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UNIFICATION ON GENERALIZED (φ, ψ)-WEAK RATIONAL TYPE CONTRACTION MAPPINGS AND FIXED POINTS

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Abstract. In this paper, sufficient conditions for the existence and uniqueness of a fixed point in complete metric spaces under weak rational type contraction condition for two pairs of discontinuous weak compatible mappings are obtained. Also, an example is provided to illustrate the usability of our results. Our results generalize well-known results in the literature.

1. Introduction and preliminaries

The classical Banach contraction principle is an effective tool which assures the existence and uniqueness of fixed points of contraction self mappings on complete metric spaces. Besides offering a constructive procedure to compute the fixed points of the underlying mappings this principle has played a major role in the development of nonlinear analysis. Due to its importance in various branches of mathematics (pure as well as applied mathematics), and in economics, life sciences, physical sciences, engineering, computer science,

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and others, generalizations of Banach's contraction principle have been scrutinized by several authors. Fixed point theorems for various types of nonlinear contractive mappings have been scrutinized extensively by various researchers (see [1]-[16] and references cited therein).

Let (ζ, η) be a metric space. A self mapping $\kappa : \zeta \to \zeta$ is said to be the contraction if there exists $0 \leq \sigma < 1$ such that for all $u_1, u_2 \in \zeta$,

$$\eta(\kappa u_1, \kappa u_2) \leq \sigma \eta(u_1, u_2). \tag{1.1}$$

Any self mapping of a complete metric space satisfying (1.1) will gives the assurity of the existence a unique fixed point.

A question comes over here is that whether we can get contractive conditions which are not continuous but still leads to the existence of fixed point in a complete metic space. Kannan(see [12], [13]) scrutinized the below mentioned result, giving an endorsing reply to the above question.

Theorem 1.1. *If $\kappa : \zeta \to \zeta$, where (ζ, η) is a complete metric, satisfies*

$$\eta(\kappa u_1, \kappa u_2) \leq \sigma[\eta(u_1, \kappa u_1) + \eta(u_2, \kappa u_2)], \tag{1.2}$$

where $0 \leq \sigma < 1/2$ and $u_1, u_2 \in \zeta$, then κ has a unique fixed point.

Definition 1.2. ([14]) A function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if it satisfies the following conditions:

(i) φ is monotone increasing and continuous,
(ii) $\varphi(s) = 0$ if and only if $s = 0$.

Definition 1.3. ([10]) A pair of self-mappings S and T of a metric space (ζ, η) is said to be weakly compatible if they commute at their coincidence points. In other words, $St = Tt$ for some $t \in \zeta$, then $STt = TSt$.

There are many results on nonlinear rational type contraction mappings (see [4]-[9] and references cited therein) in various abstract spaces. In this paper, we obtained sufficient conditions for the existence and uniqueness of a fixed point for a certain rational type operators without the continuity of mappings on the setting of complete metric spaces. An example is also provided to illustrate the usability of our results.

2. Main results

In this section, we prove some common fixed point results for two pairs of weak compatible mappings in the framework of metric spaces.
For brevity, we denote Ψ the family of functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(s) > 0$, a lower semi continuous mapping for all $s > 0$, ψ is a discontinuous

mapping at $s = 0$ with $\psi(0) = 0$ and Φ the family of an altering distance functions.

Theorem 2.1. *Let (ζ, η) be a complete metric space. Suppose that $P, Q, R, S : \zeta \to \zeta$ satisfy the following hypotheses:*

$$S(\zeta) \subseteq Q(\zeta) \text{ and } R(\zeta) \subseteq P(\zeta); \tag{2.1}$$

$$\{P, S\} \text{ and } \{Q, R\} \text{ are weak compatible pairs}; \tag{2.2}$$

$$\varphi(\eta(S\mu, R\nu)) \leq \varphi(M(\mu, \nu)) - \psi(N(\mu, \nu)), \text{ for all } \mu, \nu \in \zeta, \text{ with } \mu \neq \nu, \tag{2.3}$$

where

$$M(\mu,\nu) = \max\left\{\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \frac{\eta(P\mu, S\mu)\eta(Q\nu, R\nu)}{1 + \eta(P\mu, Q\nu)}, \frac{1 + \eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{1 + \eta(P\mu, S\mu) + \eta(Q\nu, R\nu)}\eta(P\mu, S\mu)\right\},$$

$$N(\mu,\nu) = \min\left\{\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \frac{\eta(P\mu, S\mu)\eta(Q\nu, R\nu)}{1 + \eta(P\mu, Q\nu)}, \frac{1 + \eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{1 + \eta(P\mu, S\mu) + \eta(Q\nu, R\nu)}\eta(P\mu, S\mu)\right\},$$

$\psi \in \Psi$ and $\varphi \in \Phi$. Then P, Q, R and S have a unique fixed point in ζ.

Proof. Let μ_0 be an arbitrary point and $\nu_1 = S\mu_0$. Since $S\mu_0 \in S(\zeta) \subseteq Q(\zeta)$, there exist $\mu_1 \in \zeta$ such that $\nu_1 = S\mu_0 = Q\mu_1$. Let $\nu_2 = R\mu_1$. Since $R\mu_1 \in R(\zeta) \subseteq P(\zeta)$, there exist $\mu_2 \in \zeta$ such that $\nu_2 = R\mu_1 = P\mu_2$. Let $\nu_3 = S\mu_2$. On generalising, we can find two sequences $\{\mu_n\}$ and $\{\nu_n\}$ in ζ such that $\nu_{2n+1} = Q\mu_{2n+1} = S\mu_{2n}$ and $\nu_{2n+2} = P\mu_{2n+2} = R\mu_{2n+1}$ for all $n \in N_0 = N \cup \{0\}$. Suppose that

$$\nu_{2n} \neq \nu_{2n+1} \quad \text{for all } n \in N_0 = N \cup \{0\}. \tag{2.4}$$

First of all, we shall show that $\eta(\nu_{2n}, \nu_{2n+1}) \to 0$ as $n \to \infty$ for all $n \in N_0 = N \cup \{0\}$. Letting $\mu = \mu_{2n}, \nu = \mu_{2n+1}$, in equation (2.3), we get

$$\begin{aligned} \varphi(\eta(S\mu_{2n}, R\mu_{2n+1})) &= \varphi(\eta(\nu_{2n+1}, \nu_{2n+2})) \\ &\leq \psi(M(\mu_{2n}, \mu_{2n+1})) - \psi(N(\mu_{2n}, \mu_{2n+1})), \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&M(\mu_{2n},\mu_{2n+1})\\
&=\max\Big\{\eta(P\mu_{2n},Q\mu_{2n+1}),\eta(P\mu_{2n},S\mu_{2n}),\eta(Q\mu_{2n+1},R\mu_{2n+1}),\\
&\qquad\frac{\eta(P\mu_{2n},R\mu_{2n+1})+\eta(S\mu_{2n},Q\mu_{2n+1})}{2},\\
&\qquad\frac{\eta(P\mu_{2n},S\mu_{2n})\eta(Q\mu_{2n+1},R\mu_{2n+1})}{1+\eta(P\mu_{2n},Q\mu_{2n+1})},\\
&\qquad\frac{1+\eta(P\mu_{2n},R\mu_{2n+1})+\eta(S\mu_{2n},Q\mu_{2n+1})}{1+\eta(P\mu_{2n},S\mu_{2n})+\eta(Q\mu_{2n+1},R\mu_{2n+1})}\eta(P\mu_{2n},S\mu_{2n})\Big\}\\
&=\max\Big\{\eta(\nu_{2n},\nu_{2n+1}),\eta(\nu_{2n},\nu_{2n+1}),\eta(\nu_{2n+1},\nu_{2n+2}),\\
&\qquad\frac{\eta(\nu_{2n},\nu_{2n+2})+\eta(\nu_{2n+1},\nu_{2n+1})}{2},\frac{\eta(\nu_{2n},\nu_{2n+1})\eta(\nu_{2n+1},\nu_{2n+2})}{1+\eta(\nu_{2n},\nu_{2n+1})},\\
&\qquad\frac{1+\eta(\nu_{2n},\nu_{2n+2})+\eta(\nu_{2n+1},\nu_{2n+1})}{1+\eta(\nu_{2n},\nu_{2n+1})+\eta(\nu_{2n+1},\nu_{2n+2})}\eta(\nu_{2n},\nu_{2n+1})\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&N(\mu_{2n},\mu_{2n+1})\\
&=\min\Big\{\eta(P\mu_{2n},Q\mu_{2n+1}),\eta(P\mu_{2n},S\mu_{2n}),\eta(Q\mu_{2n+1},R\mu_{2n+1}),\\
&\qquad\frac{\eta(P\mu_{2n},R\mu_{2n+1})+\eta(S\mu_{2n},Q\mu_{2n+1})}{2},\\
&\qquad\frac{\eta(P\mu_{2n},S\mu_{2n})\eta(Q\mu_{2n+1},R\mu_{2n+1})}{1+\eta(P\mu_{2n},Q\mu_{2n+1})},\\
&\qquad\frac{1+\eta(P\mu_{2n},R\mu_{2n+1})+\eta(S\mu_{2n},Q\mu_{2n+1})}{1+\eta(P\mu_{2n},S\mu_{2n})+\eta(Q\mu_{2n+1},R\mu_{2n+1})}\eta(P\mu_{2n},S\mu_{2n})\Big\}\\
&=\min\Big\{\eta(\nu_{2n},\nu_{2n+1}),\eta(\nu_{2n},\nu_{2n+1}),\eta(\nu_{2n+1},\nu_{2n+2}),\\
&\qquad\frac{\eta(\nu_{2n},\nu_{2n+2})+\eta(\nu_{2n+1},\nu_{2n+1})}{2},\frac{\eta(\nu_{2n},\nu_{2n+1})\eta(\nu_{2n+1},\nu_{2n+2})}{1+\eta(\nu_{2n},\nu_{2n+1})},\\
&\qquad\frac{1+\eta(\nu_{2n},\nu_{2n+2})+\eta(\nu_{2n+1},\nu_{2n+1})}{1+\eta(\nu_{2n},\nu_{2n+1})+\eta(\nu_{2n+1},\nu_{2n+2})}\eta(\nu_{2n},\nu_{2n+1})\Big\}.
\end{aligned}
$$

By triangular inequality, we have

$$\begin{aligned}
&M(\mu_{2n}, \mu_{2n+1})\\
&= \max\Bigg\{\eta(\nu_{2n}, \nu_{2n+1}), \eta(\nu_{2n}, \nu_{2n+1}), \eta(\nu_{2n+1}, \nu_{2n+2}),\\
&\qquad \frac{\eta(\nu_{2n}, \nu_{2n+1}) + \eta(\nu_{2n+1}, \nu_{2n+2})}{2}, \eta(\nu_{2n+1}, \nu_{2n+2}),\\
&\qquad \frac{1 + \eta(\nu_{2n}, \nu_{2n+1}) + \eta(\nu_{2n+1}, \nu_{2n+2})}{1 + \eta(\nu_{2n}, \nu_{2n+1}) + \eta(\nu_{2n+1}, \nu_{2n+2})}\eta(\nu_{2n}, \nu_{2n+1})\Bigg\}\\
&= \max\Bigg\{\eta(\nu_{2n}, \nu_{2n+1}), \eta(\nu_{2n}, \nu_{2n+1}), \eta(\nu_{2n+1}, \nu_{2n+2}),\\
&\qquad \frac{\eta(\nu_{2n}, \nu_{2n+1}) + \eta(\nu_{2n+1}, \nu_{2n+2})}{2}, \eta(\nu_{2n+1}, \nu_{2n+2}), \eta(\nu_{2n}, \nu_{2n+1})\Bigg\}\\
&= \max\Bigg\{\eta(\nu_{2n}, \nu_{2n+1}), \eta(\nu_{2n+1}, \nu_{2n+2}), \frac{\eta(\nu_{2n}, \nu_{2n+1}) + \eta(\nu_{2n+1}, \nu_{2n+2})}{2}\Bigg\}.
\end{aligned}$$

If

$$\eta(\nu_{2n}, \nu_{2n+1}) < \eta(\nu_{2n+1}, \nu_{2n+2}), \tag{2.6}$$

then

$$M(\mu_{2n}, \mu_{2n+1}) \leq \eta(\nu_{2n+1}, \nu_{2n+2}). \tag{2.7}$$

Also, equation (2.5) implies

$$\begin{aligned}
\varphi(\eta(\nu_{2n+1}, \nu_{2n+2})) &\leq \varphi(M(\mu_{2n}, \mu_{2n+1})) - \psi(N(\mu_{2n}, \mu_{2n+1}))\\
&\leq \varphi(M(\mu_{2n}, \mu_{2n+1})),
\end{aligned}$$

using monotonically increasing property of φ function we have

$$\eta(\nu_{2n+1}, \nu_{2n+2}) \leq M(\mu_{2n}, \mu_{2n+1}). \tag{2.8}$$

From equation (2.7) and (2.8) we have

$$\eta(\nu_{2n+1}, \nu_{2n+2}) = M(\mu_{2n}, \mu_{2n+1}). \tag{2.9}$$

Since

$$0 < |\eta(\nu_{2n+1}, \nu_{2n+2}) - \eta(\nu_{2n}, \nu_{2n+1})| \leq \eta(\nu_{2n}, \nu_{2n+2}), \tag{2.10}$$

we have $N(\mu_{2n}, \mu_{2n+1}) > 0$, then from equations (2.5), (2.9) and the property of ψ and φ we have

$$\begin{aligned}
\varphi(\eta(\nu_{2n+1}, \nu_{2n+2})) &\leq \varphi(\eta(\nu_{2n+1}, \nu_{2n+2})) - \psi(N(\mu_{2n}, \mu_{2n+1}))\\
&< \varphi(\eta(\nu_{2n+1}, \nu_{2n+2})),
\end{aligned}$$

which is a contradiction. Thus, we get

$$\eta(\nu_{2n+1}, \nu_{2n+2}) \leq \eta(\nu_{2n}, \nu_{2n+1}). \tag{2.11}$$

So, we get

$$M(\mu_{2n}, \mu_{2n+1}) = \eta(\nu_{2n}, \nu_{2n+1}). \tag{2.12}$$

Substituting equation (2.12) in equation (2.5), we get

$$\varphi(\eta(\nu_{2n+1}, \nu_{2n+2})) \leq \varphi(\eta(\nu_{2n}, \nu_{2n+1})) - \psi(N(\mu_{2n}, \mu_{2n+1})). \tag{2.13}$$

Therefore, $\eta(\nu_{2n}, \nu_{2n+1})$ is montonically decreasing sequence of non negative real numbers and hence there exist a number $t > 0$ such that

$$\lim_{n\to\infty} \eta(\nu_{2n}, \nu_{2n+1}) = t > 0. \tag{2.14}$$

By virtue of equation (2.4) and (2.10), we have $N(\mu_{2n}, \mu_{2n+1}) > 0$. Taking $n \to \infty$ in equation (2.13) and using equation (2.14), we get

$$\lim_{n\to\infty} \varphi(\eta(\nu_{2n+1}, \nu_{2n+2})) \leq \lim_{n\to\infty} \varphi(\eta(\nu_{2n}, \nu_{2n+1})) - \lim_{n\to\infty} \psi(N(\mu_{2n}, \mu_{2n+1})).$$

This implies

$$\varphi(t) \leq \varphi(t) - \lim_{n\to\infty} \psi(N(\mu_{2n}, \mu_{2n+1})) < \varphi(t),$$

which is not possible. Hence,

$$\lim_{n\to\infty} \eta(\nu_{2n}, \nu_{2n+1}) = 0.$$

Letting $\mu = \mu_{2n+1}$ and $\nu = \mu_{2n+2}$ in equation (2.3) and proceeding as above we get

$$\lim_{n\to\infty} \eta(\nu_{2n+1}, \nu_{2n+2}) = 0.$$

Therefore, for all $n \in N_0 = N \cup \{0\}$, we have

$$\lim_{n\to\infty} \eta(\nu_n, \nu_{n+1}) = 0. \tag{2.15}$$

Next, We shall prove that $\{\nu_n\}$ is a Cauchy sequence. Assume that $\{\nu_n\}$ is not a Cauchy sequence. To show this it is sufficient to show that $\{\nu_{2n}\}$ is not a Cauchy sequence. If $\{\nu_{2n}\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and the sequence of natural numbers $2n(k)$ and $2m(k)$ such that $2n(k) > 2m(k) > 2k$ for $k \in N$ and

$$\eta(\nu_{2m(k)}, \nu_{2n(k)}) \geq \epsilon. \tag{2.16}$$

Corresponding to $2m(k)$ we can choose $2n(k)$ to be the smallest such that equation (2.16) is satisfied. Then we have

$$\eta(\nu_{2m(k)}, \nu_{2n(k)-1}) < \epsilon. \tag{2.17}$$

Substituting $\mu = \mu_{2m(k)-1}$ and $\nu = \mu_{2n(k)-1}$ in equation (2.3), where for all $k \in N$,

$$\begin{aligned}&\varphi(\eta(\nu_{2m(k)}, \nu_{2n(k)})) \\ &\leq \varphi(M(\mu_{2m(k)-1}, \nu_{2n(k)-1})) - \psi(N(\mu_{2m(k)-1}, \mu_{2n(k)-1}))\end{aligned} \tag{2.18}$$

where,

$$\begin{aligned}&M(\mu_{2m(k)-1}, \mu_{2n(k)-1}) \\ &= \max\Bigg\{\eta(\nu_{2m(k)-1}, \nu_{2n(k)-1}), \eta(\nu_{2n(k)-1}, \nu_{2n(k)}), \\ &\qquad \frac{\eta(\nu_{2m(k)-1}, \nu_{2m(k)}) + \eta(\nu_{2n(k)-1}, \nu_{2n(k)})}{2}\Bigg\}\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&N(\mu_{2m(k)-1}, \mu_{2n(k)-1}) \\ &= \min\Bigg\{\eta(\nu_{2m(k)-1}, \nu_{2n(k)-1}), \eta(\nu_{2n(k)-1}, \nu_{2n(k)}), \\ &\qquad \frac{\eta(\nu_{2m(k)-1}, \nu_{2m(k)}) + \eta(\nu_{2n(k)-1}, \nu_{2n(k)})}{2}\Bigg\}\end{aligned} \tag{2.20}$$

using triangle inequality, we get

$$\eta(\nu_{2m(k)}, \nu_{2n(k)}) \leq \eta(\nu_{2m(k)}, \nu_{2n(k)-1}) + \eta(\nu_{2n(k)-1}, \nu_{2n(k)}).$$

Taking limit as $k \to \infty$, we get

$$\lim_{n\to\infty} \eta(\nu_{2m(k)}, \nu_{2n(k)}) = \epsilon. \tag{2.21}$$

Again for all k, we get

$$\begin{aligned}&\eta(\nu_{2m(k)-1}, \nu_{2n(k)-1}) \\ &\leq \eta(\nu_{2m(k)}, \nu_{2m(k)-1}) + \eta(\nu_{2m(k)}, \nu_{2n(k)}) + \eta(\nu_{2n(k)-1}, \nu_{2n(k)})\end{aligned}$$

and

$$\begin{aligned}&\eta(\nu_{2m(k)}, \nu_{2n(k)}) \\ &\leq \eta(\nu_{2m(k)}, \nu_{2m(k)-1}) + \eta(\nu_{2m(k)-1}, \nu_{2n(k)-1}) + \eta(\nu_{2n(k)-1}, \nu_{2n(k)}).\end{aligned}$$

Taking limit as $k \to \infty$ and using equation (2.15) and (2.19), we get

$$\lim_{k\to\infty} \eta(\nu_{2m(k)-1}, \nu_{2n(k)-1}) = \epsilon. \tag{2.22}$$

Taking limit as $k \to \infty$ in equation (2.19) and (2.20) and using equation (2.15) and (2.22), we get

$$\lim_{k\to\infty} M(\mu_{2m(k)-1}, \mu_{2n(k)-1}) = \epsilon$$

and

$$\lim_{k\to\infty} N(\mu_{2m(k)-1}, \mu_{2n(k)-1}) = 0.$$

Taking limit as $k \to \infty$ in equation (2.18), we get

$$\varphi(\epsilon) \le \varphi(\epsilon) - \lim_{k\to\infty} \psi(N(\mu_{2m(k)-1}, \mu_{2n(k)-1})).$$

Using discontinuity of ψ at $s = 0$ and $\psi > 0$ for $s > 0$, we conclude that in the above inequality the second term is non zero. Thus, we get $\varphi(\epsilon) < \varphi(\epsilon)$, which is a contradiction. Hence $\{\nu_n\}$ is a Cauchy sequence. We know that every Cauchy sequence $\{\nu_n\}$ is a convergent sequence therefore the above sequence $\{\nu_n\}$ coverges to a point p_1(say) in ζ. Hence, the subsequences of the sequence $\{\nu_n\}$ also converges to p_1 in ζ;

$$S\mu_{2n} \to p_1, \ \ Q\mu_{2n+1} \to p_1, \ \ R\mu_{2n+1} \to p_1 \ \text{ and } \ P\mu_{2n+2} \to p_1.$$

Now, we claim that common fixed point of P, Q, R and S is p_1. Since $R(\zeta) \subset P(\zeta)$, there exist $l \in \zeta$ such that $p_1 = Pl$. Let $\eta(p_1, Sl) \neq 0$ and substituting $\mu = l$ and $\nu = \mu_{2n+1}$ in equation (2.3), we get

$$\varphi(\eta(Sl, R\mu_{2n+1})) \le \varphi(M(l, \mu_{2n+1})) - \psi(N(l, \mu_{2n+1})), \tag{2.23}$$

where

$$\begin{aligned} &M(l, \mu_{2n+1}) \\ &= \max\Bigg\{ \eta(Pl, Q\mu_{2n+1}), \eta(Pl, Sl), \eta(Q\mu_{2n+1}, R\mu_{2n+1}), \\ &\qquad \frac{\eta(Pl, R\mu_{2n+1}) + \eta(Sl, Q\mu_{2n+1})}{2}, \frac{\eta(Pl, Sl)\eta(Q\mu_{2n+1}, R\mu_{2n+1})}{1 + \eta(Pl, Q\mu_{2n+1})}, \\ &\qquad \frac{1 + \eta(Pl, R\mu_{2n+1}) + \eta(Sl, Q\mu_{2n+1})}{1 + \eta(Pl, Sl) + \eta(Q\mu_{2n+1}, R\mu_{2n+1})} \eta(Pl, Sl) \Bigg\} \end{aligned}$$

and

$$\begin{aligned} &N(l, \mu_{2n+1}) \\ &= \min\Bigg\{ \eta(Pl, Q\mu_{2n+1}), \eta(Pl, Sl), \eta(Q\mu_{2n+1}, R\mu_{2n+1}), \\ &\qquad \frac{\eta(Pl, R\mu_{2n+1}) + \eta(Sl, Q\mu_{2n+1})}{2}, \frac{\eta(Pl, Sl)\eta(Q\mu_{2n+1}, R\mu_{2n+1})}{1 + \eta(Pl, Q\mu_{2n+1})}, \\ &\qquad \frac{1 + \eta(Pl, R\mu_{2n+1}) + \eta(Sl, Q\mu_{2n+1})}{1 + \eta(Pl, Sl) + \eta(Q\mu_{2n+1}, R\mu_{2n+1})} \eta(Pl, Sl) \Bigg\}. \end{aligned}$$

Taking $n \to \infty$ and using $p_1 = Pl$, we have

$$M(l, p_1) = \max\Big\{\eta(p_1, p_1), \eta(p_1, Sl), \eta(p_1, p_1), \frac{\eta(p_1, p_1) + \eta(Sl, p_1)}{2}, \frac{\eta(p_1, Sl)\eta(p_1, p_1)}{1 + \eta(p_1, p_1)}, \frac{1 + \eta(p_1, p_1) + \eta(Sl, p_1)}{1 + \eta(p_1, Sl) + \eta(p_1, p_1)}\eta(p_1, Sl)\Big\}$$

and

$$\begin{aligned} M(l, p_1) &= \max\Big\{0, \eta(p_1, Sl), 0, \frac{\eta(Sl, p_1)}{2}, 0, \eta(p_1, Sl)\Big\} \\ &= \max\Big\{0, \eta(p_1, Sl), \frac{\eta(Sl, p_1)}{2}\Big\} \\ &= \eta(p_1, Sl). \end{aligned}$$

Also

$$\varphi(\eta(Sl, p_1)) \leq \varphi(\eta(p_1, Sl)) - \lim_{n\to\infty} \psi(N(l, \mu_{2n+1})).$$

Using discontinuity of ψ at $s = 0$ and $\psi > 0$ for $s > 0$, we conclude that the second term of the above inequality is non zero. Therefore, we get

$$\varphi(\eta(Sl, p_1)) < \varphi(\eta(p_1, Sl)),$$

which is not possible. Therefore, using the properties of φ function, we have

$$\eta(p_1, Sl) = 0.$$

This implies

$$Sl = p_1 = Pl.$$

Since $\{P, S\}$ is a weakly compatible pair of mappings, therefore it commutes at their coincidence point l that is $PSl = SPl$ this implies $Pp_1 = Sp_1$.
Now we shall show that $Pp_1 = Sp_1 = p_1$. For this letting $\mu = p_1$ and $\nu = \mu_{2n+1}$ in equation (2.3), we get

$$\varphi(\eta(Sp_1, R\mu_{2n+1})) \leq \varphi(M(p_1, \mu_{2n+1})) - \psi(N(p_1, \mu_{2n+1})), \tag{2.24}$$

where

$$\begin{aligned}
&M(p_1, \mu_{2n+1}) \\
&= \max\Bigg\{\eta(Pp_1, Q\mu_{2n+1}), \eta(Pp_1, Sp_1), \eta(Q\mu_{2n+1}, R\mu_{2n+1}), \\
&\qquad \frac{\eta(Pp_1, R\mu_{2n+1}) + \eta(Sp_1, Q\mu_{2n+1})}{2}, \\
&\qquad \frac{\eta(Pp_1, Sp_1)\eta(Q\mu_{2n+1}, R\mu_{2n+1})}{1 + \eta(Pp_1, Q\mu_{2n+1})}, \\
&\qquad \frac{1 + \eta(Pp_1, R\mu_{2n+1}) + \eta(Sp_1, Q\mu_{2n+1})}{1 + \eta(Pp_1, Sp_1) + \eta(Q\mu_{2n+1}, R\mu_{2n+1})}\eta(Pp_1, Sp_1)\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
&N(p_1, \mu_{2n+1}) \\
&= \min\Bigg\{\eta(Pp_1, Q\mu_{2n+1}), \eta(Pp_1, Sp_1), \eta(Q\mu_{2n+1}, R\mu_{2n+1}), \\
&\qquad \frac{\eta(Pp_1, R\mu_{2n+1}) + \eta(Sp_1, Q\mu_{2n+1})}{2}, \\
&\qquad \frac{\eta(Pp_1, Sp_1)\eta(Q\mu_{2n+1}, R\mu_{2n+1})}{1 + \eta(Pp_1, Q\mu_{2n+1})}, \\
&\qquad \frac{1 + \eta(Pp_1, R\mu_{2n+1}) + \eta(Sp_1, Q\mu_{2n+1})}{1 + \eta(Pp_1, Sp_1) + \eta(Q\mu_{2n+1}, R\mu_{2n+1})}\eta(Pp_1, Sp_1)\Bigg\}.
\end{aligned}$$

Taking limit as $n \to \infty$ and using $Pp_1 = Sp_1$, we get

$$\begin{aligned}
&M(p_1, p_1) \\
&= \max\Bigg\{\eta(Pp_1, p_1), \eta(Sp_1, Sp_1), \eta(p_1, p_1), \frac{\eta(Sp_1, p_1) + \eta(Sp_1, p_1)}{2}, \\
&\qquad \frac{\eta(Sp_1, Sp_1)\eta(p_1, p_1)}{1 + \eta(Pp_1, p_1)}, \frac{1 + \eta(Sp_1, p_1) + \eta(Sp_1, p_1)}{1 + \eta(Sp_1, p_1) + \eta(p_1, p_1)}\eta(Sp_1, Sp_1)\Bigg\} \\
&= \max\Bigg\{\eta(Sp_1, p_1), 0, 0, \eta(Sp_1, p_1), 0, 0\Bigg\} \\
&= \eta(Sp_1, p_1).
\end{aligned}$$

Therefore, we have $M(p_1, p_1) = \eta(Sp_1, p_1)$.
Now equation (2.24) implies that

$$\varphi(\eta(Sp_1, p_1)) \leq \varphi(\eta(Sp_1, p_1)) - \lim_{n\to\infty} \psi(N(p_1, \mu_{2n+1})).$$

Using discontinuity of ψ at $t = 0$ and $\psi > 0$ for $t > 0$, we conclude that the second term of the above inequality is non zero. Therefore, we get

$$\varphi(\eta(Sp_1, p_1)) < \varphi(\eta(Sp_1, p_1)),$$

which is a contradiction. Therefore, $\eta(Sp_1, p_1) = 0$ this implies $Sp_1 = p_1$ this imples $Sp_1 = Pp_1 = p_1$. Similarly, we can show that $Qp_1 = Rp_1 = p_1$. Hence $Pp_1 = Sp_1 = Qp_1 = Rp_1 = p_1$. □

On the similar lines of Theorem 2.1, we have the following results.

Theorem 2.2. *Let (ζ, η) be a complete metric space. Suppose that $P, Q, R, S : \zeta \to \zeta$ satisfy the following hypotheses:*

$$S(\zeta) \subseteq Q(\zeta) \quad \textit{and} \quad R(\zeta) \subseteq P(\zeta), \tag{2.25}$$

$$\{P, S\} \textit{ and } \{Q, R\} \textit{ are weak compatible pairs,} \tag{2.26}$$

$$\begin{aligned} &\varphi(\eta(S\mu, R\nu)) \\ &\leq \varphi(M_1(\mu, \nu)) - \psi(N_1(\mu, \nu)), \textit{ for all } \mu, \nu \in \zeta \textit{ with } \mu \neq \nu, \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} M_1(\mu, \nu) = \max\Bigg\{ &\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \\ &\frac{1 + \eta(P\mu, S\mu)}{1 + \eta(P\mu, Q\nu)}\eta(Q\nu, R\nu), \frac{1 + \eta(Q\nu, R\nu)}{1 + \eta(P\mu, Q\nu)}\eta(P\mu, S\mu), \\ &\frac{1 + \eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{1 + \eta(P\mu, S\mu) + \eta(Q\nu, R\nu)}\eta(Q\nu, R\nu) \Bigg\}, \end{aligned}$$

$$\begin{aligned} N_1(\mu, \nu) = \min\Bigg\{ &\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \\ &\frac{1 + \eta(P\mu, S\mu)}{1 + \eta(P\mu, Q\nu)}\eta(Q\nu, R\nu), \frac{1 + \eta(Q\nu, R\nu)}{1 + \eta(P\mu, Q\nu)}\eta(P\mu, S\mu), \\ &\frac{1 + \eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{1 + \eta(P\mu, S\mu) + \eta(Q\nu, R\nu)}\eta(Q\nu, R\nu) \Bigg\}, \end{aligned}$$

$\psi \in \Psi$ and $\varphi \in \Phi$. Then P, Q, R and S have a unique fixed point in ζ.

Theorem 2.3. *Let (ζ, η) be a complete metric space. Suppose that $P, Q, R, S : \zeta \to \zeta$ satisfy the following hypotheses:*

$$S(\zeta) \subseteq Q(\zeta) \quad \textit{and} \quad R(\zeta) \subseteq P(\zeta), \tag{2.28}$$

$$\{P, S\} \textit{ and } \{Q, R\} \textit{ are weak compatible pairs,} \tag{2.29}$$

$$\begin{aligned} &\varphi(\eta(S\mu, R\nu)\\ &\leq \varphi(M_1(\mu,\nu)) - \psi(N_1(\mu,\nu)), \ \textit{for all } \mu,\nu \in \zeta \textit{ with } \mu \neq \nu, \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} M_1(\mu,\nu) = \max\bigg\{ &\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2},\\ &\frac{\eta(P\mu, R\nu)\eta(S\mu, Q\nu)}{1+\eta(P\mu, Q\nu)}, \frac{\eta(P\mu, S\mu)\eta(Q\nu, R\nu)}{1+\eta(P\mu, Q\nu)},\\ &\frac{1+\eta(P\mu, R\nu)+\eta(S\mu, Q\nu)}{1+\eta(P\mu, S\mu)+\eta(Q\nu, R\nu)}\eta(P\mu, S\mu)\bigg\},\\ N_1(\mu,\nu) = \min\bigg\{ &\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2},\\ &\frac{\eta(P\mu, R\nu)\eta(S\mu, Q\nu)}{1+\eta(P\mu, Q\nu)}, \frac{\eta(P\mu, S\mu)\eta(Q\nu, R\nu)}{1+\eta(P\mu, Q\nu)},\\ &\frac{1+\eta(P\mu, R\nu)+\eta(S\mu, Q\nu)}{1+\eta(P\mu, S\mu)+\eta(Q\nu, R\nu)}\eta(P\mu, S\mu)\bigg\}, \end{aligned}$$

$\psi \in \Psi$ *and* $\varphi \in \Phi$*. Then* P, Q, R *and* S *have a unique fixed point in* ζ*.*

Theorem 2.4. *Let* (ζ, η) *be a complete metric space. Suppose that* $P, Q, R, S : \zeta \to \zeta$ *satisfy the following hypotheses:*

$$S(\zeta) \subseteq Q(\zeta) \quad \textit{and} \quad R(\zeta) \subseteq P(\zeta), \tag{2.31}$$

$$\{P, S\} \textit{ and } \{Q, R\} \textit{ are weak compatible pairs,} \tag{2.32}$$

$$\begin{aligned} &\varphi(\eta(S\mu, R\nu)\\ &\leq \varphi(M_1(\mu,\nu)) - \psi(N_1(\mu,\nu)), \ \textit{for all } \mu,\nu \in \zeta \textit{ with } \mu \neq \nu, \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} M_1(\mu,\nu) = \max\bigg\{ &\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu),\\ &\frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \frac{\eta(P\mu, R\nu)\eta(S\mu, Q\nu)}{1+\eta(P\mu, Q\nu)}\bigg\},\\ N_1(\mu,\nu) = \min\bigg\{ &\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu),\\ &\frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \frac{\eta(P\mu, R\nu)\eta(S\mu, Q\nu)}{1+\eta(P\mu, Q\nu)}\bigg\}, \end{aligned}$$

$\psi \in \Psi$ *and* $\varphi \in \Phi$. *Then* P, Q, R *and* S *have a unique fixed point in* ζ.

Theorem 2.5. *Let* (ζ, η) *be a complete metric space. Suppose that* $P, Q, R, S : \zeta \to \zeta$ *satisfy the following hypotheses:*

$$S(\zeta) \subseteq Q(\zeta) \quad \textit{and} \quad R(\zeta) \subseteq P(\zeta), \tag{2.34}$$

$$\{P, S\} \textit{ and } \{Q, R\} \textit{ are weak compatible pairs,} \tag{2.35}$$

$$\begin{aligned} &\varphi(\eta(S\mu, R\nu) \\ &\leq \varphi(M_1(\mu, \nu)) - \psi(N_1(\mu, \nu)), \textit{ for all } \mu, \nu \in \zeta \textit{ with } \mu \neq \nu, \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} M_1(\mu, \nu) = \max\Bigg\{ &\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \\ &\frac{\eta(S\mu, P\mu) + \eta(Q\nu, R\nu)}{2}, \frac{\eta(P\mu, S\mu)\eta(Q\nu, R\nu)}{1 + \eta(P\mu, Q\nu)}, \\ &\frac{1 + \eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{1 + \eta(P\mu, S\mu) + \eta(Q\nu, R\nu)}\eta(P\mu, S\mu) \Bigg\}, \end{aligned}$$

$$\begin{aligned} N_1(\mu, \nu) = \min\Bigg\{ &\eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \\ &\frac{\eta(S\mu, P\mu) + \eta(Q\nu, R\nu)}{2}, \frac{\eta(P\mu, S\mu)\eta(Q\nu, R\nu)}{1 + \eta(P\mu, Q\nu)}, \\ &\frac{1 + \eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{1 + \eta(P\mu, S\mu) + \eta(Q\nu, R\nu)}\eta(P\mu, S\mu) \Bigg\}, \end{aligned}$$

$\psi \in \Psi$ *and* $\varphi \in \Phi$. *Then* P, Q, R *and* S *have a unique fixed point in* ζ.

On taking $P = Q = I$ identity map, we get the following results.

Corollary 2.6. *Let* (ζ, η) *be a complete metric space. Suppose that* $R, S : \zeta \to \zeta$ *satisfy the following hypotheses:*

$$\begin{aligned} &\varphi(\eta(S\mu, R\nu) \\ &\leq \varphi(M_1(\mu, \nu)) - \psi(N_1(\mu, \nu)), \textit{ for all } \mu, \nu \in \zeta \textit{ with } \mu \neq \nu, \end{aligned} \tag{2.37}$$

where

$$\begin{aligned} M_1(\mu, \nu) = \max\Bigg\{ &\eta(\mu, \nu), \eta(\mu, S\mu), \eta(\nu, R\nu), \frac{\eta(\mu, R\nu) + \eta(S\mu, \nu)}{2}, \\ &\frac{\eta(\mu, S\mu)\eta(\nu, R\nu)}{1 + \eta(\mu, \nu)}, \frac{1 + \eta(\mu, R\nu) + \eta(S\mu, \nu)}{1 + \eta(\mu, S\nu) + \eta(\nu, R\nu)}\eta(\mu, S\mu) \Bigg\}, \end{aligned}$$

$$N_1(\mu,\nu) = \min\Bigg\{\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \frac{\eta(\mu,S\mu)\eta(\nu,R\nu)}{1+\eta(\mu,\nu)}, \frac{1+\eta(\mu,R\nu)+\eta(S\mu,\nu)}{1+\eta(\mu,S\nu)+\eta(\nu,R\nu)}\eta(\mu,S\mu)\Bigg\},$$

$\psi \in \Psi$ and $\varphi \in \Phi$. Then R and S have a unique fixed point in ζ.

Corollary 2.7. *Let (ζ,η) be a complete metric space. Suppose that $R,S : \zeta \to \zeta$ satisfy the following hypotheses:*

$$\begin{aligned}&\varphi(\eta(S\mu,R\nu)\\ &\leq \varphi(M_1(\mu,\nu)) - \psi(N_1(\mu,\nu)), \text{ for all } \mu,\nu \in \zeta \text{ with } \mu \neq \nu,\end{aligned} \tag{2.38}$$

where

$$M_1(\mu,\nu) = \max\Bigg\{\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \frac{1+\eta(\mu,S\mu)}{1+\eta(\mu,\nu)}\eta(\nu,R\nu), \frac{1+\eta(\nu,R\nu)}{1+\eta(\mu,v)}\eta(\mu,S\mu), \frac{1+\eta(\mu,R\nu)+\eta(S\mu,\nu)}{1+\eta(\mu,S\nu)+\eta(\nu,R\nu)}\eta(\nu,R\nu)\Bigg\},$$

$$N_1(\mu,\nu) = \min\Bigg\{\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \frac{1+\eta(\mu,S\mu)}{1+\eta(\mu,\nu)}\eta(\nu,R\nu), \frac{1+\eta(\nu,R\nu)}{1+\eta(\mu,v)}\eta(\mu,S\mu), \frac{1+\eta(\mu,R\nu)+\eta(S\mu,\nu)}{1+\eta(\mu,S\nu)+\eta(\nu,R\nu)}\eta(\nu,R\nu)\Bigg\},$$

$\psi \in \Psi$ and $\varphi \in \Phi$. Then R and S have a unique fixed point in ζ.

Corollary 2.8. *Let (ζ,η) be a complete metric space. Suppose that $R,S : \zeta \to \zeta$ satisfy the following hypotheses:*

$$\begin{aligned}&\varphi(\eta(S\mu,R\nu)\\ &\leq \varphi(M_1(\mu,\nu)) - \psi(N_1(\mu,\nu)), \text{ for all } \mu,\nu \in \zeta \text{ with } \mu \neq \nu,\end{aligned} \tag{2.39}$$

where

$$M_1(\mu,\nu) = \max\left\{\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \frac{\eta(\mu,R\nu)\eta(S\mu,\nu)}{1+\eta(\mu,\nu)}, \frac{\eta(\mu,S\mu)\eta(\nu,R\nu)}{1+\eta(\mu,\nu)}, \frac{1+\eta(\mu,R\nu)+\eta(S\mu,\nu)}{1+\eta(\mu,S\mu)+\eta(\nu,R\nu)}\eta(\mu,S\mu)\right\},$$

$$N_1(\mu,\nu) = \min\left\{\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \frac{\eta(\mu,R\nu)\eta(S\mu,\nu)}{1+\eta(\mu,\nu)}, \frac{\eta(\mu,S\mu)\eta(\nu,R\nu)}{1+\eta(\mu,\nu)}, \frac{1+\eta(\mu,R\nu)+\eta(S\mu,\nu)}{1+\eta(\mu,S\mu)+\eta(\nu,R\nu)}\eta(\mu,S\mu)\right\},$$

$\psi \in \Psi$ *and* $\varphi \in \Phi$. *Then* R *and* S *have a unique fixed point in* ζ.

Corollary 2.9. *Let* (ζ, η) *be a complete metric space. Suppose that* $R, S : \zeta \to \zeta$ *satisfy the following hypotheses:*

$$\varphi(\eta(S\mu, R\nu) \leq \varphi(M_1(\mu,\nu)) - \psi(N_1(\mu,\nu)), \text{ for all } \mu,\nu \in \zeta \text{ with } \mu \neq \nu, \tag{2.40}$$

where

$$M_1(\mu,\nu) = \max\left\{\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \frac{\eta(\mu,R\nu)\eta(S\mu,\nu)}{1+\eta(\mu,\nu)}\right\},$$

$$N_1(\mu,\nu) = \min\left\{\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \frac{\eta(\mu,R\nu)\eta(S\mu,\nu)}{1+\eta(\mu,\nu)}\right\},$$

$\psi \in \Psi$ *and* $\varphi \in \Phi$. *Then* R *and* S *have a unique fixed point in* ζ.

Corollary 2.10. *Let (ζ, η) be a complete metric space. Suppose that $R, S : \zeta \to \zeta$ satisfy the following hypotheses:*

$$\begin{aligned} &\varphi(\eta(S\mu, R\nu) \\ &\leq \varphi(M_1(\mu,\nu)) - \psi(N_1(\mu,\nu)), \text{ for all } \mu,\nu \in \zeta \text{ with } \mu \neq \nu, \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} M_1(\mu,\nu) = \max\Bigg\{ &\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \\ &\frac{\eta(S\mu,\mu)+\eta(\nu,R\nu)}{2}, \frac{\eta(\mu,S\mu)\eta(\nu,R\nu)}{1+\eta(\mu,\nu)}, \\ &\frac{1+\eta(\mu,R\nu)+\eta(S\mu,\nu)}{1+\eta(\mu,S\mu)+\eta(\nu,R\nu)}\eta(\mu,S\mu) \Bigg\}, \\ N_1(\mu,\nu) = \min\Bigg\{ &\eta(\mu,\nu), \eta(\mu,S\mu), \eta(\nu,R\nu), \frac{\eta(\mu,R\nu)+\eta(S\mu,\nu)}{2}, \\ &\frac{\eta(S\mu,\mu)+\eta(\nu,R\nu)}{2}, \frac{\eta(\mu,S\mu)\eta(\nu,R\nu)}{1+\eta(\mu,\nu)}, \\ &\frac{1+\eta(\mu,R\nu)+\eta(S\mu,\nu)}{1+\eta(\mu,S\mu)+\eta(\nu,R\nu)}\eta(\mu,S\mu) \Bigg\}, \end{aligned}$$

$\psi \in \Psi$ and $\varphi \in \Phi$. Then R and S have a unique fixed point in ζ.

On taking $\varphi(t) = t$ in Theorems 2.1-2.5, we get the following corollaries.

Corollary 2.11. *Let (ζ, η) be a complete metric space. Suppose that $P, Q, R, S : \zeta \to \zeta$ satisfy the following hypotheses:*

$$S(\zeta) \subseteq Q(\zeta) \quad \text{and} \quad R(\zeta) \subseteq P(\zeta), \tag{2.42}$$

$$\{P, S\} \text{ and } \{Q, R\} \text{ are weak compatible pairs}, \tag{2.43}$$

$$\begin{aligned} &\eta(S\mu, R\nu) \\ &\leq M_1(\mu,\nu) - \psi(N_1(\mu,\nu)), \text{ for all } \mu,\nu \in \zeta \text{ with } \mu \neq \nu, \end{aligned} \tag{2.44}$$

where

$$\begin{aligned} M_1(\mu,\nu) = \max\Bigg\{ &\eta(P\mu,Q\nu), \eta(P\mu,S\mu), \eta(Q\nu,R\nu), \frac{\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{2}, \\ &\frac{\eta(P\mu,S\mu)\eta(Q\nu,R\nu)}{1+\eta(P\mu,Q\nu)}, \frac{1+\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{1+\eta(P\mu,S\mu)+\eta(Q\nu,R\nu)}\eta(P\mu,S\mu) \Bigg\}, \end{aligned}$$

$$N_1(\mu,\nu)=\min\Bigg\{\eta(P\mu,Q\nu),\eta(P\mu,S\mu),\eta(Q\nu,R\nu),\frac{\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{2},$$
$$\frac{\eta(P\mu,S\mu)\eta(Q\nu,R\nu)}{1+\eta(P\mu,Q\nu)},\frac{1+\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{1+\eta(P\mu,S\mu)+\eta(Q\nu,R\nu)}\eta(P\mu,S\mu)\Bigg\},$$

and $\psi\in\Psi$. *Then* P,Q,R *and* S *have a unique fixed point in* ζ.

Corollary 2.12. *Let* (ζ,η) *be a complete metric space. Suppose that* P,Q,R, $S:\zeta\to\zeta$ *satisfy the following hypotheses:*

$$S(\zeta)\subseteq Q(\zeta)\quad \textit{and}\quad R(\zeta)\subseteq P(\zeta), \tag{2.45}$$

$$\{P,S\}\ \textit{and}\ \{Q,R\}\ \textit{are weak compatible pairs}, \tag{2.46}$$

$$\eta(S\mu,R\nu) \le M_1(\mu,\nu)-\psi(N_1(\mu,\nu)),\ \textit{for all}\ \mu,\nu\in\zeta\ \textit{with}\ \mu\neq\nu, \tag{2.47}$$

where

$$M_1(\mu,\nu)=\max\Bigg\{\eta(P\mu,Q\nu),\eta(P\mu,S\mu),\eta(Q\nu,R\nu),\frac{\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{2},$$
$$\frac{1+\eta(P\mu,S\mu)}{1+\eta(P\mu,Q\nu)}\eta(Q\nu,R\nu),\frac{1+\eta(Q\nu,R\nu)}{1+\eta(P\mu,Q\nu)}\eta(P\mu,S\mu),$$
$$\frac{1+\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{1+\eta(P\mu,S\mu)+\eta(Q\nu,R\nu)}\eta(Q\nu,R\nu)\Bigg\},$$

$$N_1(\mu,\nu)=\min\Bigg\{\eta(P\mu,Q\nu),\eta(P\mu,S\mu),\eta(Q\nu,R\nu),\frac{\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{2},$$
$$\frac{1+\eta(P\mu,S\mu)}{1+\eta(P\mu,Q\nu)}\eta(Q\nu,R\nu),\frac{1+\eta(Q\nu,R\nu)}{1+\eta(P\mu,Q\nu)}\eta(P\mu,S\mu),$$
$$\frac{1+\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{1+\eta(P\mu,S\mu)+\eta(Q\nu,R\nu)}\eta(Q\nu,R\nu)\Bigg\}$$

and $\psi\in\Psi$. *Then* P,Q,R *and* S *have a unique fixed point in* ζ.

Corollary 2.13. *Let* (ζ,η) *be a complete metric space. Suppose that* P,Q,R, $S:\zeta\to\zeta$ *satisfy the following hypotheses:*

$$S(\zeta)\subseteq Q(\zeta)\quad \textit{and}\quad R(\zeta)\subseteq P(\zeta), \tag{2.48}$$

$$\{P,S\}\ \textit{and}\ \{Q,R\}\ \textit{are weak compatible pairs}, \tag{2.49}$$

$$\eta(S\mu,R\nu) \le M_1(\mu,\nu)-\psi(N_1(\mu,\nu)),\ \textit{for all}\ \mu,\nu\in\zeta\ \textit{with}\ \mu\neq\nu, \tag{2.50}$$

where

$$M_1(\mu,\nu)=\max\Bigg\{\eta(P\mu,Q\nu),\eta(P\mu,S\mu),\eta(Q\nu,R\nu),\frac{\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{2},\frac{\eta(P\mu,R\nu)\eta(S\mu,Q\nu)}{1+\eta(P\mu,Q\nu)},\frac{\eta(P\mu,S\mu)\eta(Q\nu,R\nu)}{1+\eta(P\mu,Q\nu)},\frac{1+\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{1+\eta(P\mu,S\mu)+\eta(Q\nu,R\nu)}\eta(P\mu,S\mu)\Bigg\},$$

$$N_1(\mu,\nu)=\min\Bigg\{\eta(P\mu,Q\nu),\eta(P\mu,S\mu),\eta(Q\nu,R\nu),\frac{\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{2},\frac{\eta(P\mu,R\nu)\eta(S\mu,Q\nu)}{1+\eta(P\mu,Q\nu)},\frac{\eta(P\mu,S\mu)\eta(Q\nu,R\nu)}{1+\eta(P\mu,Q\nu)},\frac{1+\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{1+\eta(P\mu,S\mu)+\eta(Q\nu,R\nu)}\eta(P\mu,S\mu)\Bigg\}$$

and $\psi\in\Psi$. Then P,Q,R and S have a unique fixed point in ζ.

Corollary 2.14. *Let (ζ,η) be a complete metric space. Suppose that $P,Q,R,$ $S:\zeta\to\zeta$ satisfy the following hypotheses:*

$$S(\zeta)\subseteq Q(\zeta)\quad\text{and}\quad R(\zeta)\subseteq P(\zeta), \tag{2.51}$$

$$\{P,S\}\ \text{and}\ \{Q,R\}\ \text{are weak compatible pairs}, \tag{2.52}$$

$$\eta(S\mu,R\nu)\le M_1(\mu,\nu)-\psi(N_1(\mu,\nu)),\ \text{for all}\ \mu,\nu\in\zeta\ \text{with}\ \mu\neq\nu, \tag{2.53}$$

where

$$M_1(\mu,\nu)=\max\Bigg\{\eta(P\mu,Q\nu),\eta(P\mu,S\mu),\eta(Q\nu,R\nu),\frac{\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{2},\frac{\eta(P\mu,R\nu)\eta(S\mu,Q\nu)}{1+\eta(P\mu,Q\nu)}\Bigg\},$$

$$N_1(\mu,\nu)=\min\Bigg\{\eta(P\mu,Q\nu),\eta(P\mu,S\mu),\eta(Q\nu,R\nu),\frac{\eta(P\mu,R\nu)+\eta(S\mu,Q\nu)}{2},\frac{\eta(P\mu,R\nu)\eta(S\mu,Q\nu)}{1+\eta(P\mu,Q\nu)}\Bigg\}$$

and $\psi\in\Psi$. Then P,Q,R and S have a unique fixed point in ζ.

Corollary 2.15. *Let (ζ, η) be a complete metric space. Suppose that $P, Q, R, S : \zeta \to \zeta$ satisfy the following hypotheses:*

$$S(\zeta) \subseteq Q(\zeta) \quad \textit{and} \quad R(\zeta) \subseteq P(\zeta), \tag{2.54}$$

$$\{P, S\} \textit{ and } \{Q, R\} \textit{ are weak compatible pairs}, \tag{2.55}$$

$$\eta(S\mu, R\nu) \leq M_1(\mu, \nu) - \psi(N_1(\mu, \nu)), \textit{ for all } \mu, \nu \in \zeta \textit{ with } \mu \neq \nu, \tag{2.56}$$

where

$$\begin{aligned} M_1(\mu,\nu) = \max\Bigg\{ & \eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \\ & \frac{\eta(S\mu, P\mu) + \eta(Q\nu, R\nu)}{2}, \frac{\eta(P\mu, S\mu)\eta(Q\nu, R\nu)}{1 + \eta(P\mu, Q\nu)}, \\ & \frac{1 + \eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{1 + \eta(P\mu, S\mu) + \eta(Q\nu, R\nu)}\eta(P\mu, S\mu) \Bigg\}, \end{aligned}$$

$$\begin{aligned} N_1(\mu,\nu) = \min\Bigg\{ & \eta(P\mu, Q\nu), \eta(P\mu, S\mu), \eta(Q\nu, R\nu), \frac{\eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{2}, \\ & \frac{\eta(S\mu, P\mu) + \eta(Q\nu, R\nu)}{2}, \frac{\eta(P\mu, S\mu)\eta(Q\nu, R\nu)}{1 + \eta(P\mu, Q\nu)}, \\ & \frac{1 + \eta(P\mu, R\nu) + \eta(S\mu, Q\nu)}{1 + \eta(P\mu, S\mu) + \eta(Q\nu, R\nu)}\eta(P\mu, S\mu) \Bigg\} \end{aligned}$$

and $\psi \in \Psi$. Then P, Q, R and S have a unique fixed point in ζ.

Example 2.16. Let $H = [0, 2]$ be endowed with Euclidean metric $\eta(h_1, h_2) = |h_1 - h_2|$. P, Q, R and $S : H \to H$ be defined by

$$S(h) = \begin{cases} 0 & \text{if } h = 0 \\ h/5 & \text{otherwise} \end{cases}, \quad R(h) = \begin{cases} 0 & \text{if } h = 0 \\ 2h/5 & \text{otherwise} \end{cases},$$

$$P(h) = \begin{cases} 0 & \text{if } h = 0 \\ 3h/5 & \text{otherwise} \end{cases}, \quad Q(h) = \begin{cases} 0 & \text{if } h = 0 \\ 4h/5 & \text{otherwise} \end{cases},$$

where $h_1, h_2 \in H$, $S(H) = [0, \frac{2}{5}]$, $S(H) = [0, \frac{4}{5}]$, $P(H) = [0, \frac{6}{5}]$, $Q(H) = [0, \frac{8}{5}]$. Here $S(H) \subseteq Q(H)$ and $R(H) \subseteq P(H)$, (P, S) and (Q, R) are weakly compatible maps at $h = 0$ but not compatible.
Take

$$\varphi(t) = t \quad \text{and} \quad \psi(t) = \begin{cases} t/2 & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

To apply Theorem 2.1 we need to check the inequality of the theorem for the various cases.

Case (i) If $h_1 = 0$ and $h_2 = 0$,

$\varphi(\eta(Sh_1, Rh_2)) = \varphi(|Sh_2 - Rh_2|) = 0,\ \varphi(M(h_1, h_2)) = 0$ and $\psi(N(h_1, h_2)) = 0$,

hence

$$\varphi(\eta(Sh_1, Rh_2)) = \varphi(M(h_1, h_2)) - \psi(N(h_1, h_2)).$$

Case (ii) If $h_1 = 0$ and $h_2 \neq 0$,

$$\varphi(\eta(Sh_1, Rh_2)) = \varphi(|Sh_1 - Rh_2|) = \varphi(|0 - 2h_2/5|) = \varphi(|2h_2/5|) = |2h_2/5|$$

and

$$\begin{aligned} &M(h_1, h_2) \\ &= \max\left\{|Ph_1 - Qh_2|, |Ph_1 - Sh_2|, |Qh_2 - Rh_2|, \frac{|Ph_1 - Rh_2| + |Sh_1 - Qh_2|)}{2},\right. \\ &\qquad \left.\frac{|Ph_1 - Sh_1||Qh_2 - Rh_2|}{1 + |Ph_1 - Qh_2|}, \frac{1 + |Ph_1 - Rh_2| + |Sh_1 - Qh_2|}{1 + |Ph_1 - Sh_1| + |Qh_2 - Rh_2|}|Ph_1 - Sh_1|\right\} \\ &= \max\left\{|\frac{4h_2}{5}|, 0, |\frac{4h_2}{5} - \frac{2h_2}{5}|, \frac{|\frac{2h_2}{5}| + |\frac{4h_2}{5}|}{2}, 0, 0\right\} = |\frac{4h_2}{5}| \end{aligned}$$

and

$$\begin{aligned} &N(h_1, h_2) \\ &= \min\left\{|Ph_1 - Qh_2|, |Ph_1 - Sh_2|, |Qh_2 - Rh_2|, \frac{|Ph_1 - Rh_2| + |Sh_1 - Qh_2|}{2},\right. \\ &\qquad \left.\frac{|Ph_1 - Sh_1||Qh_2 - Rh_2|}{1 + |Ph_1 - Qh_2|}, \frac{1 + |Ph_1 - Rh_2| + |Sh_1 - Qh_2|}{1 + |Ph_1 - Sh_1| + |Qh_2 - Rh_2|}|Ph_1 - Sh_1|\right\} \\ &= \min\left\{|\frac{4h_2}{5}|, 0, |\frac{4h_2}{5} - \frac{2h_2}{5}|, \frac{|\frac{2h_2}{5}| + |\frac{4h_2}{5}|}{2}, 0, 0\right\} = 0, \end{aligned}$$

hence

$$\varphi(M(h_1, h_2)) - \psi(N(h_1, h_2)) = |\frac{4h_2}{5}| \geq |\frac{2h_2}{5}| = \varphi(\eta(Sh_1, Rh_2)).$$

Case (iii) If $h_1 \neq 0$ and $h_2 = 0$,

$$\varphi(\eta(Sh_1, Rh_2)) = \varphi(|Sh_1 - Rh_2|) = \varphi(|h_1/5 - 0|) = \varphi(|h_1/5|) = |h_1/5|,$$

$$
\begin{aligned}
&M(h_1,h_2)\\
&=\max\Big\{|Ph_1-Qh_2|,|Ph_1-Sh_2|,|Qh_2-Rh_2|,\frac{|Ph_1-Rh_2|+|Sh_1-Qh_2|}{2},\\
&\qquad\frac{|Ph_1-Sh_1||Qh_2-Rh_2|}{1+|Ph_1-Qh_2|},\frac{1+|Ph_1-Rh_2|+|Sh_1-Qh_2|}{1+|Ph_1-Sh_1|+|Qh_2-Rh_2|}|Ph_1-Sh_1|,\Big\}\\
&=\max\Big\{|\frac{3h_1}{5}|,|\frac{3h_1}{5}-\frac{h_1}{5}|,0,\frac{|\frac{3h_1}{5}|+|\frac{h_1}{5}|}{2},0,\frac{1+|\frac{3h_1}{5}|+|\frac{h_1}{5}|}{1+|\frac{3h_1}{5}-\frac{h_1}{5}|+0}|\frac{3h_1}{5}-\frac{h_1}{5}|\Big\}\\
&=|\frac{3h_1}{5}|
\end{aligned}
$$

and

$$
\begin{aligned}
&N(h_1,h_2)\\
&=\min\Big\{|Ph_1-Qh_2|,|Ph_1-Sh_2|,|Qh_2-Rh_2|,\frac{|Ph_1-Rh_2|+|Sh_1-Qh_2|}{2},\\
&\qquad\frac{|Ph_1-Sh_1||Qh_2-Rh_2|}{1+|Ph_1-Qh_2|},\frac{1+|Ph_1-Rh_2|+|Sh_1-Qh_2|}{1+|Ph_1-Sh_1|+|Qh_2-Rh_2|}|Ph_1-Sh_1|\Big\}\\
&=\min\Big\{|\frac{3h_1}{5}|,|\frac{3h_1}{5}-\frac{h_1}{5}|,0,\frac{|\frac{3h_1}{5}|+|\frac{h_1}{5}|}{2},0,\frac{1+|\frac{3h_1}{5}|+|\frac{h_1}{5}|}{1+|\frac{3h_1}{5}-\frac{h_1}{5}|+0}|\frac{3h_1}{5}-\frac{h_1}{5}|\Big\}\\
&=0,
\end{aligned}
$$

hence

$$\varphi(M(h_1,h_2))-\psi(N(h_1,h_2))=|\frac{3h_1}{5}|\geq|\frac{h_1}{5}|=\varphi(\eta(Sh_1,Rh_2)).$$

Case (iv) If $h_1\neq 0$ and $h_2\neq 0$,

$$\varphi(\eta(Sh_1,Rh_2))=\varphi(|Sh_1-Rh_2|)=\varphi(|h_1/5-2h_2/5|)=|h_1/5-2h_2/5|$$

and

$$
\begin{aligned}
&M(h_1,h_2)\\
&=\max\Big\{|Ph_1-Qh_2|,|Ph_1-Sh_2|,|Qh_2-Rh_2|,\frac{|Ph_1-Rh_2|+|Sh_1-Qh_2|}{2},\\
&\qquad\frac{|Ph_1-Sh_1||Qh_2-Rh_2|}{1+|Ph_1-Qh_2|},\frac{1+|Ph_1-Rh_2|+|Sh_1-Qh_2|}{1+|Ph_1-Sh_1|+|Qh_2-Rh_2|}|Ph_1-Sh_1|\Big\}
\end{aligned}
$$

$$= \max\Big\{|\frac{3h_1}{5}-\frac{4h_2}{5}|, |\frac{3h_1}{5}-\frac{h_1}{5}|, |\frac{4h_2}{5}-\frac{2h_2}{5}|, \frac{|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{2},$$

$$\frac{|\frac{3h_1}{5}-\frac{h_1}{5}||\frac{4h_2}{5}-\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|}, \frac{|1+\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{h_1}{5}|+|\frac{4h_2}{5}-\frac{2h_2}{5}|}|\frac{3h_1}{5}-\frac{h_1}{5}|\Big\}$$

$$= \max\Big\{|\frac{3h_1}{5}-\frac{4h_2}{5}|, |\frac{2h_1}{5}|, |\frac{2h_2}{5}|, \frac{|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{2},$$

$$\frac{|\frac{2h_1}{5}||\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|}, \frac{1+|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{1+|\frac{2h_1}{5}|+|\frac{2h_2}{5}|}|\frac{2h_1}{5}|\Big\}$$

$$= \begin{cases} |\frac{3h_1}{5}-\frac{4h_2}{5}| & \text{if} \quad 0<h_2<\frac{2h_1}{5}, \\ \frac{1+|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{1+|\frac{2h_1}{5}|+|\frac{2h_2}{5}|}|\frac{2h_1}{5}| & \text{if} \quad h_2=\frac{2h_1}{5}, \\ \frac{|\frac{2h_1}{5}||\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|} & \text{if} \quad \frac{2h_1}{5}<h_2<h_1, \end{cases}$$

$N(h_1, h_2)$

$$= \min\Big\{|Ph_1-Qh_2|, |Ph_1-Sh_2|, |Qh_2-Rh_2|, \frac{|Ph_1-Rh_2|+|Sh_1-Qh_2|)}{2},$$

$$\frac{|Ph_1-Sh_1||Qh_2-Rh_2|}{1+|Ph_1-Qh_2|}, \frac{1+|Ph_1-Rh_2|+|Sh_1-Qh_2|}{1+|Ph_1-Sh_1|+|Qh_2-Rh_2|}|Ph_1-Sh_1|\Big\}$$

$$= \min\Big\{|\frac{3h_1}{5}-\frac{4h_2}{5}|, |\frac{3h_1}{5}-\frac{h_1}{5}|, |\frac{4h_2}{5}-\frac{2h_2}{5}|, \frac{|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{2},$$

$$\frac{|\frac{3h_1}{5}-\frac{h_1}{5}||\frac{4h_2}{5}-\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|}, \frac{|1+\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{h_1}{5}|+|\frac{4h_2}{5}-\frac{2h_2}{5}|}|\frac{3h_1}{5}-\frac{h_1}{5}|\Big\}$$

$$= \min\Big\{|\frac{3h_1}{5}-\frac{4h_2}{5}|, |\frac{2h_1}{5}|, |\frac{2h_2}{5}|, \frac{|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{2},$$

$$\frac{|\frac{2h_1}{5}||\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|}, \frac{1+|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{1+|\frac{2h_1}{5}|+|\frac{2h_2}{5}|}|\frac{2h_1}{5}|\Big\}$$

$$= \begin{cases} \frac{|\frac{2h_1}{5}||\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|} & \text{if} \quad 0<h_2<\frac{2h_1}{5}, \\ |\frac{2h_1}{5}| & \text{if} \quad h_2=\frac{2h_1}{5}, \\ |\frac{3h_1}{5}-\frac{4h_2}{5}| & \text{if} \quad \frac{2h_1}{5}<h_2<h_1, \end{cases}$$

therefore

$$\varphi(M(h_1,h_2))-\psi(N(h_1,h_2))$$
$$=\begin{cases}\varphi(|\frac{3h_1}{5}-\frac{4h_2}{5}|)-\psi(\frac{|\frac{2h_1}{5}||\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|}) & \text{if} \quad 0<h_2<\frac{2h_1}{5},\\ \varphi(\frac{1+|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{1+|\frac{2h_1}{5}|+|\frac{2h_2}{5}|}|\frac{2h_1}{5}|)-\psi(|\frac{2h_1}{5}|) & \text{if} \quad h_2=\frac{2h_1}{5},\\ \varphi(\frac{|\frac{2h_1}{5}||\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|})-\psi(|\frac{3h_1}{5}-\frac{4h_2}{5}|) & \text{if} \quad \frac{2h_1}{5}<h_2<h_1,\end{cases}$$

$$\varphi(M(h_1,h_2))-\psi(N(h_1,h_2))$$
$$=\begin{cases}|\frac{3h_1}{5}-\frac{4h_2}{5}|-\frac{1}{2}\frac{|\frac{2h_1}{5}||\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|} & \text{if} \quad 0<h_2<\frac{2h_1}{5},\\ \frac{1+|\frac{3h_1}{5}-\frac{2h_2}{5}|+|\frac{h_1}{5}-\frac{4h_2}{5}|}{1+|\frac{2h_1}{5}|+|\frac{2h_2}{5}|}|\frac{2h_1}{5}|-\frac{1}{2}|\frac{2h_1}{5}| & \text{if} \quad h_2=\frac{2h_1}{5},\\ \frac{|\frac{2h_1}{5}||\frac{2h_2}{5}|}{1+|\frac{3h_1}{5}-\frac{4h_2}{5}|}-\frac{1}{2}|\frac{3h_1}{5}-\frac{4h_2}{5}| & \text{if} \quad \frac{2h_1}{5}<h_2<h_1.\end{cases}$$

Thus

$$\varphi(\eta(Sh_1,Rh_2))\le\varphi(M(h_1,h_2))-\psi(N(h_1,h_2)).$$

So the inequalities hold in each of the cases. Hence in all the cases condition of Theorem 2.1 holds and hence P,Q,R and S have the unique common fixed point at $h=0$ in H.

3. Conclusion

Our results generalize the results of Abbas and Dorić [1] as we do not require the closedness of mappings $P(\zeta)$ or $Q(\zeta)$ or $R(\zeta)$ or $S(\zeta)$. Also, the continuity of self mappings P,Q,R and S is not required in the theorems proved.

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