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POSITIVE SOLUTIONS FOR DISCRETE FOURTH-ORDER M-POINT BOUNDARY VALUE PROBLEMS WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, by using fixed point index theorems, the existence of positive solutions are obtained for discrete nonlinear fourth-order m-point boundary value problems with variable coefficients.

1. Introduction

The theory of nonlinear difference equations has been widely used to study discrete models in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. In recent years, a great deal of work has been done in the study of the existence of solutions for discrete boundary value problem. For the background and recent results, we refer the reader to the monographs [1-4,8,13,14,16-18] and the references therein.

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Anderson and Minhós [1] studied the existence, multiplicity, and nonexistence of nontrivial solutions for fourth-order boundary value problem with explicit parameters β and λ given by

$$\begin{cases}\triangle^4 u(t-2) - \beta\triangle^2 u(t-1) = \lambda f(t,u(t)), \quad t\in \mathbf{Z}[a+1,b+1],\\ u(a) = \triangle^2 u(a-1) = 0, \quad u(b+2) = \triangle^2 u(b+1) = 0.\end{cases} \tag{1.1}$$

In this paper, we consider more general m-point boundary value problem with variable coefficients as follows:

$$\begin{cases}\triangle^4 u(t-2) + B(t)\triangle^2 u(t-1) - A(t)u(t) = f(t,u(t)), \quad t\in \mathbf{Z}[a+1,b+1],\\ u(a) = \sum_{i=1}^{m-2} a_i u(l_i), \quad u(b+2) = \sum_{i=1}^{m-2} b_i u(l_i),\\ \triangle^2 u(a-1) = \sum_{i=1}^{m-2} a_i \triangle^2 u(l_i-1), \quad \triangle^2 u(b+1) = \sum_{i=1}^{m-2} b_i \triangle^2 u(l_i-1),\end{cases} \tag{1.2}$$

where $\triangle$ denotes the forward difference operator defined by

$$\triangle u(t) = u(t+1) - u(t), \triangle^n u(t) = \triangle(\triangle^{n-1} u(t)), \ \mathbf{Z}[a+1,b+1]$$

is the discrete interval given by $\{a+1, a+2, \cdots, b+1\}$ with a and b $(a<b)$ integers, $l_i \in \mathbf{Z}[a+1,b+1]$, $a_i, b_i \in [0,+\infty)$ for $i = 1,2,\cdots,m-2$ are given constants, $A(t), B(t) : \mathbf{Z}[a+1,b+1] \to (-\infty,+\infty)$, $f : \mathbf{Z}[a+1,b+1]\times [0,+\infty) \to [0,+\infty)$ is continuous.

The study of multipoint BVPs for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [9]. Then Gupta [6] studied three-point BVPs for nonlinear ordinary differential equations. Since then, the more general nonlinear multipoint BVPs for ordinary differential equations have been studied by many authors, for example, see [11,12,15,19]. However, few results have been seen in literature for fourth-order difference equations with multi-point boundary condition. So, in this paper, motivated by [1,5,10-12], we aim to study the existence of positive solutions for BVP (1.2).

By a solution u of BVP (1.2), we mean a real sequence u which is defined on $\mathbf{Z}[a-1,b+3]$ and satisfies the difference equation as well as the boundary conditions in (1.2). A solution $\{u(t)\}_{t=a-1}^{b+3}$ of (1.2) is called to be positive if $u(t) > 0$ for $t \in \mathbf{Z}[a+1,b+1]$.

Let $\alpha = \min_{t\in\mathbf{Z}[a+1,b+1]} A(t), \beta = \min_{t\in\mathbf{Z}[a+1,b+1]} B(t)$. We make the following assumptions for convenience:

(H_1) $\beta < 8\sin^2\dfrac{\pi}{2(b-a+2)}, \alpha \geq 0, \alpha + 4\beta\sin^2\dfrac{\pi}{2(b-a+2)} < 16\sin^4\dfrac{\pi}{2(b-a+2)}$,

(H_{1a}) $\beta < 4\sin^2\dfrac{\pi}{2(b-a+2)}, \alpha \geq 0, \alpha + 4\beta\sin^2\dfrac{\pi}{2(b-a+2)} < 16\sin^4\dfrac{\pi}{2(b-a+2)}$.

The proofs of the main theorems of this paper are based on the fixed point index theory. Let E be a real Bananch space with cone P. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $P\cap\Omega = \emptyset$. Let $A : P\cap\overline{\Omega} \to P$ be a completely continuous operator. If $Ax \neq x$ for all $x \in P\cap\overline{\Omega}$, then the fixed point index $i(A, P\cap\Omega, P)$ has definition. One important fact is that if $i(A, P\cap\Omega, P)\neq 0$, then A has a fixed point in $P\cap\Omega$. The following three well-known lemmas in [7] are needed in our argument.

Lemma 1.1. Let $A: P\to P$ be a completely continuous operator. If $\mu Ax\neq x$ for all $x\in P\cap\partial\Omega,\ 0<\mu\leq 1$, then the fixed point index $i(A, P_r, P)=1$.

Lemma 1.2. Let $A: P\to P$ be a completely continuous operator. If $\inf_{x\in\partial P_r}\|Ax\|>0$ and $\mu Ax\neq x$ for $x\in\partial P_r,\ \mu\geq 1$, then the fixed point index $i(A, P_r, P)=0$.

Lemma 1.3. Let $A: P\to P$ be a completely continuous operator, $x_0\in P\backslash\{\theta\}$. If $x-Ax\neq\mu x_0$ for $x\in P\cap\partial\Omega,\ \mu\geq 0$, then the fixed point index $i(A, P\cap\Omega, P)=0$.

2. Preliminaries

In order to obtain our main results, we present some preliminary results in this section. Let

$$X=\{u:\mathbf{Z}[a+1,b+1]\to R\},\ X_+=\{u\in X: u(t)\geq 0,\ t\in\mathbf{Z}[a+1,b+1]\}.$$

It is well known that X is a Banach space equipped with the norm

$$\|u\|_\infty=\max_{t\in\mathbf{Z}[a+1,b+1]}\{|u(t)|\}.$$

Let

$$E=\left\{u:\mathbf{Z}[a,b+2]\to\mathbf{R},\ u(a)=\sum_{i=1}^{m-2}a_iu(l_i),\ u(b+2)=\sum_{i=1}^{m-2}b_iu(l_i)\right\},$$

$$E_+=\{u\in E: u(t)\geq 0,\ t\in\mathbf{Z}[a,b+2]\}.$$

For any $u\in E$, set

$$\|u\|_\infty=\max_{t\in\mathbf{Z}[a+1,b+1]}\{|u(t)|\},$$

$$\|u\|_\lambda=\max_{t\in\mathbf{Z}[a+1,b+1]}\{|\triangle^2u(t-1)|+\lambda|u(t)|\}(\lambda\geq 0)$$

and

$$\|u\|_E=\max\{\|u\|_\infty,\ \|\triangle^2u\|_\infty\},$$

where

$$\|\triangle^2u\|_\infty=\max_{t\in\mathbf{Z}[a+1,b+1]}|\triangle^2u(t-1)|.$$

It is easy to verity that $\|\cdot\|_\infty,\ \|\cdot\|_\lambda(\lambda>0)$ and $\|\cdot\|_E$ are all norms on E. Obviously, $(E,\|\cdot\|_\infty),\ (E,\|\cdot\|_\lambda)(\lambda>0)$ and $(E,\|\cdot\|_E)$ are all Banach spaces. From the following remark 2.1, we know that $\|u\|_0=\|\triangle^2u\|_\infty$ is also a norm on E.

Lemma 2.1. Let (H_1) holds. Then there exist unique $\varphi_i, \psi_i,\ i=1,2$ satisfying

$$\begin{cases} -\triangle^2\varphi_i(t-1)+\lambda_i\varphi_i(t)=0, & t\in \mathbf{Z}[a+1,b+1],\\ \varphi_i(a)=0, \quad \varphi_i(b+2)=1; \end{cases}$$

$$\begin{cases} -\triangle^2\psi_i(t-1)+\lambda_i\psi_i(t)=0, & t\in \mathbf{Z}[a+1,b+1],\\ \psi_i(a)=1, \quad \psi_i(b+2)=0; \end{cases}$$

respectively. And on $\mathbf{Z}[a,b+2],\ \varphi_i\geq 0,\ \psi\geq 0,\ i=1,2$, where $\lambda_1,\ \lambda_2$ are the roots of the polynomial $P(\lambda)=\lambda^2+\beta\lambda-\alpha$, namely,

$$\lambda_1=\frac{-\beta+\sqrt{\beta^2+4\alpha}}{2}, \quad \lambda_2=\frac{-\beta-\sqrt{\beta^2+4\alpha}}{2}.$$

Proof. We can obtain by calculation that $\varphi_i,\ \psi_i,\ i=1,2$ are explicitly given by

$$(i)\ \varphi_i(t)=\frac{\sin(t-a)\theta}{\sin(b+2-a)\theta}, \quad \psi_i=\frac{\sin(b+2-t)\theta}{\sin(b+2-a)\theta},$$

where $\theta := \arctan\frac{\sqrt{-\lambda_i(\lambda_i+4)}}{\lambda_i+2}\in(0,\frac{\pi}{b+2-a})$, when $-4\sin^2\frac{\pi}{2(b+2-a)}<\lambda_i<0$;

$$(ii)\ \varphi_i(t)=\frac{t-a}{b+2-a},\ \psi_i(t)=\frac{b+2-t}{b+2-a}, \text{when } \lambda_i=0;$$

$$(iii)\ \varphi_i(t)=\frac{\gamma^{t-a}-\gamma^{a-t}}{\gamma^{b+2-a}-\gamma^{a-b-2}},\ \psi_i(t)=\frac{\gamma^{b+2-t}-\gamma^{t-b-2}}{\gamma^{b+2-a}-\gamma^{a-b-2}},$$

where $\gamma := \frac{\lambda_i+2+\sqrt{\lambda_i(\lambda_i+4)}}{2}$, when $\lambda_i>0$.

It is obviously that on $\mathbf{Z}[a,b+2],\ \varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2\geq 0$ and $\triangle\varphi_1(a),\ \triangle\varphi_2(a)>0$. The proof is complete. □

Let $G_i(t,s)(i=1,2)$ be the Green's function of the linear boundary value problem

$$\begin{cases} -\triangle^2u(t-1)+\lambda_iu(t)=0, & t\in \mathbf{Z}[a+1,b+1],\\ u(a)=u(b+2)=0. \end{cases}$$

Then $G_i(t,s)(i=1,2)$ can be expressed by

$$G_i(t,s)=\frac{1}{\triangle\varphi_i(a)}\begin{cases} \varphi_i(t)\psi_i(s), & a\leq t\leq s\leq b+2,\\ \varphi_i(s)\psi_i(t), & a\leq s\leq t\leq b+2. \end{cases} \tag{2.1}$$

Lemma 2.2. $G_i(t,s),\ \varphi_i,\ \psi_i(i=1,2)$ have the following properties:

(i) $G_i(t,s)>0, \forall t,s\in\mathbf{Z}[a+1,b+1]$;

(ii) $\delta_iG_i(t,t)G_i(s,s)\leq G_i(t,s)\leq C_iG_i(s,s), \forall t,s\in\mathbf{Z}[a+1,b+1]$;

(iii) $\delta_iG_i(t,t)\leq\varphi_i(t),\ \psi_i(t)\leq C_i, \forall t,s\in\mathbf{Z}[a+1,b+1]$, where $C_i = \max\{\max_{a+1\le t\le s\le b+2} \frac{\varphi_i(t)}{\varphi_i(s)},\ \max_{a\le s\le t\le b+1} \frac{\psi_i(t)}{\psi_i(s)}\} > 0$ and

$$\delta_i = \min\left\{ \min_{a+1\le t\le s\le b+1} \frac{\triangle\varphi_i(a)}{\psi_i(t)\varphi_i(s)},\ \min_{a+1\le s\le t\le b+1} \frac{\triangle\varphi_i(a)}{\varphi_i(t)\psi_i(s)},\right.$$

$$\left. \min_{a+1\le t\le b+1} \frac{\triangle\varphi_i(a)}{\psi_i(t)},\ \min_{a+1\le t\le b+1} \frac{\triangle\varphi_i(a)}{\varphi_i(t)} \right\} > 0.$$

The proof is simple and is omitted.

For convenience, let

$$\nabla_k = \begin{vmatrix} -\sum_{i=1}^{m-2} b_i\psi_k(l_i) & 1-\sum_{i=1}^{m-2} b_i\varphi_k(l_i) \\ 1-\sum_{i=1}^{m-2} a_i\psi_k(l_i) & -\sum_{i=1}^{m-2} a_i\varphi_k(l_i) \end{vmatrix},\ k=1,2, \tag{2.2}$$

$$A_k(h) = \frac{1}{\nabla_k} \begin{vmatrix} \sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} b_iG_k(l_i,s)h(s) & 1-\sum_{i=1}^{m-2} b_i\varphi_k(l_i) \\ \sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} a_iG_k(l_i,s)h(s) & -\sum_{i=1}^{m-2} a_i\varphi_k(l_i) \end{vmatrix},\ k=1,2,\ h\in X, \tag{2.3}$$

$$B_k(h) = \frac{1}{\nabla_k} \begin{vmatrix} -\sum_{i=1}^{m-2} b_i\psi_k(l_i) & \sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} b_iG_k(l_i,s)h(s) \\ 1-\sum_{i=1}^{m-2} a_i\psi_k(l_i) & \sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} a_iG_k(l_i,s)h(s) \end{vmatrix},\ k=1,2,\ h\in X. \tag{2.4}$$

Lemma 2.3. Let (H_1) holds. Assume that

(H_{2k}) $\nabla_k \neq 0,\ k=1,2.$

Then for any $h\in X$, the BVP

$$\begin{cases} -\triangle^2 u(t-1)+\lambda_k u(t) = h(t), & t\in \mathbf{Z}[a+1,b+1], \\ u(a) = \sum_{i=1}^{m-2} a_iu(l_i), & u(b+2) = \sum_{i=1}^{m-2} b_iu(l_i) \end{cases} \tag{2.5}$$

has a unique solution

$$u(t) = A_k(h)\psi_k(t) + B_k(h)\varphi_k(t) + \sum_{s=a+1}^{b+1} G_k(t,s)h(s), \quad t\in \mathbf{Z}[a,b+2].$$

Proof. It is easy to see that the linear boundary value problem

$$-\triangle^2 u(t-1)+\lambda_k u(t) = h(t), \quad t\in \mathbf{Z}[a+1,b+1], \quad u(a)=u(b+2)=0$$

has a unique solution $u(t) = \sum_{s=a+1}^{b+1} G_k(t,s)h(s),\ t\in \mathbf{Z}[a,b+1]$. And notice that $\varphi_k,\ \psi_k$ are two linearly independent solutions of the problem

$$-\triangle^2 u(t-1)+\lambda_k u(t) = 0.$$

The proof follows by routine calculations. □

In the rest of the paper, we make the following assumption:

$$(H_{3k})\ \nabla_k < 0, \quad 1 - \sum_{i=1}^{m-2} a_i\psi_k(l_i) > 0, \quad 1 - \sum_{i=1}^{m-2} b_i\varphi_k(l_i) > 0, \quad k = 1, 2.$$

Lemma 2.4. Let (H_1) and (H_{3k}) hold. Then for any $h : \mathbf{Z}[a+1, b+1] \to [0, +\infty)$, the unique solution u of the problem (2.5) satisfies $u(t) \geq 0,\ t \in \mathbf{Z}[a, b+2]$.

Proof. Since $\nabla_k < 0$, and $G_k \geq 0$ on $\mathbf{Z}[a, b+2] \times \mathbf{Z}[a, b+2]$, we obtain that $A_k(h) \geq 0$ and $B_k(h) \geq 0$. By Lemma 2.3, $u(t) \geq 0,\ t \in \mathbf{Z}[a, b+2]$. □

Lemma 2.5. Let (H_{3k}) holds. Then

(i) For any $h \in X_+,\ A_k(h),\ B_k(h)$ are two linear functionals and nondecreasing in h.

(ii) For any $h \in X,\ |A_k(h)| \leq A_k(1)\|h\|_\infty,\ |B_k(h)| \leq B_k(1)\|h\|_\infty$.

Now notice that

$$\begin{aligned}\triangle^4 u(t-2) + \beta\triangle^2 u(t-1) - \alpha u(t) &= (-\triangle^2 L + \lambda_2)(-\triangle^2 L + \lambda_1)u(t)\\ &= (-\triangle^2 L + \lambda_1)(-\triangle^2 L + \lambda_2)u(t),\end{aligned}$$

where $Lu(t) = u(t-1)$. Then we can easily get

Lemma 2.6. Let $(H_1), (H_{31})$ and (H_{32}) hold. Then for any $h \in X$, the BVP

$$\begin{cases} \triangle^4 u(t-2) + \beta\triangle^2 u(t-1) - \alpha u(t) = h(t), \quad t \in \mathbf{Z}[a+1, b+1],\\ u(a) = \sum_{i=1}^{m-2} a_i u(l_i), \quad u(b+2) = \sum_{i=1}^{m-2} b_i u(l_i),\\ \triangle^2 u(a-1) = \sum_{i=1}^{m-2} a_i \triangle^2 u(l_i - 1), \quad \triangle^2 u(b+1) = \sum_{i=1}^{m-2} b_i \triangle^2 u(l_i - 1)\end{cases} \tag{2.6}$$

has a unique solution $\{u(t)\}_{t=a-1}^{b+3}$ with

$$u(t) = A_2(v)\psi_2(t) + B_2(v)\varphi_2(t) + \sum_{s=a+1}^{b+1} G_2(t,s)v(s),\ t \in \mathbf{Z}[a, b+2] \tag{2.7}$$

and

$$\begin{aligned} u(a-1) &= \sum_{i=1}^{m-2} a_i\triangle^2 u(l_i - 1) - u(a+1) + 2\sum_{i=1}^{m-2} a_i u(l_i),\\ u(b+3) &= \sum_{i=1}^{m-2} b_i\triangle^2 u(l_i - 1) - u(b+1) + 2\sum_{i=1}^{m-2} b_i u(l_i),\end{aligned}$$

where $G_i,\ A_i,\ B_i (i = 1, 2)$ are defined as in (2.1), (2.3), (2.4) and

$$v(t) = A_1(h)\psi_1(t) + B_1(h)\varphi_1(t) + \sum_{s=a+1}^{b+1} G_1(t,s)h(s),\ t \in \mathbf{Z}[a, b+2]. \tag{2.8}$$

Moreover, if $h \in X_+$, then $u(t) \geq 0,\ t \in \mathbf{Z}[a, b+2]$.

Denote

$$G_0(t,s) = \frac{1}{b+2-a}\begin{cases}(t-a)(b+2-s), & a \leq t \leq s \leq b+2,\\ (s-a)(b+2-t), & a \leq s \leq t \leq b+2,\end{cases} \tag{2.9}$$

$$\nabla_0 = \begin{vmatrix} -\sum_{i=1}^{m-2} b_i \frac{b+2-l_i}{b+2-a} & 1-\sum_{i=1}^{m-2} b_i \frac{l_i-a}{b+2-a} \\ 1-\sum_{i=1}^{m-2} a_i \frac{b+2-l_i}{b+2-a} & -\sum_{i=1}^{m-2} a_i \frac{l_i-a}{b+2-a} \end{vmatrix}, \tag{2.10}$$

$$\begin{aligned} D_0 = {} & \frac{1}{|\nabla_0|}\left(\sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} b_i G_0(l_i,s)\sum_{i=1}^{m-2} a_i \frac{l_i-a}{b+2-a} + \left|1-\sum_{i=1}^{m-2} b_i \frac{l_i-a}{b+2-a}\right|\sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} a_i G_0(l_i,s)\right) \\ & + \frac{1}{|\nabla_0|}\left(\sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} a_i G_0(l_i,s)\sum_{i=1}^{m-2} b_i \frac{b+2-l_i}{b+2-a} + \left|1-\sum_{i=1}^{m-2} a_i \frac{b+2-l_i}{b+2-a}\right|\sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} b_i G_0(l_i,s)\right) \\ & + \max_{t\in\mathbf{Z}[a+1,b+1]}\sum_{s=a+1}^{b+1} G_0(t,s). \end{aligned} \tag{2.11}$$

A simple computation shows that $D_0 > 1$. By Lemma 2.3 with $\lambda_k = 0$ and $h(t) = -\triangle^2 u(t-1)$, we have the following.

Lemma 2.7. Let (H_1) holds. Assume that
(H_4) $\nabla_0 \neq 0$.
Then for any $u \in E$,

$$\begin{aligned} u(t) = {} & A_0(-\triangle^2 u)\frac{b+2-t}{b+2-a} + B_0(-\triangle^2 u)\frac{t-a}{b+2-a} \\ & + \sum_{s=a+1}^{b+1} G_0(t,s)(-\triangle^2 u(s-1)), \quad t \in \mathbf{Z}[a,b+2], \end{aligned} \tag{2.12}$$

where

$$A_0(-\triangle^2 u) = \frac{1}{\nabla_0}\begin{vmatrix} \sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} b_i G_0(l_i,s)(-\triangle^2 u(s-1)) & 1-\sum_{i=1}^{m-2} b_i \frac{l_i-a}{b+2-a} \\ \sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} a_i G_0(l_i,s)(-\triangle^2 u(s-1)) & -\sum_{i=1}^{m-2} a_i \frac{l_i-a}{b+2-a} \end{vmatrix},$$

$$B_0(-\triangle^2 u) = \frac{1}{\nabla_0}\begin{vmatrix} -\sum_{i=1}^{m-2} b_i \frac{b+2-l_i}{b+2-a} & \sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} b_i G_0(l_i,s)(-\triangle^2 u(s-1)) \\ 1-\sum_{i=1}^{m-2} a_i \frac{b+2-l_i}{b+2-a} & \sum_{i=1}^{m-2}\sum_{s=a+1}^{b+1} a_i G_0(l_i,s)(-\triangle^2 u(s-1)) \end{vmatrix}.$$

Hence,

$$\|u\|_\infty \leq D_0 \|\triangle^2 u\|_\infty,\ \forall u \in E. \tag{2.13}$$

Remark 2.1. Let (H_4) holds, then it follows from (2.13) that $\|u\|_0 = \|\triangle^2 u\|_\infty$ is a norm on E. Moreover, for given $\lambda \geq 0$, the norm $\|\cdot\|_\lambda$ is equivalent to the norm $\|\cdot\|_E$, that is,

$$(1+\lambda)^{-1}\|u\|_\lambda \leq \|u\|_E \leq D_0\|u\|_\lambda,\ \forall u \in E. \tag{2.14}$$

In fact, $\forall u \in E,\ t \in \mathbf{Z}[a+1, b+1]$,

$$|\triangle^2 u(t-1)| + \lambda|u(t)| \leq \|\triangle^2 u\|_\infty + \lambda\|u\|_\infty \leq (1+\lambda)\|u\|_E.$$

Thus,

$$\|u\|_\lambda \leq (1+\lambda)\|u\|_E.$$

On the other hand, $\forall u \in E,\ t \in \mathbf{Z}[a+1, b+1]$,

$$|\triangle^2 u(t-1)| \leq |\triangle^2 u(t-1)| + \lambda|u(t)| \leq \|u\|_\lambda,$$

and so $\|\triangle^2 u\|_\infty \leq \|u\|_\lambda$. By (2.13), we have

$$\|u\|_\infty \leq D_0\|\triangle^2 u\|_\infty \leq D_0\|u\|_\lambda.$$

Hence, $\|u\|_E \leq D_0\|u\|_\lambda$. Then $\|\cdot\|_E$ is equivalent to the norm $\|\cdot\|_\lambda$.

For any $h \in X$, the linear BVP (2.6) has a unique solution $\{u\}_{t=a-1}^{b+3}$. Let $(Th)(t) = u(t),\ t \in \mathbf{Z}[a, b+2]$. From Lemma 2.6, the operator T can be expressed by

$$(Th)(t) = A_2(v)\psi_2(t) + B_2(v)\varphi_2(t) + \sum_{s=a+1}^{b+1} G_2(t,s)v(s),\ t \in \mathbf{Z}[a, b+2], \tag{2.15}$$

where v is defined by (2.8). And $Th \in E_+$ for $h \in X_+$.

For $k = 1, 2$, let

$$E_k = \max_{t\in\mathbf{Z}[a+1,b+1]} \psi_k(t), \quad F_k = \max_{t\in\mathbf{Z}[a+1,b+1]} \varphi_k(t), \tag{2.16}$$

$$M_k = \max_{t\in\mathbf{Z}[a+1,b+1]} \sum_{s=a+1}^{b+1} G_k(t,s), \tag{2.17}$$

$$D_k = A_k(1)E_k + B_k(1)F_k + M_k. \tag{2.18}$$

Lemma 2.8. Assume that $(H_1), (H_{31}), (H_{32})$ and (H_4) hold, then $T : (X, \|\cdot\|_\infty) \to (E, \|\cdot\|_{\lambda_1})$ is linear completely continuous, and $\|T\| \leq D_2$.

Proof. It follows from (2.15) that T maps X into E and is linear. Since E is finite dimensional, we only need to prove that $T : (X, \|\cdot\|_\infty) \to (E, \|\cdot\|_{\lambda_1})$ is continuous. For any $\{h(t)\}_{t=a+1}^{b+1} \in X$, let

$$\{u(t)\}_{t=a}^{b+2} = \{(Th)(t)\}_{t=a}^{b+2},$$

$$u(a-1)=\sum_{i=1}^{m-2}a_i\triangle^2u(l_i-1)-u(a+1)+2\sum_{i=1}^{m-2}a_iu(l_i),$$

$$u(b+3)=\sum_{i=1}^{m-2}b_i\triangle^2u(l_i-1)-u(b+1)+2\sum_{i=1}^{m-2}b_iu(l_i).$$

Then

$$u(a)=\sum_{i=1}^{m-2}a_iu(l_i),\quad u(b+2)=\sum_{i=1}^{m-2}b_iu(l_i),$$

$$\triangle^2u(a-1)=\sum_{i=1}^{m-2}a_i\triangle^2u(l_i-1),\quad \triangle^2u(b+1)=\sum_{i=1}^{m-2}b_i\triangle^2u(l_i-1).$$

Let $v(t)=-\triangle^2u(t-1)+\lambda_2u(t),\ t\in\mathbf{Z}[a,b+2]$. Then $v(a)=\sum_{i=1}^{m-2}a_iv(l_i)$, $v(b+2)=\sum_{i=1}^{m-2}b_iv(l_i)$. Hence, $\{v(t)\}_{t=a}^{b+2}$ satisfies the following BVP:

$$\begin{cases}-\triangle^2v(t-1)+\lambda_2v(t)=h(t),\quad t\in\mathbf{Z}[a+1,b+1],\\ v(a)=\sum_{i=1}^{m-2}a_iv(l_i),\quad v(b+2)=\sum_{i=1}^{m-2}b_iv(l_i).\end{cases}$$

From Lemma 2.3, we obtain

$$v(t)=A_1(h)\psi_1(t)+B_1(h)\varphi_1(t)+\sum_{s=a+1}^{b+1}G_1(t,s)h(s),\ t\in\mathbf{Z}[a,b+2].$$

That is,

$$-\triangle^2u(t-1)+\lambda_2u(t)=A_1(h)\psi_1(t)+B_1(h)\varphi_1(t)+\sum_{s=a+1}^{b+1}G_1(t,s)h(s),\ t\in\mathbf{Z}[a,b+2].\tag{2.19}$$

Similarly, we also have

$$-\triangle^2u(t-1)+\lambda_1u(t)=A_2(h)\psi_2(t)+B_2(h)\varphi_2(t)+\sum_{s=a+1}^{b+1}G_2(t,s)h(s),\ t\in\mathbf{Z}[a,b+2].\tag{2.20}$$

By (2.19) and (2.15), we have

$$\begin{aligned}\triangle^2u(t-1)&=\lambda_2u(t)-A_1(h)\psi_1(t)-B_1(h)\varphi_1(t)-\sum_{s=a+1}^{b+1}G_1(t,s)h(s)\\ &=\lambda_2\left(A_2(v)\psi_2(t)+B_2(v)\varphi_2(t)+\sum_{s=a+1}^{b+1}G_2(t,s)v(s)\right)\end{aligned}$$

$$-A_1(h)\psi_1(t) - B_1(h)\varphi_1(t) - \sum_{s=a+1}^{b+1} G_1(t,s)h(s),\ t \in \mathbf{Z}[a, b+2].$$

It follows from (2.16)-(2.18) and Lemma 2.5 that

$$\begin{aligned}&\left|A_1(h)\psi_1(t) + B_1(h)\varphi_1(t) + \sum_{s=a+1}^{b+1} G_1(t,s)h(s)\right|\\ &\quad\le (A_1(1)E_1 + B_1(1)F_1 + M_1)\|h\|_\infty = D_1\|h\|_\infty,\\ &\left|A_2(v)\psi_2(t) + B_2(v)\varphi_2(t) + \sum_{s=a+1}^{b+1} G_2(t,s)v(s)\right|\\ &\quad\le (A_2(1)E_2 + B_2(1)F_2 + M_2)\|v\|_\infty = D_1D_2\|h\|_\infty.\end{aligned}$$

Thus, we get

$$|\triangle^2 u(t-1)| \le (|\lambda_2|D_2 + 1)D_1\|h\|_\infty,\ t \in \mathbf{Z}[a+1, b+1].$$

Then, $\|\triangle^2 u\|_\infty \le (|\lambda_2|D_2 + 1)D_1\|h\|_\infty$. By (2.13), we have

$$\begin{aligned}\|Th\|_E &= \|u\|_E = \max\{\|u\|_\infty, \|\triangle^2 u\|_\infty\}\\ &\le D_0\|\triangle^2 u\|_\infty \le (|\lambda_2|D_2 + 1)D_1\|h\|_\infty,\end{aligned}$$

which implies that $T : (X, \|\cdot\|_\infty) \to (E, \|\cdot\|_E)$ is continuous. Since the norms $\|\cdot\|_E$ and $\|\cdot\|_{\lambda_1}$ are equivalent from Remark 2.1, $T : (X, \|\cdot\|_\infty) \to (E, \|\cdot\|_{\lambda_1})$ is also continuous.

Now, we show that $\|T\| \le D_2$. For any $h \in X_+$, let $u = Th$, by Lemma 2.6, $u(t) \ge 0,\ t \in \mathbf{Z}[a, b+2]$. It follows from (H_1) that $\lambda_1 \ge 0 \ge \lambda_2$. From (2.19) and Lemma 2.5, we obtain that $\triangle^2 u(t-1) \le 0,\ t \in \mathbf{Z}[a, b+2]$. Thus, by (2.20), we immediately have

$$\begin{aligned}&|-\triangle^2 u(t-1)| + |\lambda_1 u(t)| = -\triangle^2 u(t-1) + \lambda_1 u(t)\\ &\quad= A_2(h)\psi_2(t) + B_2(h)\varphi_2(t) + \sum_{s=a+1}^{b+1} G_2(t,s)h(s),\ t \in \mathbf{Z}[a, b+2].\end{aligned}$$

For any $h \in X$, let $h = h_1 - h_2, u_1 = Th_1, u_2 = Th_2$, where h_1 and h_2 are the positive and negative part of h, respectively. Let $u = Th$. Then $u = u_1 - u_2$. From the discuss above, we have $u_k(t) \ge 0,\ \triangle^2 u_k(t-1) \ge 0,\ t \in \mathbf{Z}[a, b+2],\ k = 1, 2$. Hence

$$\begin{aligned}|-\triangle^2 u_k(t-1)| + |\lambda_1 u_k(t)| &= -\triangle^2 u_k(t-1) + \lambda_1 u_k(t)\\ &= A_2(h_k)\psi_2(t) + B_2(h_k)\varphi_2(t) + \sum_{s=a+1}^{b+1} G_2(t,s)h_k(s)\\ &=: Hh_k,\ t \in \mathbf{Z}[a, b+2],\ k = 1, 2.\end{aligned}$$

Then

$$|\triangle^2 u(t-1)| + \lambda_1|u(t)| = |\triangle^2 u_1(t-1) - \triangle^2 u_2(t-1)| + \lambda_1|u_1(t) - u_2(t)|$$

$$\le |\triangle^2 u_1(t-1)| + \lambda_1|u_1(t)| + |\triangle^2 u_2(t-1)| + \lambda_1|u_2(t)|$$
$$= Hh_1 + Hh_2 = H|h|$$
$$\le (A_2(1)E_2 + B_2(1)F_2 + M_2)\|h\|_\infty = D_2\|h\|_\infty.$$

Thus $\|Th\|_{\lambda_1} \le D_2\|h\|_\infty$, and so $\|T\| \le D_2$. The proof is completed. □

In the rest of the paper, we make the following notations:

$$\underline{f}_{0,\xi} = \liminf_{x\to 0^+} \min_{t\in \mathbf{Z}[a+1,b+1]} \frac{f(t,x)}{x^\xi}, \quad \overline{f}_{0,\eta} = \limsup_{x\to 0^+} \max_{t\in \mathbf{Z}[a+1,b+1]} \frac{f(t,x)}{x^\eta};$$

$$\underline{f}_{\infty,\xi} = \liminf_{x\to +\infty} \min_{t\in \mathbf{Z}[a+1,b+1]} \frac{f(t,x)}{x^\xi}, \quad \overline{f}_{\infty,\eta} = \limsup_{x\to +\infty} \max_{t\in \mathbf{Z}[a+1,b+1]} \frac{f(t,x)}{x^\eta};$$

$$\lambda^* = 16\sin^4\frac{\pi}{2(b-a+2)} - 4\beta\sin^2\frac{\pi}{2(b-a+2)} - \alpha; \tag{2.21}$$

$$K = \max_{t\in \mathbf{Z}[a+1,b+1]} [A(t) - \alpha + B(t) - \beta]; \tag{2.22}$$

$$V_k = A_k(1) + B_k(1) + \sum_{s=a+1}^{b+1} G_k(s,s), \ k = 1,2; \tag{2.23}$$

$$U_k = \min_{t\in \mathbf{Z}[a+1,b+1]} G_k(t,t), \ k = 1,2. \tag{2.24}$$

3. Main results

Now with the aid of the lemmas in Section 2, we are in position to state and prove our main results.

Theorem 3.1. Assume that $(H_{1a}), (H_{31}), (H_{32})$ and (H_4) hold, and $L = KD_0D_2 < 1$. If one of the following conditions are satisfied

(i) $\underline{f}_{0,\xi} \in (\lambda^*, +\infty]$, $\overline{f}_{\infty,\eta} \in [0, \lambda_*)$ with $\xi = 1, \eta = 1$;

(ii) $\underline{f}_{\infty,\xi} \in (\lambda^*, +\infty]$, $\overline{f}_{0,\eta} \in [0, \lambda_*)$ with $\xi = 1, \eta = 1$,

then, BVP (1.2) has at least one positive solution, where $\lambda_* = (1-L)(C_1C_2V_1V_2)^{-1}$, C_1, C_2 are given in Lemma 2.2, V_1, V_2 are defined as in (2.23), λ^*, K are defined as in (2.21), and D_0, D_2 are defined as in (2.11) and (2.18), respectively.

Proof. For any $h \in X$, consider the linear BVP:

$$\begin{cases} \triangle^4 u(t-2) + B(t)\triangle^2 u(t-1) - A(t)u(t) = h(t), \ t \in \mathbf{Z}[a+1, b+1], \\ u(a) = \sum_{i=1}^{m-2} a_i u(l_i), \ u(b+2) = \sum_{i=1}^{m-2} b_i u(l_i), \\ \triangle^2 u(a-1) = \sum_{i=1}^{m-2} a_i \triangle^2 u(l_i - 1), \ \triangle^2 u(b+1) = \sum_{i=1}^{m-2} b_i \triangle^2 u(l_i - 1). \end{cases}$$

It is easy to see that the above BVP is equivalent to the following BVP:

$$\begin{cases} \triangle^4 u(t-2) + \beta\triangle^2 u(t-1) - \alpha u(t) \\ \quad = -(B(t)-\beta)\triangle^2 u(t-1) + (A(t)-\alpha)u(t) + h(t), t \in \mathbf{Z}[a+1,b+1], \\ u(a) = \sum_{i=1}^{m-2} a_i u(l_i), \quad u(b+2) = \sum_{i=1}^{m-2} b_i u(l_i), \\ \triangle^2 u(a-1) = \sum_{i=1}^{m-2} a_i \triangle^2 u(l_i - 1), \quad \triangle^2 u(b+1) = \sum_{i=1}^{m-2} b_i \triangle^2 u(l_i-1), \end{cases} \tag{3.1}$$

For any $v \in E$, let $(Gv)(t) = -(B(t)-\beta)\triangle^2 u(t-1) + (A(t)-\alpha)u(t),\ t \in \mathbf{Z}[a+1,b+1]$. Obviously, the operator $G: E \to X$ is linear. Owing to (2.14), one has that for $v \in E,\ t \in \mathbf{Z}[a+1,b+1]$,

$$|(Gv)(t)| \le [(B(t)-\beta) + (A(t)-\alpha)] \max\{\|v\|_\infty,\ \|\triangle^2 v\|_\infty\} \le KD_0\|v\|_{\lambda_1}.$$

Hence, $\|(Gv)\|_\infty \le KD_0\|v\|_{\lambda_1}$, and so $\|G\| \le KD_0$. On the other hand, $\{u(t)\}_{t=a-1}^{b+3}$ is a solution of (3.1) if and only if $u = \{u(t)\}_{t=a}^{b+2} \in E$ satisfies $u = T(Gu+h)$, i.e.,

$$(I - TG)u = Th. \tag{3.2}$$

It follows from $T: X \to E$ and $G: E \to X$ that $I - TG$ maps E into E. By $\|T\| \le D_2$ (see Lemma 2.8), $\|G\| \le KD_0$ and condition $KD_0D_2 < 1$, we obtain that $(I-TG)^{-1}$, the inverse mapping of $I-TG$, exists and is bounded.

Let $S = (I-TG)^{-1}T$. Then (3.2) is equivalent to $u = Sh$ and S can be expressed by

$$S = (I + TG + \cdots + (TG)^n + \cdots)T = T + (TG)T + \cdots + (TG)^n T + \cdots. \tag{3.3}$$

The complete continuity of T together with the continuity of $(I-TG)^{-1}$ implies that the operator $S: X \to E$ is completely continuous. For any $h \in X_+$, let $u = Th$. Then the definition of T and Lemma 2.6 yield that $u \in E$ and $u(t) \ge 0,\ t \in \mathbf{Z}[a,b+2]$. From (2.19), Lemma 2.5 and $\lambda_2 \le 0$, we obtain that $\triangle^2 u(t-1) \le 0,\ t \in \mathbf{Z}[a,b+2]$. So we have

$$(Gu)(t) = -(B(t)-\beta)\triangle^2 u(t-1) + (A(t)-\alpha)u(t) \ge 0,\ t \in \mathbf{Z}[a+1,b+1].$$

Hence for any $h \in X_+,\ (GTh)(t) \ge 0,\ t \in \mathbf{Z}[a+1,b+1]$, and so $(TG)(Th)(t) \ge 0,\ t \in Z[a,b+2]$. It follows from mathematical induction that

$$(TG)^n(Th)(t) \ge 0,\ \forall h \in X_+,\ t \in \mathbf{Z}[a,b+2],\ n = 1,2,\cdots \tag{3.4}$$

By (3.3) and (3.4), we have

$$\begin{aligned} (Sh)(t) &= (Th)(t) + (TG)(Th)(t) + \cdots + (TG)^n(Th)(t) + \cdots \\ &\ge (Th)(t),\ \forall h \in X_+,\ t \in \mathbf{Z}[a,b+2]. \end{aligned} \tag{3.5}$$

Then $S: X_+ \to E_+$. On the other hand, we have that for any $h \in X_+$,

$$\begin{aligned} (Sh)(t) &\le (Th)(t) + \|TG\|(Th)(t) + \cdots + \|(TG)^n\|(Th)(t) + \cdots \\ &\le (1 + L + \cdots + L^n + \cdots)(Th)(t) \\ &= (1-L)^{-1}(Th)(t), \quad t \in \mathbf{Z}[a,b+2]. \end{aligned} \tag{3.6}$$

Thus,

$$\|Sh\|_\infty \le (1-L)^{-1}\|Th\|_\infty. \tag{3.7}$$

Define operators $\mathbf{f} : X \to X$, $\mathbf{p} : E \to X$, respectively, by

$$(\mathbf{f}u)(t) = f(t, u(t)), \quad \forall u \in X, \quad t \in \mathbf{Z}[a+1, b+1];$$

$$\mathbf{p}\{u(t)\}_{t=a}^{b+2} = \{u(t)\}_{a+1}^{b+1}, \quad \forall \{u(t)\}_{t=a}^{b+2} \in E.$$

The continuity of f means that $\mathbf{f} : X_+ \to X_+$ is continuous. It is easy to see that $\{u(t)\}_{t=a-1}^{b+3}$ is a positive solution of BVP (1.2) if and only if $u = \{u(t)\}_{t=a}^{b+2} \in E_+$ is a nonzero solution of the operator equation $u = Wu$, where $W := S\mathbf{fp}$. Obviously, $W : E_+ \to E_+$ is completely continuous. We next show that the operator W has at least fixed point in E_+.

Set

$$P = \{u \in E_+ : u(t) \ge \sigma G_2(t,t)\|u\|_\infty,\ t \in \mathbf{Z}[a+1, b+1]\},$$

where

$$\begin{aligned}\sigma &= \delta_1\delta_2(1-L)(C_1C_2V_2)^{-1}\\ &\times \left(A_2(G_1(t,t)) + B_2(G_1(t,t)) + \sum_{s=a+1}^{b+1} G_1(s,s)G_2(s,s)\right).\end{aligned} \tag{3.8}$$

It is easy to see that P is a cone. Now, we show $W(P) \subset P$.

For any $u \in E_+$, then $\mathbf{fp}u \in X_+$. By the definition of T, we have

$$\begin{aligned}(T\mathbf{fp}u)(t) &= A_2(w)\psi_2(t) + B_2(w)\varphi_2(t)\\ &\quad + \sum_{s=a+1}^{b+1} G_2(t,s)w(s),\ \forall u \in P,\ t \in \mathbf{Z}[a, b+2],\end{aligned} \tag{3.9}$$

where $w(t) = A_1(\mathbf{fp}u)\psi_1(t) + B_1(\mathbf{fp}u)\varphi_1(t) + \sum_{k=a+1}^{b+1} G_1(t,k)(\mathbf{fp}u)(k),\ t \in \mathbf{Z}[a, b+2]$. By Lemmas 2.2 and 2.5, we have

$$\begin{aligned}&A_2(w)\psi_2(t) + B_2(w)\varphi_2(t)\\ &\le C_2(A_2(1) + B_2(1))\|w\|_\infty\\ &\le C_1C_2(A_2(1) + B_2(1))\left(A_1(\mathbf{fp}u) + B_1(\mathbf{fp}u) + \sum_{k=a+1}^{b+1} G_1(k,k)(\mathbf{fp}u)(k)\right),\end{aligned}$$

$$\sum_{s=a+1}^{b+1} G_2(t,s)w(s) \le C_1C_2 \sum_{s=a+1}^{b+1} G_2(s,s)\left(A_1(\mathbf{fp}u) + B_1(\mathbf{fp}u) + \sum_{k=a+1}^{b+1} G_1(k,k)(\mathbf{fp}u)(k)\right).$$

Then,

$$(T\mathbf{fp}u)(t) \le C_1C_2V_2\left(A_1(\mathbf{fp}u) + B_1(\mathbf{fp}u) + \sum_{k=a+1}^{b+1} G_1(k,k)(\mathbf{fp}u)(k)\right),\ t \in \mathbf{Z}[a+1, b+1].$$

This gives

$$A_1(\mathbf{fp}u) + B_1(\mathbf{fp}u) + \sum_{k=a+1}^{b+1} G_1(k,k)(\mathbf{fp}u)(k) \geq (C_1C_2V_2)^{-1}\|T\mathbf{fp}u\|_\infty. \tag{3.10}$$

Similarly, we also have

$$(T\mathbf{fp}u)(t) \geq \delta_1\delta_2 G_2(t,t)\left(A_2(G_1(t,t)) + B_2(G_1(t,t)) + \sum_{s=a+1}^{b+1} G_1(s,s)G_2(s,s)\right) \times \left(A_1(\mathbf{fp}u) + B_1(\mathbf{fp}u) + \sum_{s=a+1}^{b+1} G_1(s,s)(\mathbf{fp}u)(s)\right). \tag{3.11}$$

This together with (3.5), (3.10) and (3.7) gives

$$(Wu)(t) = (S\mathbf{fp}u)(t) \geq (T\mathbf{fp}u)(t) \geq \sigma G_2(t,t)\|S\mathbf{fp}u\|_\infty,\ t \in \mathbf{Z}[a+1, b+1].$$

Hence, $W(P) \subset P$. Obviously, $T(P) \subset P$.

Let $\omega_k = \min_{a+1 \leq t,s \leq b+1} G_k(t,s)$. Obviously, $\omega_k > 0 (k = 1, 2)$, and moreover

$$u(t) \geq \sigma\omega_2\|u\|_\infty,\ \forall u \in P,\ t \in Z[a+1, b+1]. \tag{3.12}$$

Suppose that condition (i) holds. By $\underline{f}_{0,1} > \lambda^*$, we can choose $\varepsilon > 0$ such that $\underline{f}_{0,1} > \lambda^* + \varepsilon$. Then there exists $r > 0$ such that $f(t,x) > (\lambda^* + \varepsilon)x$ for $x \in (0, r],\ t \in \mathbf{Z}[a+1, b+1]$. Let $\Omega_r = \{u \in P : \|u\|_\infty \leq r\}$. For any $u \in \partial\Omega_r$, it follows from (3.12) that

$$f(t, u(t)) > (\lambda^* + \varepsilon)u(t) > (\lambda^* + \varepsilon)\sigma\omega_2 r,\ t \in \mathbf{Z}[a+1, b+1]. \tag{3.13}$$

For any $u \in \partial\Omega_r$, by (3.5), (3.9) and (3.13), we have

$$\|Wu\|_\infty \geq (Wu)(a+1) \geq (T\mathbf{fp}u)(a+1) \geq \sum_{s=a+1}^{b+1}\sum_{k=a+1}^{b+1} G_2(a+1,s)G_1(s,k)(\mathbf{fp}u)(k) \geq (\lambda^* + \varepsilon)\sigma\omega_1\omega_2^2 r(b-a+1)^2 > 0.$$

Therefore,

$$\inf_{u \in \partial\Omega_r} \|Wu\|_\infty > 0. \tag{3.14}$$

Now we shall prove

$$Wu \neq \mu u,\ \forall u \in \partial\Omega_r,\ \mu \in (0, 1]. \tag{3.15}$$

Suppose the contrary, then there exist $u_0 \in \partial\Omega_r$ and $\mu_0 \in (0, 1]$ such that $Wu_0 = \mu_0 u_0$. By (3.5), we have

$$u_0(t) \geq \mu_0 u_0(t) = (Wu_0)(t) \geq (T\mathbf{fp}u_0)(t) := v_0(t),\ t \in \mathbf{Z}[a, b+2]. \tag{3.16}$$

Let

$$v_0(a-1) = \sum_{i=1}^{m-2} a_i \triangle^2 v_0(l_i - 1) - v_0(a+1) + 2\sum_{i=1}^{m-2} a_i v_0(l_i),$$

$$v_0(b+3) = \sum_{i=1}^{m-2} b_i \triangle^2 v_0(l_i - 1) - v_0(b+1) + 2\sum_{i=1}^{m-2} b_i v_0(l_i).$$

Then, $\{v_0(t)\}_{t=a-1}^{b+3}$ satisfies BVP (2.6) with $\{h(t)\}_{t=a+1}^{b+1} = \{\mathbf{fp}u_0(t)\}_{t=a+1}^{b+1}$. That is,

$$\begin{cases} \triangle^4 v_0(t-2) + \beta\triangle^2 v_0(t-1) - \alpha v_0(t) = f(t, u_0(t)), \\ t \in \mathbf{Z}[a+1, b+1], \\ v_0(a) = \sum_{i=1}^{m-2} a_i v_0(l_i), \quad v_0(b+2) = \sum_{i=1}^{m-2} b_i v_0(l_i), \\ \triangle^2 v_0(a-1) = \sum_{i=1}^{m-2} a_i \triangle^2 v_0(l_i - 1), \\ \triangle^2 v_0(b+1) = \sum_{i=1}^{m-2} b_i \triangle^2 v_0(l_i - 1). \end{cases} \tag{3.17}$$

For $x, y : \mathbf{Z} \to \mathbf{Z}$, a simple computation shows

$$\begin{aligned} &\sum_{t=a+1}^{b+1} y(t)\triangle^2 x(t-1) \\ &= -x(a+1)y(a) + x(a)y(a+1) + x(b+2)y(b+1) \\ &\quad - x(b+1)y(b+2) + \sum_{t=a+1}^{b+1} x(t)\triangle^2 y(t-1). \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\sum_{t=a+1}^{b+1} y(t)\triangle^4 x(t-2) \\ &= -y(a)\triangle^2 x(a) + y(a+1)\triangle^2 x(a-1) - x(a+1)\triangle^2 y(a-1) \\ &\quad + x(a)\triangle^2 y(a) + x(b+2)\triangle^2 y(b) - x(b+1)\triangle^2 y(b+1) \\ &\quad + y(b+1)\triangle^2 x(b+1) - y(b+2)\triangle^2 x(b) + \sum_{t=a+1}^{b+1} x(t)\triangle^4 y(t-2). \end{aligned} \tag{3.19}$$

Multiplying the first equation of (3.17) by $e(t) := \sin\frac{t-a}{b-a+2}\pi$ and summing from $a+1$ to $b+1$, it follows from (3.18), (3.19) and the boundary conditions in (3.17) that

$$e(a+1)\triangle^2 v_0(a-1) + v_0(a)[\triangle^2 e(a) + \beta e(a+1)] + v_0(b+2)[\triangle^2 e(b) + \beta e(b+1)]$$

$$+e(b+1)\triangle^2 v_0(b+1) + \sum_{t=a+1}^{b+1} [\triangle^4 e(t-2) + \beta\triangle^2 e(t-1) - \alpha e(t)]v_0(t) = \sum_{t=a+1}^{b+1} f(t, u_0(t))e(t).$$

That is,

$$\sin\frac{\pi}{b-a+2}\left(\sum_{i=1}^{m-2}(a_i+b_i)\triangle^2 v_0(l_i-1)+[-4\sin^2\frac{\pi}{2(b-a+2)}+\beta]\sum_{i=1}^{m-2}(a_i+b_i)v_0(l_i)\right)$$

$$+\lambda^*\sum_{t=a+1}^{b+1}v_0(t)e(t)=\sum_{t=a+1}^{b+1}f(t,u_0(t))e(t). \tag{3.20}$$

It follows from Lemma 2.6 that $v_0(t)\geq 0,\ t\in\mathbf{Z}[a,b+2]$. Similarly to (2.19), we have

$$-\triangle^2 v_0(t-1)+\lambda_2 v_0(t)=A_1(\mathbf{fp}u_0)\psi_1(t)+B_1(\mathbf{fp}u_0)\varphi_1(t)$$

$$+\sum_{s=a+1}^{b+1}G_1(t,s)(\mathbf{fp}u_0)(s),\quad t\in\mathbf{Z}[a,b{+}2].$$

Bearing in mind that $\lambda_2\leq 0$, we obtain that $\triangle^2 v_0(t-1)\leq 0,\ t\in\mathbf{Z}[a,b+2]$. By (3.13), (3.20), (H_{1a}) and (3.16), we get

$$(\lambda^*{+}\varepsilon)\sum_{t=a+1}^{b+1}u_0(t)e(t)\leq\sum_{t=a+1}^{b+1}f(t,u_0(t))e(t)\leq\lambda^*\sum_{t=a+1}^{b+1}v_0(t)e(t)\leq\lambda^*\sum_{t=a+1}^{b+1}u_0(t)e(t).$$

Since $u_0(t)\geq\sigma\omega_2\|u_0\|_\infty=\sigma\omega_2 r>0,\ t\in\mathbf{Z}[a+1,b+1]$, we have

$$\sum_{t=a+1}^{b+1}u_0(t)e(t)>0.$$

Then $\lambda^*+\varepsilon<\lambda^*$, which is a contradiction. This proves (3.15). It follows from (3.14), (3.15) and Lemma 1.2 that

$$i(W,\Omega_r,P)=0. \tag{3.21}$$

From $\overline{f}_{\infty,1}<\lambda_*$, we can choose $\varepsilon=(0,\lambda_*)$ such that $\overline{f}_{\infty,1}<\lambda_*-\varepsilon$. Then there exists $R_0>0$ such that $f(t,x)<(\lambda_*-\varepsilon)x$ for $x>R_0,\ t\in\mathbf{Z}[a{+}1,b{+}1]$. Let $C=\sup_{t\in\mathbf{Z}[a+1,b+1],\ x\in[0,R_0]}f(t,x)$. Obviously,

$$f(t,x)\leq(\lambda_*-\varepsilon)x+C,\ \forall x\in[0,+\infty),\ t\in\mathbf{Z}[a+1,b+1].$$

Take $R>\max\{r,\varepsilon^{-1}C\}$, and let $\Omega_R=\{u\in P:\|u\|_\infty\leq R\}$. We next show $Wu\neq\mu u,\ \forall u\in\partial\Omega_R,\ \mu\geq 0$. In fact, if there exist $u_0\in\partial\Omega_R$ and $\mu_0\geq 1$ such that $Wu_0 = \mu_0 u_0$, then by (3.7) and (3.10), we obtain

$$\begin{aligned}
&(Wu_0)(t)\\
&= (S\mathbf{fp}u_0)(t)\\
&\le (1-L)^{-1}\|T\mathbf{fP}u_0\|_\infty\\
&\le (1-L)^{-1}C_1C_2V_2\left(A_1(\mathbf{fp}u_0) + B_1(\mathbf{fp}u_0) + \sum_{k=a+1}^{b+1} G_1(k,k)(\mathbf{fp}u_0)(k)\right)\\
&\le \frac{1}{\lambda_*}\|\mathbf{fp}u_0\|_\infty \le \left(1-\frac{\varepsilon}{\lambda_*}\right)\|u_0\|_\infty + \frac{1}{\lambda_*}C,\ t\in \mathbf{Z}[a+1,b+1].
\end{aligned}$$

Then

$$u_0(t) \le \mu_0 u_0(t) = (Wu_0)(t) \le \left(1-\frac{\varepsilon}{\lambda_*}\right)\|u_0\|_\infty + \frac{1}{\lambda_*}C,\ t\in \mathbf{Z}[a+1,b+1],$$

which implies $\|u_0\|_\infty \le \left(1-\frac{\varepsilon}{\lambda_*}\right)\|u_0\|_\infty + \frac{1}{\lambda_*}C$. Thus $R = \|u_0\|_\infty \le \frac{C}{\varepsilon}$, which contradicts the choice of R. By Lemma 1.1, we have $i(W,\Omega_R,P) = 1$. Taking (3.21) into account, we have $i(W,\Omega_R\setminus\Omega_r,P) = 1$. Then W has at least one fixed point in $\Omega_R\setminus\Omega_r$, which means BVP (1.1) has at least one positive solution. This completes the proof of (i).

The proof of (ii) is similar and will be omitted here. □

Theorem 3.2. Assume that $(H_1),(H_{31}),(H_{32})$ and (H_4) hold, and $L = KD_0D_2 < 1$. If one of the following conditions are satisfied

(i) $\underline{f}_{\infty,\xi} \in (0,+\infty]$, $\overline{f}_{0,\eta} \in [0,+\infty)$ with $\xi > 1, \eta > 1$;

(ii) $\underline{f}_{0,\xi} \in (0,+\infty]$, $\overline{f}_{\infty,\eta} \in [0,+\infty)$ with $0 < \xi < 1, 0 < \eta < 1$,

then, BVP (1.2) has at least one positive solution.

Proof. According to the proof of Theorem 3.1, it suffices to prove that the operator W has at least fixed point in E_+.

First, suppose that the condition (i) holds. Define the cone P_1 in E by

$$P_1 = \{u\in E_+ : u(t) \ge \delta_1\delta_2(1-L)U_1U_2(C_1C_2)^{-1}\|u\|_\infty,\ t\in\mathbf{Z}[a+1,b+1]\},$$

where $C_1, C_2, \delta_1, \delta_2$ are given in Lemma 2.2, U_1, U_2 are defined as in (2.24). By (3.5), (3.11), (3.10) and (3.7), we have, for $u \in P_1$ and $t \in \mathbf{Z}[a+1,b+1]$,

$$(Wu)(t) = (S\mathbf{fp}u)(t) \ge (T\mathbf{fp}u)(t) \ge \delta_1\delta_2(1-L)U_1U_2(C_1C_2)^{-1}\|Wu\|_\infty.$$

Hence, $W(P_1)\subset P_1$.

Let $\widetilde{u}_1 = Sh_1$, where $h_1 = \{1\}_{a+1}^{b+1} \in X_+$. Then by (3.5), (3.6), (2.15), Lemma 2.2 and Lemma 2.5, one has, for $t \in \mathbf{Z}[a, b+2]$,

$$\begin{aligned}\delta_1\delta_2 U_1 U_2 V_1 V_2 \le (Th_1)(t) \le \widetilde{u}_1(t) &= (Sh_1)(t)\\ &\le (1-L)^{-1}(Th_1)(t)\\ &\le (1-L)^{-1} C_1 C_2 V_1 V_2.\end{aligned}$$

Set

$$u_1(t) \equiv \delta_1\delta_2 U_1 U_2 V_1 V_2$$

for $t \in \mathbf{Z}[a+1, b+1]$,

$$u_1(a) = \delta_1\delta_2 U_1 U_2 V_1 V_2 \sum_{i=1}^{m} a_i$$

and

$$u_1(b+2) = \delta_1\delta_2 U_1 U_2 V_1 V_2 \sum_{i=1}^{m} b_i.$$

Then $u_1 \in P_1 \setminus \{\theta\}$, and

$$\delta_1\delta_2 U_1 U_2 V_1 V_2 = u_1(t) \le (1-L)^{-1} C_1 C_2 V_1 V_2,\ t \in \mathbf{Z}[a+1, b+1]. \tag{3.22}$$

By $\underline{f}_{\infty,\xi} \in (0, +\infty]$ with $\xi > 1$, there exist $\varepsilon_1 > 0$ and $\nu_1 > 0$ such that

$$f(t, x) \ge \nu_1 x^{\xi},\ t \in \mathbf{Z}[a+1, b+1],\ x \ge \varepsilon_1. \tag{3.23}$$

Choose ε_2 such that

$$\begin{aligned}\varepsilon_2 > \max\{&\varepsilon_1 C_1 C_2[\delta_1\delta_2(1-L)U_1U_2]^{-1},\\ &\nu_1^{-\frac{1}{\xi-1}}(1-L)^{-2}C_1^2C_2^2(\delta_1\delta_2U_1U_2)^{-\frac{2\xi-1}{\xi-1}}(V_1V_2)^{-\frac{1}{\xi-1}}\},\end{aligned}$$

and let $\Omega_{\varepsilon_2} = \{u \in P_1 : \|u\|_\infty \le \varepsilon_2\}$. If there exists $u_0 \in \partial\Omega_{\varepsilon_2}$ such that $u_0 - Wu_0 = 0$, then the conclusion holds, so suppose that $u - Wu \ne 0, \forall u \in \partial\Omega_{\varepsilon_2}$. We claim that

$$u - Wu \ne su_1,\ \forall u \in \partial\Omega_{\varepsilon_2},\ s \ge 0. \tag{3.24}$$

Suppose the contrary, then there exist $u_2 \in \partial\Omega_{\varepsilon_2}$ and $s_0 \ge 0$ such that $u_2 - Wu_2 = s_0u_1$. By the assumption that $u - Wu \ne 0, \forall u \in \partial\Omega_{\varepsilon_2}$, we obtain that $s_0 > 0$.

Notice that

$$u_2(t) = Wu_2(t) + s_0u_1(t) \ge s_0u_1(t),\ t \in \mathbf{Z}[a+1, b+1].$$

Let $s^* = \sup\{s : u_2(t) \ge su_1(t),\ t \in \mathbf{Z}[a+1, b+1]\}$. Then $s_0 \le s^* < +\infty$ and $u_2(t) \ge s^*u_1(t),\ t \in \mathbf{Z}[a+1, b+1]$. By $u_2 \in \partial\Omega_{\varepsilon_2}$ and (3.22), we have, for $t \in \mathbf{Z}[a+1, b+1]$,

$$u_2(t) \ge \delta_1\delta_2(1-L)U_1U_2(C_1C_2)^{-1}\varepsilon_2$$

$$\ge C_1C_2\nu_1^{-\frac{1}{\xi-1}}(1-L)^{-1}(\delta_1\delta_2U_1U_2)^{-\frac{\xi}{\xi-1}}(V_1V_2)^{-\frac{1}{\xi-1}}$$

$$=\nu_1^{-\frac{1}{\xi-1}}(1-L)^{-1}(\delta_1\delta_2U_1U_2V_1V_2)^{-\frac{\xi}{\xi-1}}C_1C_2V_1V_2$$

$$\ge \nu_1^{-\frac{1}{\xi-1}}(\delta_1\delta_2U_1U_2V_1V_2)^{-\frac{\xi}{\xi-1}}u_1(t).$$

From the definition of s^*, it follows that

$$s^*\ge \nu_1^{-\frac{1}{\xi-1}}(\delta_1\delta_2U_1U_2V_1V_2)^{-\frac{\xi}{\xi-1}}. \tag{3.25}$$

Taking into account that

$$u_2(t)\ge \delta_1\delta_2(1-L)U_1U_2(C_1C_2)^{-1}\varepsilon_2>\varepsilon_1,\ \forall t\in \mathbf{Z}[a+1,b+1],$$

we have, by (3.5), (3.11), (3.23) and (3.25), for $t\in\mathbf{Z}[a+1,b+1]$,

$$\begin{aligned}
&u_2(t)\\
&=(Wu_2)(t)+s_0u_1(t)\\
&\ge \delta_1\delta_2U_1U_2V_2\times\left(A_1(\nu_1u_2^{\xi})+B_1(\nu_1u_2^{\xi})+\sum_{s=a+1}^{b+1}G_1(s,s)(\nu_1u_2^{\xi})(s)\right)\\
&\quad+s_0u_1(t)\\
&\ge \nu_1\delta_1\delta_2U_1U_2V_2\times\left(A_1(s^*u_2^{\xi})+B_1(s^*u_2^{\xi})+\sum_{s=a+1}^{b+1}G_1(s,s)(s^*u_2^{\xi})(s)\right)\\
&\quad+s_0u_1(t)\\
&=\nu_1\delta_1\delta_2U_1U_2V_1V_2(s^*\delta_1\delta_2U_1U_2V_1V_2)^{\xi}+s_0u_1(t)\\
&=[\nu_1(s^*\delta_1\delta_2U_1U_2V_1V_2)^{\xi}+s_0]u_1(t)\\
&\ge (s^*+s_0)u_1(t),
\end{aligned}$$

which contradicts the definition of s^*, and so (3.24) holds. It follows from Lemma 1.3 that $i(W,\Omega_{\varepsilon_2},P_1)=0$.

On the other hand, by $\overline{f}_{0,\eta}\in[0,+\infty)$ with $\eta>1$, there exist $\varepsilon_3>0$ and $\nu_2>0$ such that $0\le f(t,x)\le \nu_2x^{\eta},\ t\in\mathbf{Z}[a+1,b+1],\ 0\le x\le\varepsilon_3$. Choose ε_4 such that

$$0<\varepsilon_4<\min\left\{\varepsilon_2,\varepsilon_3,[\nu_2(1-L)C_1C_2V_1V_2]^{-\frac{1}{\eta-1}}\right\},$$

and let $\Omega_{\varepsilon_4}=\{u\in P_1:\|u\|_\infty\le\varepsilon_4\}$. We next show $Wu\ne su,\ \forall u\in\partial\Omega_{\varepsilon_4},\ s\ge 1$. In fact, if there exist $u_3\in\partial\Omega_{\varepsilon_4}$ and $s_1\ge1$ such that $Wu_3=s_1u_3$, then by

(3.7) and (3.10), we obtain that for $t \in \mathbf{Z}[a+1, b+1]$,

$$
\begin{aligned}
(Wu_3)(t) &= (S\mathbf{fp}u_3)(t) \\
&\leq (1-L)^{-1}\|T\mathbf{fp}u_3\|_\infty \\
&\leq \nu_2(1-L)^{-1}C_1C_2V_2 \times \left(A_1(u_3^\eta) + B_1(u_3^\eta) + \sum_{s=a+1}^{b+1} G_1(s,s)(u_3^\eta)(s)\right) \\
&\leq \nu_2(1-L)^{-1}C_1C_2V_1V_2\varepsilon_4^\eta.
\end{aligned}
$$

Then, $\varepsilon_4 \leq s_1\varepsilon_4 = s_1\|u_3\|_\infty = \|Wu_3\|_\infty \leq \nu_2(1-L)^{-1}C_1C_2V_1V_2\varepsilon_4^\eta$. That is,

$$\varepsilon_4 \geq [\nu_2(1-L)C_1C_2V_1V_2]^{-\frac{1}{\eta-1}},$$

which contradicts the choice of ε_4. By Lemma 1.1, we have $i(W, \Omega_{\varepsilon_4}, P_1) = 1$. Then we have $i(W, \Omega_{\varepsilon_2} \setminus \Omega_{\varepsilon_4}, P_1) = -1$. Hence W has at least one fixed point in $\Omega_{\varepsilon_2} \setminus \Omega_{\varepsilon_4}$, which means BVP (1.2) has at least positive solution. This completes the proof of (i).

The proof of (ii) is similar and will be omitted here. □

References

[1] D.R. Anderson and F. Minhós, A discrete fourth-order Lidstone problem with parameters, Appl. Math. Comput. 214 (2009) 523-533.

[2] R.P. Agarwal, Difference Equations and Inequalities: Theory, Methods, and Applications, second edition, Marcel Dekker, New York, 2000.

[3] R.P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, Nonlinear Anal. 58 (2004) 69-73.

[4] A. Cabada and N.D. Dimitrov, Multiplicity results for nonlinear periodic fourth order difference equations with parameter dependence and singularities, J. Math. Anal. Appl. 371 (2010) 518-533.

[5] G.Q. Chai, Existence of positive solutions for fourth-order boundary value problem with variable parameters, Nonlinear Anal. 66 (2007) 870-880.

[6] C.P. Gupta, Solvability of a three-point nonlocal boundary value problem for a second order ordinary differential equation, J. Math. Anal. Appl. 168 (1992) 540-551.

[7] D.J. Guo and V. Lakskmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.

[8] Z.M. He and J.S. Yu, On the existence of positive solutions of fourth-order difference equations, Appl. Math. Comput. 161 (2005) 139-148.

[9] V.A. Il'in and E. I. Moiseev, Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, Differen. Equat. 23 1987 803-810.

[10] Y.X. Li, Positive solutions of fourth-order boundary value problems with two parameters, J. Math. Anal. Appl. 281 (2003) 477-484.

[11] H.L. Ma, Positive solution for m-point boundary value problems of fourth-order, J. Math. Anal. Appl. 321 (2006) 37-49.

[12] R.Y. Ma and B. Thompson, Positive solutions for nonlinear m-point eigenvalue problems, J. Math. Anal. Appl. 297 (2004) 24-37.

[13] D.B. Wang and W. Guan, Three positive solutions of boundary value problems for p-Laplacian difference equations, Comput. Math. Appl. 55 (2008) 1943-1949.

[14] P.J.Y. Wong and L. Xie, Three symmetric solutions of Lidstone boundary value problems for difference and partial difference equations, Comput. Math. Appl. 45 (2003) 1445-1460.

[15] J.B. Yang and Z.L. Wei, Existence of positive solutions for fourth-order m-point boundary value problems with a one-dimensional p-Laplacian operator, Nonlinear Anal. 71 (2009) 2985-2996.

[16] J.S. Yu and Z.M. Guo, On boundary value problems for a discrete generalized Emden-Fowler equation, J. Differential Equations, 231 (2006) 18-31.

[17] G. Zhang and Z.L. Yang, Positive solutions of a general discrete boundary value problem, J. Math. Anal. Appl. 339 (2008) 469-481.

[18] B.G. Zhang, L.J. Kong, Y.J. Sun and X.H. Deng, Existence of positive solutions for BVPs of fourth-order difference equations, Appl. Math. Comput. 131 (2002) 583-591.

[19] M.C. Zhang and Z.L. Wei, Existence of positive solutions for fourth-order m-point boundary value problem with variable parameters, Appl. Math. Comput. 190 (2007) 1417-1431.