by the second sub-step of method (1.2) for $n = 0$. We can write

$$z_0 = y_0 + (\frac{1}{2}F'(x_0)^{-1} - (3F'(y_0) - F'(x_0))^{-1})F(x_0). \qquad (2.19)$$

In view of (2.2), (2.4), (2.13), (2.15), (2.16), (2.18) and (2.19), we get in turn that

$$\begin{aligned}
\|z_0 - x^*\| &\leq \|y_0 - x^*\| + \frac{3}{4}\|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times\|(3F'(y_0)-F'(x_0))^{-1}F'(x^*)\|[\|F'(x^*)^{-1}(F'(y_0)-F'(x^*))\| \\
&\quad +\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|]\|F'(x^*)^{-1}F(x_0)\| \\
&\leq \frac{L\|x_0 - x^*\|^2}{1 - L_0\|x_0 - x^*\|} + \frac{3L_0(\|y_0 - x^*\| + \|x_0 - x^*\|)M\|x_0 - x^*\|}{4(1 - L_0\|x_0 - x^*\|)(1 - p(\|x_0 - x^*\|))} \\
&\leq \frac{L\|x_0 - x^*\|^2}{1 - L_0\|x_0 - x^*\|} + \frac{3L_0M(1 + g_1(\|x_0 - x^*\|))\|x_0 - x^*\|^2}{4(1 - L_0\|x_0 - x^*\|)(1 - p(\|x_0 - x^*\|))} \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&< \|x_0 - x^*\| < r_3, \qquad (2.20)
\end{aligned}$$

which shows (2.10) for $n = 0$ and $z_0 \in U(x^*, r_3)$. Next, we show that $bF'(x_0)+cF'(y_0)$ is invertible. Indeed, using (2.2), (2.4) and (2.16), we obtain that

$$\begin{aligned}
&\|((b + c)F'(x^*))^{-1}(bF'(x_0) + cF'(y_0) - (b + c)F'(x^*))\| \\
&\leq |b + c|^{-1}[|b|\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\
&\quad +|c|\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\|] \\
&\leq |b + c|^{-1}L_0[|b|\|x_0 - x^*\| + |c|\|y_0 - x^*\|] \\
&\leq |b + c|^{-1}L_0[|b| + |c|g_1(\|x_0 - x^*\|)]\|x_0 - x^*\| \\
&= q(\|x_0 - x^*\|) < 1. \qquad (2.21)
\end{aligned}$$

It follows from (2.21) that $bF'(x_0) + cF'(y_0)$ is invertible and

$$\|(bF'(x_0) + cF'(y_0))^{-1}F'(x^*)\| \leq \frac{1}{|b + c|(1 - q(\|x_0 - x^*\|))}. \qquad (2.22)$$

Hence, $x_1$ is well defined by the third sub-step of method (1.2) for $n = 0$. Using (2.3), (2.5), (2.13), (2.15) (for $x_0 = z_0$), (2.22) and the third sub-step of method (1.2) for $n = 0$, we get in turn that

$$\begin{aligned}\|x_1 - x^*\| &\leq \|z_0 - x^*\| + \|F'(x_0)^{-1}F'(x^*)\|\|(bF'(x_0) + cF'(y_0))^{-1}F'(x^*)\| \\ &\quad \times[\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| + |a|\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| \\ &\quad + |1+a|\|F'(x^*)^{-1}F'(x^*)\|]\|F'(x^*)^{-1}F(z_0)\| \\ &\leq [1 + \frac{(L_0\|x_0 - x^*\| + |a|L_0\|y_0 - x^*\| + |1+a|)M}{|b+c|(1 - L_0\|x_0 - x^*\|)(1 - q(\|x_0 - x^*\|))}] \\ &\quad \times g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &< \|x_0 - x^*\| < r_3, \end{aligned} \tag{2.23}$$

which shows (2.11) for $n = 0$ and $x_1 \in U(x^*, r_3)$. By simply replacing $x_0, y_0, z_0, x_1$ by $x_k, y_k, z_k, x_{k+1}$ in the preceding estimates we arrive at (2.9)–(2.11). Using the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r_3$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r_3)$. Finally, to show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.5) we get that

$$\begin{aligned}\|F'(x^*)^{-1}(Q - F'(x^*))\| &\leq \int_0^1 L_0\|y^* + \theta(x^* - y^*) - x^*\|d\theta \\ &\leq \frac{L_0}{2}\|x^* - y^*\| = \frac{L_0}{2}T < 1.\end{aligned}$$

It follows that linear operator $Q$ is invertible. Then, from the identity

$$0 = F(x^*) - F(y^*) = Q(x^* - y^*),$$

we conclude that $x^* = y^*$. □

**Remark 2.2.** In Theorem 2.1 we used the hypothesis $b + c \neq 0$ (see also (2.22)). To cover the case $b + c = 0$, function $q$ must be replaced by $\bar{q}$ defined by

$$\bar{q}(t) = L_0(1 + g_1(t))t.$$

Then as in (2.21), we get that

$$\begin{aligned}&\|(F'(x^*))^{-1}(F'(x_0) - F'(y_0))\| \\ &\leq \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| \\ &\leq L_0[\|x_0 - x^*\| + \|y_0 - x^*\|] \\ &\leq L_0[1 + g_1(\|x_0 - x^*\|)]\|x_0 - x^*\| \\ &= \bar{q}(\|x_0 - x^*\|).\end{aligned}$$

Moreover, replace function $g_3$ by $\bar{g}_3$ defined by

$$\bar{g}_3(t) = \left[1 + \frac{(L_0t + |a|L_0g_1(t)t + |1+a|)M}{|b|(1 - L_0t)(1 - \bar{q}(t))}\right] g_2(t).$$

Then, instead of (2.23), we obtain that

$$\begin{aligned}
&\|x_1 - x^*\| \\
&\leq \left[1 + \frac{(L_0\|x_0 - x^*\| + |a|L_0 g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + |1+a|)M}{|b|(1 - L_0\|x_0 - x^*\|)(1 - \bar{q}(\|x_0 - x^*\|))}\right] \\
&\quad \times g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= \bar{g}_3(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&< \|x_0 - x^*\| < \bar{r}_3.
\end{aligned}$$

Then, the conclusions of Theorem 2.1 hold, if we drop the hypothesis $b+c \neq 0$, and replace $q, g_3, r_3$ by $\bar{q}, \bar{g}_3$ and $\bar{r}_3$, respectively.

**Remark 2.3.** (a) The radius $r_1$ was obtained by Argyros in [4] as the convergence radius for Newton's method under condition (2.4)-(2.7). Notice that the convergence radius for Newton's method given independently by Rheinboldt [18] and Traub [20] is given by

$$\rho = \frac{2}{3L} < r_1.$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$. Set $D = U(0, 1)$. Then, we have that $L_0 = e - 1 < l = e$, so $\rho = 0.24252961 < r_1 = 0.324947231$.

Moreover, the new error bounds [4, 5, 6, 7] are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [15, 17]

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by $r_3$ to be larger than $r_1$ since $r_3 \leq r_1$.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [4, 5, 6, 7].

(c) The results can be also be used to solve equations where the operator $F'$ satisfies the autonomous differential equation [5, 7, 15, 17]:

$$F'(x) = P(F(x)),$$

where $P$ is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution $x^*$. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x+1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [19]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \sup \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \sup \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1.

(e) In view of (2.5) and the estimate

$$\begin{aligned} |F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x) - F'(x^*)) + I| \\ &\leq 1 + |F'(x^*)^{-1}(F'(x) - F'(x^*))| \\ &\leq 1 + L_0|x - x^*| \end{aligned}$$

condition (2.7) can be dropped and $M$ can be replaced by

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

## 3. Numerical examples

We present numerical examples in this section.

**Example 3.1.** Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$. Define $F$ on $D$ for $v = (x, y, z)^T$ by

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T. \tag{3.1}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1 < L = e$, $M = 2$. Then, for $a = 1, b = -2, c = 4$, the parameters are

$$r_1 = 0.3249, \quad r_p = 0.1709, \quad r_2 = 0.2286, \quad r_q = 0.2366, \quad r_3 = 0.2157.$$

**Example 3.2.** Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ be and equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function $F$ on $D$ by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\tau\varphi(\tau)^3 d\tau. \tag{3.2}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\tau\varphi(\tau)^2\xi(\tau)d\tau, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$, $M = 2$. Then, for $a = 1$, $b = -2$, $c = 4$, the parameters are

$$r_1 = 0.6667, \quad r_p = 0.0667, \quad r_2 = 0.1333, \quad r_q = 0.0861, \quad r_3 = 0.1272.$$

**Example 3.3.** Returning back to the motivation example at the introduction on this paper, we have $L = L_0 = 146.6629073\ldots, M = 2$. Then, for $a = 1$, $b = -2$, $c = 4$, the parameters are

$$r_1 = 0.0045, \quad r_p = 0.0041, \quad r_2 = 0.0136, \quad r_q = 0.0051, \quad r_3 = 0.0083.$$

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