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QUANTITATIVE APPROXIMATION BY FRACTIONAL SMOOTH GAUSS WEIERSTRASS SINGULAR OPERATORS

George A. Anastassiou[1] and Razvan A. Mezei[2]

[1]Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152 U.S.A.
e-mail: ganastss@memphis.edu

[2]Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152 U.S.A.
e-mail: rmezei@memphis.edu

Abstract. In this article we study the very general fractional smooth Gauss Wierstrass singular integral operators on the real line, regarding their convergence to the unit operator with fractional rates in the uniform norm. The related established inequalities involve the higher order module of smoothness of the associated right and left Caputo fractional derivatives of the engaged function. Furthermore we produce a fractional Voronovskaya type of result giving the fractional asymptotic expansion of the basic error of our approximation. We finish with applications. Our operators are not in general positive. We are mainly motivated by [2].

1. Background

We mention

Definition 1.1. Let $\nu \geq 0, n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number, $\lfloor \cdot \rfloor$ the integral part), $f \in C^n(\mathbb{R})$. We call left Caputo fractional derivative the function

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$$D_{*x_0}^{\nu}f(x)=\frac{1}{\Gamma(n-\nu)}\int_{x_0}^{x}(x-t)^{n-\nu-1}f^{(n)}(t)dt, \tag{1.1}$$

$\forall x\geq x_0\in\mathbb{R}$ fixed, where Γ is the gamma function $\Gamma(\nu)=\int_0^\infty e^{-t}t^{\nu-1}dt$, $\nu>0$.

We set $D_{*x_0}^{0}f(x)=f(x),\forall x\geq x_0$. We assume $D_{*x_0}^{\nu}f(x)=0$, for $x<x_0$.

We need

Lemma 1.2. **([2])** Let $\nu>0,\nu\notin\mathbb{N},n=\lceil\nu\rceil$, $f\in C^n(\mathbb{R})$, $\left\|f^{(n)}\right\|_\infty<\infty$, $x_0\in\mathbb{R}$ fixed. Then $D_{*x_0}^{\nu}f(x_0)=0$.

We need the following left Caputo fractional Taylor formula.

Theorem 1.3. **([1],[3])** Let $f\in C^m(\mathbb{R}),m=\lceil\alpha\rceil,\alpha>0$. Then

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-\zeta)^{\alpha-1}D_{*x_0}^{\alpha}f(\zeta)d\zeta, \tag{1.2}$$

$\forall x\in\mathbb{R}:x\geq x_0$.

We also mention

Definition 1.4. **([5], [6])** Let $f\in C^m(\mathbb{R})$, $\alpha>0$, $m=\lceil\alpha\rceil$. The right Caputo fractional derivative of order $\alpha>0$ is given by

$$D_{x_0-}^{\alpha}f(x)=\frac{(-1)^m}{\Gamma(m-\alpha)}\int_{x}^{x_0}(\zeta-x)^{m-\alpha-1}f^{(m)}(\zeta)d\zeta, \tag{1.3}$$

$\forall x\leq x_0\in\mathbb{R}$ fixed.

We assume $D_{x_0-}^{\alpha}f(x)=0,\forall x>x_0$.

We need

Lemma 1.5. **([2])** Let $\alpha>0,\alpha\notin\mathbb{N}$, $m=\lceil\alpha\rceil$, $f\in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_\infty<\infty$, $x_0\in\mathbb{R}$ fixed. Then $D_{x_0-}^{\alpha}f(x_0)=0$.

We need the following right Caputo fractional Taylor formula.

Theorem 1.6. **([1],[4])** Let $f\in C^m(\mathbb{R}),m=\lceil\alpha\rceil,\alpha>0$. Then

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{\Gamma(\alpha)}\int_{x}^{x_0}(\zeta-x)^{\alpha-1}D_{x_0-}^{\alpha}f(\zeta)d\zeta, \tag{1.4}$$

$\forall x\leq x_0$.

We further need

Theorem 1.7. (**[2]**) Let $g \in C_b(\mathbb{R})$ (continuous and bounded), $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define

$$L(x, x_0) = \int_{x_0}^{x} (x - t)^{c-1} g(t) dt, \text{ for } x \geq x_0,$$

and $L(x, x_0) = 0$, for $x < x_0$. Then L is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

Theorem 1.8. (**[2]**) Let $g \in C_b(\mathbb{R})$, $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define

$$K(x, x_0) = \int_{x}^{x_0} (\zeta - x)^{c-1} g(\zeta) d\zeta, \text{ for } x \leq x_0,$$

and $K(x, x_0) = 0$, for $x > x_0$. Then $K(x, x_0)$ is jointly continuous from $\mathbb{R}^2$ into $\mathbb{R}$.

Based on Theorems 1.7, 1.8 we get

Proposition 1.9. (**[2]**) Let $f \in C^m(\mathbb{R})$, with $\left\| f^{(m)} \right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $x, x_0 \in \mathbb{R}$. Then $D_{*x_0}^{\alpha} f(x), D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.

We need

Definition 1.10. Let $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $r \in \mathbb{N}$, $x, x_0 \in \mathbb{R}$. We define the difference

$$\left(\Delta_w^r \left(D_{*x_0}^{\alpha} f\right)\right)(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{*x_0}^{\alpha} f\right)(x + jw), \tag{1.5}$$

$\forall w \in \mathbb{R}$, and the rth modulus of smoothness,

$$\omega_r \left(D_{*x_0}^{\alpha} f, h\right) := \sup_{|t| \leq h} \left\| \left(\Delta_t^r \left(D_{*x_0}^{\alpha} f\right)\right)(x) \right\|_{\infty, x, \mathbb{R}}. \tag{1.6}$$

Notice that

$$\begin{aligned} \left| \left(\Delta_w^r \left(D_{*x_0}^{\alpha} f\right)\right)(x_0) \right| &\leq \left\| \left(\Delta_w^r \left(D_{*x_0}^{\alpha} f\right)\right)(x) \right\|_{\infty, x, \mathbb{R}} \\ &\leq \omega_r \left(D_{*x_0}^{\alpha} f, |w|\right). \end{aligned} \tag{1.7}$$

Similarly, we define the difference

$$\left(\Delta_w^r \left(D_{x_0-}^{\alpha} f\right)\right)(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \left(D_{x_0-}^{\alpha} f\right)(x + jw), \tag{1.8}$$

$\forall w \in \mathbb{R}$, and the rth modulus of smoothness,

$$\omega_r \left(D_{x_0-}^{\alpha} f, h\right) := \sup_{|t| \leq h} \left\| \left(\Delta_t^r \left(D_{x_0-}^{\alpha} f\right)\right)(x) \right\|_{\infty, x, \mathbb{R}}. \tag{1.9}$$

Notice again that

$$\begin{aligned} \left|\left(\Delta_w^r\left(D_{x_0-}^\alpha f\right)\right)(x_0)\right| &\leq \left\|\left(\Delta_w^r\left(D_{x_0-}^\alpha f\right)\right)(x)\right\|_{\infty,x,\mathbb{R}} \\ &\leq \omega_r\left(D_{x_0-}^\alpha f,|w|\right). \end{aligned} \tag{1.10}$$

As a related result we mention

Proposition 1.11. **([2])** Let $f:\mathbb{R}^2\to\mathbb{R}$ be jointly continuous. Consider

$$G(x)=\omega_r\left(f(\cdot,x),\delta\right)_{[x,+\infty)},\ \delta>0,x\in\mathbb{R}.$$

Then G is continuous on $\mathbb{R}$. Here ω_r is defined over $[x,+\infty)$ instead of $\mathbb{R}$.

Proposition 1.12. **([2])** Let $f:\mathbb{R}^2\to\mathbb{R}$ be jointly continuous. Consider

$$H(x)=\omega_r\left(f(\cdot,x),\delta\right)_{(-\infty,x]},\ \delta>0,x\in\mathbb{R}.$$

Then H is continuous on $\mathbb{R}$. (Here ω_r is defined over $(-\infty,x]$ instead of $\mathbb{R}$.)

From Propositions 1.9, 1.11, 1.12 we derive

Proposition 1.13. **([2])** Let $f\in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_\infty<\infty$, $m=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $\alpha>0$, $r\in\mathbb{N}$, $x\in\mathbb{R}$. Then $\omega_r\left(D_{*x}^\alpha f,h\right)_{[x,+\infty)},\omega_r\left(D_{x-}^\alpha f,h\right)_{(-\infty,x]}$ are continuous functions of $x\in\mathbb{R}$, $h>0$ fixed.
We make

Remark 1.14. **([2])** Let g continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then we know that

$$\omega_r(g,t)\leq 2^r\|g\|_\infty<\infty.$$

Assuming that $\left(D_{*x}^\alpha f\right)(t),\left(D_{x-}^\alpha f\right)(t)$, are both continuous and bounded in $(x,t)\in\mathbb{R}^2$, i.e.

$$\begin{aligned} \left\|D_{*x}^\alpha f\right\|_\infty &\leq K_1,\forall x\in\mathbb{R}; \\ \left\|D_{x-}^\alpha f\right\|_\infty &\leq K_2,\forall x\in\mathbb{R}, \end{aligned}$$

where $K_1,K_2>0$, we get

$$\begin{aligned} \omega_r\left(D_{*x}^\alpha f,\xi\right) &\leq 2^rK_1; \\ \omega_r\left(D_{x-}^\alpha f,\xi\right) &\leq 2^rK_2,\forall\xi\geq 0, \end{aligned}$$

for each $x\in\mathbb{R}$. Therefore, for any $\xi\geq 0$,

$$\sup_{x\in\mathbb{R}}\left[\max\left(\omega_r\left(D_{*x}^\alpha f,\xi\right),\omega_r\left(D_{x-}^\alpha f,\xi\right)\right)\right]\leq 2^r\max(K_1,K_2)<\infty. \tag{1.11}$$

So in our setting for $f\in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_\infty<\infty$, $m=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $\alpha>0$, by Proposition 1.9, both $\left(D_{*x}^\alpha f\right)(t),\left(D_{x-}^\alpha f\right)(t)$ are jointly continuous in (t,x) on $\mathbb{R}^2$. Assuming further that they are both bounded on $\mathbb{R}^2$ we get (1.11) valid. In particular, each of $\omega_r\left(D_{*x}^\alpha f,\xi\right),\omega_r\left(D_{x-}^\alpha f,\xi\right)$ is finite for any $\xi\geq 0$.

We need

Remark 1.15. **([2])** Again let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$; $f^{(m)}(x) = 1, \forall x \in \mathbb{R}$; $x_0 \in \mathbb{R}$. Notice $0 < m - \alpha < 1$. Then

$$D^{\alpha}_{*x_0} f(x) = \frac{(x - x_0)^{m-\alpha}}{\Gamma(m - \alpha + 1)}, \forall x \geq x_0.$$

Let us consider $x, y \geq x_0$. Then

$$\left| D^{\alpha}_{*x_0} f(x) - D^{\alpha}_{*x_0} f(y) \right| \leq \frac{|x - y|^{m-\alpha}}{\Gamma(m - \alpha + 1)}.$$

So it is not strange to assume that

$$\left| D^{\alpha}_{*x_0} f(x_1) - D^{\alpha}_{*x_0} f(x_2) \right| \leq K |x_1 - x_2|^{\beta}, \tag{1.12}$$

$K > 0$, $0 < \beta \leq 1$, $\forall x_1, x_2 \in \mathbb{R}$, any $x_0 \in \mathbb{R}$, here more generally $\left\| f^{(m)} \right\|_{\infty} < \infty$.

In general, one may assume

$$\begin{aligned} \omega_r\left(D^{\alpha}_{x-} f, \xi\right) &\leq M_1 \xi^{r-1+\beta_1}, \text{ and} \\ \omega_r\left(D^{\alpha}_{*x} f, \xi\right) &\leq M_2 \xi^{r-1+\beta_2}, \end{aligned} \tag{1.13}$$

where $0 < \beta_1, \beta_2 \leq 1$, $\forall \xi > 0$, $r \in \mathbb{N}$; $M_1, M_2 > 0$; any $x \in \mathbb{R}$.

Setting $\beta = \min(\beta_1, \beta_2)$ and $M = \max(M_1, M_2)$, in that case we obtain

$$\sup_{x \in \mathbb{R}} \left\{ \max\left(\omega_r\left(D^{\alpha}_{x-} f, \xi\right), \omega_r\left(D^{\alpha}_{*x} f, \xi\right) \right) \right\} \leq M \xi^{r-1+\beta} \to 0, \text{ as } \xi \to 0+. \tag{1.14}$$

2. Main Results

We need

Definition 2.1. **([2])** Let $r \in \mathbb{N}$, $\alpha > 0$. We mention the numbers

$$\alpha_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-\alpha}, & j = 1, \ldots, r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-\alpha}, & j = 0, \end{cases} \tag{2.1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$.

Also denote

$$\delta_k = \sum_{j=1}^{r} \alpha_j j^k, k = 1, \ldots, m - 1, \tag{2.2}$$

where $m = \lceil \alpha \rceil$.

We give

Theorem 2.2. Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\left\| f^{(m)} \right\|_{\infty} < \infty$, $x_0 \in \mathbb{R}$ fixed, $\xi > 0$. Then

i) if $t \geq 0$ we get

$$\begin{aligned} A &:= A(t,x_0) := \sum_{j=0}^{r} \alpha_j \left[f(x_0 + jt) - f(x_0) \right] - \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \left(\Delta_w^r \left(D_{*x_0}^{\alpha} f \right) \right)(x_0)\, dw, \end{aligned} \tag{2.3}$$

and

$$|A| \leq \omega_r \left(D_{*x_0}^{\alpha} f, \sqrt{\xi} \right) \left(\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{t^{\alpha+k}}{\xi^{k/2} \Gamma(\alpha+k+1)} \right). \tag{2.4}$$

ii) if $t < 0$ we obtain

$$\begin{aligned} B &:= B(t,x_0) := \sum_{j=0}^{r} \alpha_j \left[f(x_0 + jt) - f(x_0) \right] - \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} \delta_k t^k \quad (2.5) \\ &= \frac{1}{\Gamma(\alpha)} \int_t^0 (w-t)^{\alpha-1} \left(\Delta_w^r \left(D_{x_0-}^{\alpha} f \right) \right)(x_0)\, dw, \end{aligned}$$

and

$$|B| \leq \omega_r \left(D_{x_0-}^{\alpha} f, \sqrt{\xi} \right) \left(\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{|t|^{\alpha+k}}{\xi^{k/2} \Gamma(\alpha+k+1)} \right). \tag{2.6}$$

Proof. As in the proof of Theorem 2.2 of [2] by taking $\sqrt{\xi}$ in place of ξ. □

In the next, let $\xi > 0$, $x, x_0 \in \mathbb{R}$, $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, with $\left\| f^{(m)} \right\|_{\infty} < \infty$.

Consider the Lebesgue integral

$$W_{r,\xi}(f,x) = \frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) e^{-t^2/\xi} dt. \tag{2.7}$$

We assume $W_{r,\xi}(f,x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$. Notice that

$$\frac{1}{\sqrt{\pi \xi}} \int_{-\infty}^{\infty} e^{-t^2/\xi} dt = 1, \tag{2.8}$$

$$W_{r,\xi}(c,x) = c, c \text{ constant}, \tag{2.9}$$

and

$$\begin{aligned} W_{r,\xi}(f,x_0)-f(x_0) &= \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)e^{-t^2/\xi}dt\right) \\ &= \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{0}\left(\sum_{j=1}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)e^{-t^2/\xi}dt\right) \\ &\quad +\frac{1}{\sqrt{\pi\xi}}\int_{0}^{\infty}\left(\sum_{j=1}^{r}\alpha_j\left(f(x_0+jt)-f(x_0)\right)e^{-t^2/\xi}dt\right) \\ &=: \Lambda. \end{aligned} \tag{2.10}$$

We have

$$\int_0^{\infty} t^k e^{-t^2/\xi}dt=\frac{1}{2}\Gamma\left(\frac{k+1}{2}\right)\xi^{\frac{k+1}{2}},\ \text{for any } k>-1. \tag{2.11}$$

We present

Theorem 2.3. Let $f\in C^m(\mathbb{R})$, $m=\lceil\alpha\rceil$, $\alpha>0$, with $\left\|f^{(m)}\right\|_{\infty}<\infty$, $\xi>0$, $x_0\in\mathbb{R}$. Then

1)

$$\begin{aligned} &\left|W_{r,\xi}(f,x_0)-f(x_0)-\sum_{\rho=1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}f^{(2\rho)}(x_0)\frac{\delta_{2\rho}}{\rho!}\left(\frac{1}{4}\right)^{\rho}\xi^{\rho}\right| \\ \leq\ &\frac{r!}{\sqrt{\pi}}\left[\sum_{k=0}^{r}\frac{1}{(r-k)!}\frac{\Gamma\left(\frac{\alpha+k+1}{2}\right)}{\Gamma(\alpha+k+1)}\right]\xi^{\frac{\alpha}{2}} \\ &\times\max\left\{\omega_r\left(D_{x_0-}^{\alpha}f,\sqrt{\xi}\right),\omega_r\left(D_{*x_0}^{\alpha}f,\sqrt{\xi}\right)\right\}. \end{aligned} \tag{2.12}$$

(Above if $m=1,2$ the sum disappears).

2)

$$\begin{aligned} &\left\|W_{r,\xi}(f,\cdot)-f(\cdot)-\sum_{\rho=1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}f^{(2\rho)}(\cdot)\frac{\delta_{2\rho}}{\rho!}\left(\frac{1}{4}\right)^{\rho}\xi^{\rho}\right\|_{\infty} \\ \leq\ &\frac{r!}{\sqrt{\pi}}\left[\sum_{k=0}^{r}\frac{1}{(r-k)!}\frac{\Gamma\left(\frac{\alpha+k+1}{2}\right)}{\Gamma(\alpha+k+1)}\right]\xi^{\frac{\alpha}{2}} \\ &\times\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\alpha}f,\sqrt{\xi}\right),\omega_r\left(D_{*x}^{\alpha}f,\sqrt{\xi}\right)\right]\right\}. \end{aligned} \tag{2.13}$$

We further give

Theorem 2.4. All as in Theorem 2.3. Additionally assume that $\left\|f^{(2\rho)}\right\|_{\infty} < \infty$, $\rho = 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor$. Then

$$\begin{aligned} \|W_{r,\xi}(f,\cdot) - f(\cdot)\|_{\infty} &\leq \sum_{\rho=1}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \left\|f^{(2\rho)}\right\|_{\infty} \frac{|\delta_{2\rho}|}{\rho!} \left(\frac{1}{4}\right)^{\rho} \xi^{\rho} \\ &+ \frac{r!}{\sqrt{\pi}} \left[\sum_{k=0}^{r} \frac{1}{(r-k)!} \frac{\Gamma\left(\frac{\alpha+k+1}{2}\right)}{\Gamma(\alpha+k+1)}\right] \xi^{\frac{\alpha}{2}} \qquad (2.14) \\ &\times \sup_{x\in\mathbb{R}} \left\{ \max\left[\omega_r\left(D_{x-}^{\alpha} f, \sqrt{\xi}\right), \omega_r\left(D_{*x}^{\alpha} f, \sqrt{\xi}\right)\right]\right\}. \end{aligned}$$

Assuming further that both $\left(D_{*x}^{\alpha} f\right)(t)$, $\left(D_{x-}^{\alpha} f\right)(t)$ are bounded in $(t,x) \in \mathbb{R}^2$, we get, as $\xi \to 0+$, that $W_{r,\xi} \overset{u}{\to} I$ (uniformly), see (1.11).

Or, by assuming (1.13) we get (1.14), that is from (2.14) we obtain again $W_{r,\xi} \overset{u}{\to} I$ (unit operator), as $\xi \to 0+$.

Proof of Theorem 2.3. We use here Theorem 2.2. First, from (1.2) and (1.4), we observe that

$$\begin{aligned} f(x_0 + jt) &= \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k \\ &+ \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x_0+jt} (x_0 + jt - \zeta)^{\alpha-1} D_{*x_0}^{\alpha} f(\zeta) d\zeta, \\ f(x_0 + jt) - f(x_0) &= \sum_{k=1}^{m-1} \frac{f^{(k)}(x_0)}{k!} j^k t^k \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} D_{*x_0}^{\alpha} f(wj + x_0) j^{\alpha} dw \end{aligned}$$

and

$$\begin{aligned} f(x_0 + jt) &= \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x_0 + jt - x_0)^k \\ &+ \frac{1}{\Gamma(\alpha)} \int_{x_0+jt}^{x_0} (\zeta - x_0 - jt)^{\alpha-1} D_{x_0-}^{\alpha} f(\zeta) d\zeta \end{aligned}$$

$$f(x_0+jt)-f(x_0) = \sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}j^k t^k + \frac{1}{\Gamma(\alpha)}\int_t^0 (w-t)^{\alpha-1} D_{x_0-}^{\alpha} f(wj+x_0) j^{\alpha} dw.$$

Therefore (see (2.10)), from Lemma 1.2 and Lemma 1.5 we have

$$\begin{aligned}
\Lambda &= \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{0}\left(\sum_{j=1}^{r}\alpha_j\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}j^k t^k\right.\right. \\
&\quad \left.\left.+\frac{1}{\Gamma(\alpha)}\int_t^0 (w-t)^{\alpha-1} D_{x_0-}^{\alpha} f(wj+x_0) j^{\alpha} dw\right) e^{-t^2/\xi}dt\right) \\
&\quad +\frac{1}{\sqrt{\pi\xi}}\int_{0}^{\infty}\left(\sum_{j=1}^{r}\alpha_j\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}j^k t^k\right.\right. \\
&\quad \left.\left.+\frac{1}{\Gamma(\alpha)}\int_0^t (t-w)^{\alpha-1} D_{*x_0}^{\alpha} f(wj+x_0) j^{\alpha} dw\right) e^{-t^2/\xi}dt\right) \\
&= \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{0}\left(\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\sum_{j=1}^{r}\alpha_j j^k\right) t^k\right.\right. \\
&\quad \left.\left.+\frac{1}{\Gamma(\alpha)}\int_t^0 (w-t)^{\alpha-1}\left(\sum_{j=1}^{r}\alpha_j D_{x_0-}^{\alpha} f(wj+x_0) j^{\alpha}\right) dw\right) e^{-t^2/\xi}dt\right) \\
&\quad +\frac{1}{\sqrt{\pi\xi}}\int_{0}^{\infty}\left(\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\sum_{j=1}^{r}\alpha_j j^k\right) t^k\right.\right. \\
&\quad \left.\left.+\frac{1}{\Gamma(\alpha)}\int_0^t (t-w)^{\alpha-1}\left(\sum_{j=1}^{r}\alpha_j D_{*x_0}^{\alpha} f(wj+x_0) j^{\alpha}\right) dw\right) e^{-t^2/\xi}dt\right) \\
&= \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{0}\left(\left[\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\delta_k t^k\right] e^{-t^2/\xi}\right. \\
&\quad \left.+\left[\frac{e^{-t^2/\xi}}{\Gamma(\alpha)}\int_t^0 (w-t)^{\alpha-1}\left(\Delta_w^r\left(D_{x_0-}^{\alpha} f\right)\right)(x_0)\, dw\right]\right) dt \\
&\quad +\frac{1}{\sqrt{\pi\xi}}\int_{0}^{\infty}\left(\left[\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\delta_k t^k\right] e^{-t^2/\xi}\right.
\end{aligned}$$

$$+\left[\frac{e^{-t^2/\xi}}{\Gamma(\alpha)}\int_0^t (t-w)^{\alpha-1}\left(\Delta_w^r\left(D_{*x_0}^\alpha f\right)\right)(x_0)\,dw\right]\Bigg)\,dt$$

$$= \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{m-1}\frac{f^{(k)}(x_0)}{k!}\delta_k t^k\right]e^{-t^2/\xi}dt$$
$$+\frac{1}{\sqrt{\pi\xi}}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{0}\left[e^{-t^2/\xi}\int_t^0 (w-t)^{\alpha-1}\left(\Delta_w^r\left(D_{x_0-}^\alpha f\right)\right)(x_0)\,dw\right]dt$$
$$+\frac{1}{\sqrt{\pi\xi}}\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\left[e^{-t^2/\xi}\int_0^t (t-w)^{\alpha-1}\left(\Delta_w^r\left(D_{*x_0}^\alpha f\right)\right)(x_0)\,dw\right]dt$$

$$= \frac{1}{\sqrt{\pi\xi}}\sum_{\rho=1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{f^{(2\rho)}(x_0)\delta_{2\rho}}{(2\rho)!}\int_{-\infty}^{\infty}t^{2\rho}e^{-t^2/\xi}dt$$
$$+\frac{1}{\sqrt{\pi\xi}}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{0}\left[e^{-t^2/\xi}\int_t^0 (w-t)^{\alpha-1}\left(\Delta_w^r\left(D_{x_0-}^\alpha f\right)\right)(x_0)\,dw\right]dt$$
$$+\frac{1}{\sqrt{\pi\xi}}\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\left[e^{-t^2/\xi}\int_0^t (t-w)^{\alpha-1}\left(\Delta_w^r\left(D_{*x_0}^\alpha f\right)\right)(x_0)\,dw\right]dt$$

$$= \frac{1}{\sqrt{\pi\xi}}\sum_{\rho=1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{f^{(2\rho)}(x_0)\delta_{2\rho}}{(2\rho)!}\Gamma\left(\frac{2\rho+1}{2}\right)\xi^{\frac{2\rho+1}{2}}$$
$$+\frac{1}{\sqrt{\pi\xi}}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{0}\left[e^{-t^2/\xi}\int_t^0 (w-t)^{\alpha-1}\left(\Delta_w^r\left(D_{x_0-}^\alpha f\right)\right)(x_0)\,dw\right]dt$$
$$+\frac{1}{\sqrt{\pi\xi}}\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\left[e^{-t^2/\xi}\int_0^t (t-w)^{\alpha-1}\left(\Delta_w^r\left(D_{*x_0}^\alpha f\right)\right)(x_0)\,dw\right]dt$$

$$= \sum_{\rho=1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}f^{(2\rho)}(x_0)\frac{\delta_{2\rho}}{\rho!}\left(\frac{1}{4}\right)^{\rho}\xi^{\rho}$$
$$+\frac{1}{\sqrt{\pi\xi}}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{0}\left[e^{-t^2/\xi}\int_t^0 (w-t)^{\alpha-1}\left(\Delta_w^r\left(D_{x_0-}^\alpha f\right)\right)(x_0)\,dw\right]dt$$
$$+\frac{1}{\sqrt{\pi\xi}}\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\left[e^{-t^2/\xi}\int_0^t (t-w)^{\alpha-1}\left(\Delta_w^r\left(D_{*x_0}^\alpha f\right)\right)(x_0)\,dw\right]dt.$$

Therefore

$$\theta(x_0) \quad := \quad W_{r,\xi}(f,x_0) - f(x_0) - \sum_{\rho=1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}f^{(2\rho)}(x_0)\frac{\delta_{2\rho}}{\rho!}\left(\frac{1}{4}\right)^{\rho}\xi^{\rho}$$

$$= \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{0} e^{-t^2/\xi} \left[\frac{1}{\Gamma(\alpha)} \int_t^0 (w-t)^{\alpha-1} \left(\Delta_w^r \left(D_{x_0-}^{\alpha} f \right) \right)(x_0)\, dw \right] dt$$

$$+ \frac{1}{\sqrt{\pi\xi}} \int_0^{\infty} e^{-t^2/\xi} \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} \left(\Delta_w^r \left(D_{*x_0}^{\alpha} f \right) \right)(x_0)\, dw \right] dt.$$

From (2.3) and (2.5), we get

$$\theta(x_0) = \frac{1}{\sqrt{\pi\xi}} \left[\int_{-\infty}^{0} e^{-t^2/\xi} B(t, x_0)\, dt + \int_0^{+\infty} e^{-t^2/\xi} A(t, x_0)\, dt \right].$$

Consequently we derive (see Theorem 2.2)

$$|\theta(x_0)|$$

$$\leq \frac{1}{\sqrt{\pi\xi}} \left[\int_{-\infty}^{0} e^{-t^2/\xi} |B(t, x_0)|\, dt + \int_0^{+\infty} e^{-t^2/\xi} |A(t, x_0)|\, dt \right]$$

$$\leq \frac{1}{\sqrt{\pi\xi}} \left[\left(\int_{-\infty}^{0} e^{-t^2/\xi} \left(\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{|t|^{\alpha+k}}{\xi^{k/2}\Gamma(\alpha+k+1)} \right) dt \right) \omega_r \left(D_{x_0-}^{\alpha} f, \sqrt{\xi} \right) \right.$$

$$\left. + \left(\int_0^{+\infty} e^{-t^2/\xi} \left(\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{t^{\alpha+k}}{\xi^{k/2}\Gamma(\alpha+k+1)} \right) dt \right) \omega_r \left(D_{*x_0}^{\alpha} f, \sqrt{\xi} \right) \right]$$

$$\leq \frac{1}{\sqrt{\pi\xi}} \mathcal{M}(x_0) \left[\int_{-\infty}^{\infty} e^{-t^2/\xi} \left(\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{|t|^{\alpha+k}}{\xi^{k/2}\Gamma(\alpha+k+1)} \right) dt \right]$$

$$= \frac{1}{\sqrt{\pi\xi}} \mathcal{M}(x_0) \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\alpha+k+1)\,\xi^{k/2}} \int_{-\infty}^{\infty} e^{-t^2/\xi} |t|^{\alpha+k}\, dt \right]$$

$$= \frac{1}{\sqrt{\pi}} \mathcal{M}(x_0) \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\alpha+k+1)\,\xi^{k/2}} \Gamma\left(\frac{\alpha+k+1}{2} \right) \xi^{\frac{\alpha+k}{2}} \right]$$

$$= \xi^{\frac{\alpha}{2}} \frac{1}{\sqrt{\pi}} \mathcal{M}(x_0) \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!} \frac{\Gamma\left(\frac{\alpha+k+1}{2}\right)}{\Gamma(\alpha+k+1)} \right].$$

where,

$$\text{Call } \mathcal{M}(x_0) := \max\left\{ \omega_r \left(D_{x_0-}^{\alpha} f, \sqrt{\xi} \right), \omega_r \left(D_{*x_0}^{\alpha} f, \sqrt{\xi} \right) \right\}.$$

We got

$$|\theta(x_0)| \leq \frac{r!}{\sqrt{\pi}} \left[\sum_{k=0}^{r} \frac{1}{(r-k)!} \frac{\Gamma\left(\frac{\alpha+k+1}{2}\right)}{\Gamma(\alpha+k+1)} \right] \xi^{\frac{\alpha}{2}} \mathcal{M}(x_0),$$

that is proving (2.12). □

Next we present a Voronovskaya type result regarding fractional singular integral operators.

Theorem 2.5. Here $f \in C^m(\mathbb{R})$, $m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\left\| f^{(m)} \right\|_\infty < \infty$, and $\left\| D^\alpha_{x-} f(y) \right\|_\infty \leq M_1, \left\| D^\alpha_{*x} f(y) \right\|_\infty \leq M_2$, where $M_1, M_2 > 0$, for any $x, y \in \mathbb{R}$. Then

$$W_{r,\xi}(f,x) - f(x) - \sum_{\rho=1}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \left[f^{(2\rho)}(x)\, \delta_{2\rho} \frac{1}{\rho!} \left(\frac{1}{4} \right)^\rho \xi^\rho \right] = o\left(\xi^{\frac{\alpha-\beta}{2}} \right), \qquad (2.15)$$

$0 < \beta < \alpha$, as $\xi \to 0+$. That is,

$$W_{r,\xi}(f,x) - f(x) = \sum_{\rho=1}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \left[f^{(2\rho)}(x) \left(\sum_{j=1}^{r} \alpha_j j^{2\rho} \right) \frac{1}{\rho!} \left(\frac{1}{4} \right)^\rho \xi^\rho \right] + o\left(\xi^{\frac{\alpha-\beta}{2}} \right), \qquad (2.16)$$

where $0 < \beta < \alpha$.

(Above if $m = 1, 2$ the sum disappears.)

Proof. Since $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, by (1.2) and (1.4) we obtain

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{D^\alpha_{*x_0} f(\zeta)}{\Gamma(\alpha+1)} (x - x_0)^\alpha,$$

$\forall x \geq x_0$, here $x_0 < \zeta < x$ and

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{D^\alpha_{x_0-} f(\zeta)}{\Gamma(\alpha+1)} (x_0 - x)^\alpha,$$

$\forall x < x_0$, here $x < \zeta < x_0$. So we get $(j = 1, \ldots, r)$

$$f(x + jt) - f(x) = \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} (jt)^k + \frac{D^\alpha_{*x} f(\zeta)}{\Gamma(\alpha+1)} (jt)^\alpha,$$

for $x < \zeta < x + jt$, here $t \geq 0$. Also it holds

$$f(x + jt) - f(x) = \sum_{k=1}^{m-1} \frac{f^{(k)}(x)}{k!} (jt)^k + \frac{D^\alpha_{x-} f(\zeta)}{\Gamma(\alpha+1)} (jt)^\alpha,$$

for $x+jt<\zeta<x$, here $t<0$. Notice that

$$\begin{aligned}
& W_{r,\xi}(f,x)-f(x) \\
= & \frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}(f(x+jt)-f(x))\,e^{-t^2/\xi}dt\right) \\
= & \frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=0}^{r}\alpha_j\left[\int_{-\infty}^{0}(f(x+jt)-f(x))\,e^{-t^2/\xi}dt\right.\right. \\
& \left.\left.+\int_{0}^{\infty}(f(x+jt)-f(x))\,e^{-t^2/\xi}dt\right]\right) \\
= & \frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=0}^{r}\alpha_j\left[\int_{-\infty}^{0}\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{D_{x-}^{\alpha}f(\zeta)}{\Gamma(\alpha+1)}(jt)^{\alpha}\right)e^{-t^2/\xi}dt\right.\right. \\
& \left.\left.+\int_{0}^{\infty}\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x)}{k!}(jt)^k+\frac{D_{*x}^{\alpha}f(\zeta)}{\Gamma(\alpha+1)}(jt)^{\alpha}\right)e^{-t^2/\xi}dt\right]\right) \\
= & \frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=0}^{r}\alpha_j\left[\left(\sum_{k=1}^{m-1}\frac{f^{(k)}(x)}{k!}j^k\int_{-\infty}^{\infty}t^k e^{-t^2/\xi}dt\right)\right.\right. \\
& \left.\left.+\frac{j^{\alpha}}{\Gamma(\alpha+1)}\left(\int_{-\infty}^{0}t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-t^2/\xi}dt+\int_{0}^{\infty}t^{\alpha}\left(D_{*x}^{\alpha}f(\zeta)\right)e^{-t^2/\xi}dt\right)\right]\right) \\
= & \frac{1}{\sqrt{\pi\xi}}\left(\sum_{j=0}^{r}\alpha_j\left[\left(\sum_{\rho=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{f^{(2\rho)}(x)}{(2\rho)!}j^{2\rho}\Gamma\left(\frac{2\rho+1}{2}\right)\xi^{\frac{2\rho+1}{2}}\right)\right.\right. \\
& \left.\left.+\frac{j^{\alpha}}{\Gamma(\alpha+1)}\left(\int_{-\infty}^{0}t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-t^2/\xi}dt+\int_{0}^{\infty}t^{\alpha}\left(D_{*x}^{\alpha}f(\zeta)\right)e^{-t^2/\xi}dt\right)\right]\right) \\
= & \sum_{\rho=1}^{\lfloor\frac{m-1}{2}\rfloor}f^{(2\rho)}(x)\left(\sum_{j=0}^{r}\alpha_j j^{2\rho}\right)\frac{1}{\rho!}\left(\frac{1}{4}\right)^{\rho}\xi^{\rho} \\
& +\frac{1}{\sqrt{\pi\xi}}\frac{\sum_{j=0}^{r}\alpha_j j^{\alpha}}{\Gamma(\alpha+1)}\left(\int_{-\infty}^{0}t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-t^2/\xi}dt+\int_{0}^{\infty}t^{\alpha}\left(D_{*x}^{\alpha}f(\zeta)\right)e^{-t^2/\xi}dt\right).
\end{aligned}$$

We got

$$T:=W_{r,\xi}(f,x)-f(x)-\sum_{\rho=1}^{\lfloor\frac{m-1}{2}\rfloor}f^{(2\rho)}(x)\,\delta_{2\rho}\frac{1}{\rho!}\left(\frac{1}{4}\right)^{\rho}\xi^{\rho}$$

$$= \frac{\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}}{\sqrt{\pi\xi}\Gamma(\alpha+1)}\left[\int_{-\infty}^{0} t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-t^2/\xi}dt+\int_{0}^{\infty} t^{\alpha}(D_{*x}^{\alpha}f(\zeta))e^{-t^2/\xi}dt\right].$$

We consider

$$\Delta_{\xi} := \frac{1}{\xi^{\alpha/2}}T.$$

Then we have

$$\Delta_{\xi} = \frac{\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}}{\sqrt{\pi\xi^{\alpha+1}}\Gamma(\alpha+1)}\left[\int_{-\infty}^{0} t^{\alpha}\left(D_{x-}^{\alpha}f(\zeta)\right)e^{-t^2/\xi}dt+\int_{0}^{\infty} t^{\alpha}(D_{*x}^{\alpha}f(\zeta))e^{-t^2/\xi}dt\right]$$

$$= \frac{1}{\sqrt{\pi\xi^{\alpha+1}}\Gamma(\alpha+1)}\left[\int_{-\infty}^{0} t^{\alpha}\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left(D_{x-}^{\alpha}f(\zeta)\right)\right)e^{-t^2/\xi}dt\right.$$

$$\left.+\int_{0}^{\infty} t^{\alpha}\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left(D_{*x}^{\alpha}f(\zeta)\right)\right)e^{-t^2/\xi}dt\right].$$

Call

$$\phi_{\alpha}(x,t) = \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left(D_{x-}^{\alpha}f(\zeta)\right),$$

and

$$\psi_{\alpha}(x,t) = \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left(D_{*x}^{\alpha}f(\zeta)\right).$$

Hence

$$\Delta_{\xi} = \frac{1}{\sqrt{\pi\xi^{\alpha+1}}\Gamma(\alpha+1)}\left[\int_{-\infty}^{0} t^{\alpha}\phi_{\alpha}(x,t)e^{-t^2/\xi}dt+\int_{0}^{\infty} t^{\alpha}\psi_{\alpha}(x,t)e^{-t^2/\xi}dt\right].$$

By theorem's assumptions we derive

$$\begin{aligned} |\phi_{\alpha}(x,t)| &\le \left(\sum_{j=1}^{r}\binom{r}{j}\right)M_1 \\ &= (2^r-1)M_1, \\ |\psi_{\alpha}(x,t)| &\le (2^r-1)M_2, \end{aligned}$$

$\forall x,t \in \mathbb{R}$. Call $M_3 = \max(M_1, M_2)$. Hence

$$|\phi_{\alpha}(x,t)|, |\psi_{\alpha}(x,t)| \le (2^r-1)M_3,$$

$\forall x,t \in \mathbb{R}$. Therefore

$$|\Delta_{\xi}| \le \frac{(2^r-1)M_3}{\sqrt{\pi\xi^{\alpha+1}}\Gamma(\alpha+1)}\int_{-\infty}^{\infty}|t|^{\alpha}e^{-t^2/\xi}dt$$

$$= \frac{2(2^r-1)M_3}{\sqrt{\pi\xi^{\alpha+1}}\Gamma(\alpha+1)}\int_0^\infty t^\alpha e^{-t^2/\xi}dt$$

$$= \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\Gamma(\alpha+1)}(2^r-1)M_3.$$

That is

$$|\Delta_\xi| \le \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\Gamma(\alpha+1)}(2^r-1)M_3,$$

and

$$|T| \le \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\Gamma(\alpha+1)}(2^r-1)M_3\xi^{\alpha/2},$$

resulting into $T = O\left(\xi^{\alpha/2}\right)$.

However, let $0<\beta<\alpha$. Then easily we get

$$\frac{|T|}{\xi^{(\alpha-\beta)/2}} \le \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\Gamma(\alpha+1)}(2^r-1)M_3\xi^{\beta/2}\to 0, \text{ as } \xi\to 0+.$$

That is $|T| = o\left(\xi^{\frac{\alpha-\beta}{2}}\right)$, proving the claim. □

3. Applications

Let $\alpha=\frac{1}{2}$, $\lceil\frac{1}{2}\rceil=1$, $f\in C^1(\mathbb{R})$, $\|f'\|_\infty<\infty$, $\xi>0$, $x_0\in\mathbb{R}$. Then by Theorem 2.3, (2.12), we obtain

$$|W_{r,\xi}(f,x_0)-f(x_0)| \le \frac{r!}{\sqrt{\pi}}\left[\sum_{k=0}^r\frac{1}{(r-k)!}\frac{\Gamma\left(\frac{k+1.5}{2}\right)}{\Gamma(k+1.5)}\right]\xi^{\frac{1}{4}} \times\max\left\{\omega_r\left(D_{x_0-}^{\frac{1}{2}}f,\sqrt{\xi}\right),\omega_r\left(D_{*x_0}^{\frac{1}{2}}f,\sqrt{\xi}\right)\right\}. \quad (3.1)$$

Consequently it holds

$$\|W_{r,\xi}(f)-f\|_\infty \le \frac{r!}{\sqrt{\pi}}\left[\sum_{k=0}^r\frac{1}{(r-k)!}\frac{\Gamma\left(\frac{k+1.5}{2}\right)}{\Gamma(k+1.5)}\right]\xi^{\frac{1}{4}} \times\sup_{x\in\mathbb{R}}\left[\max\left\{\omega_r\left(D_{x-}^{\frac{1}{2}}f,\sqrt{\xi}\right),\omega_r\left(D_{*x}^{\frac{1}{2}}f,\sqrt{\xi}\right)\right\}\right]. \quad (3.2)$$

Above we assume $\left(D_{x-}^{\frac{1}{2}}f\right)(y)$, $\left(D_{*x}^{\frac{1}{2}}f\right)(y)$ are bounded in $(x,y)\in\mathbb{R}^2$, for the convergence of $W_{r,\xi}\to I$, as $\xi\to 0+$.

By fractional Voronovskaya type Theorem 2.5, (2.15), under the above assumptions we get

$$W_{r,\xi}(f,x)-f(x) = o\left(\xi^{\frac{1}{4}-\frac{\beta}{2}}\right), \quad (3.3)$$

where $0 < \beta < \frac{1}{2}$.

Note. The integrals $W_{r,\xi}$ are not in general positive operators. Take $f(t) = t^2 \geq 0$, $r = 2$, $\alpha = 2.5$, $x = 0$. Then $\alpha_1 = -2$, $\alpha_2 = 2^{-2.5}$. We find

$$W_{2,\xi}\left(t^2, 0\right) = \left(-1 + \frac{1}{2\sqrt{2}}\right)\xi < 0,$$

proving the claim. $\square$

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