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BLOW UP OF SOLUTIONS OF NONLINEAR PARABOLIC SYSTEM WITH VARIABLE EXPONENTS

A. Gurusamy[1], L. Shangerganesh[2] and K. Balachandran[3]

[1]DRDO-BU Center for Life Sciences, Bharathiar University Campus
Coimbatore - 641046, India
e-mail: guru.poy@gmail.com

[2]Department of Humanities and Sciences
National Institute of Technology Goa, Goa - 403401, India
e-mail: shangerganesh@gmail.com

[3]Department of Mathematics, Bharathiar University
Coimbatore - 641046, India
e-mail: kb.maths.bu@gmail.com

Abstract. This paper deals with the existence of solutions and the blow up of solutions of nonlinear parabolic systems with Dirichlet boundary conditions. We prove the existence of solutions using contraction principle and we establish that the corresponding solutions blow up at a finite time using the eigenfunction argument.

1. Introduction

During the past few years, many works have been devoted to the study of blow up properties of solutions to degenerate parabolic equations and systems with homogeneous Dirichlet boundary conditions, for example, see [1, 2, 3, 8, 9, 11, 20, 22, 25] and also the references therein. On the other hand, there are some important physical phenomena formulated as parabolic equations which are coupled to nonlocal boundary conditions in mathematical modeling, for example, thermo elasticity [19, 21] and also see the references therein.

Kim and Lin [11] studied the existence of solutions of the parabolic system applicable to ecology but in this work we consider the same kind of parabolic system with nonlocal nonlinearities and variable exponents. This deals with

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the existence and blow up of solutions of the following nonlinear parabolic system with nonlocal nonlinearities and variable exponents described as:

$$\begin{cases} u_t = \Delta u + \displaystyle\int_\Omega u^{p(x)} \left(a_1(x) - b_1(x)u + c_1(x)v\right) \mathrm{d}x \ \text{ in } \ \Omega_T, \\ v_t = \Delta v + \displaystyle\int_\Omega v^{q(x)} \left(a_2(x) + b_2(x)u - c_2(x)v\right) \mathrm{d}x \ \text{ in } \ \Omega_T, \\ u(x,0) = u_0(x), \ \ v(x,0) = v_0(x) \ \text{ in } \ \Omega, \\ u(x,t) = 0, \ \ v(x,t) = 0 \ \text{ on } \ \Sigma_T, \end{cases} \tag{1.1}$$

where $\Omega_T = \Omega \times (0,T], \Sigma_T = \partial\Omega \times [0,T]$. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $0 < T < \infty$.

As far as the blow up of solutions for parabolic equations with nonlocal nonlinearities is concerned, the literature is very vast. For example, see [1, 2, 14, 15, 20, 22, 26] and also see the references therein. In particular, for variable exponent problems, Antontsev and Shmarev [1, 2] established the blow up of solutions for parabolic equations with nonlinear diffusion and also recently Li and Xie [14, 15] studied the blow up of solutions for p-Laplacian type equations with nonlocal nonlinearities. Moreover Pinasco [20] proved the existence and blow up of solutions for parabolic and hyperbolic problems with variable exponents. Samarskii et al. [22] established the blow up of solutions of quasilinear parabolic equations and Tsutsumi [25] studied the existence and nonexistence of global solutions for nonlinear parabolic equations. Further Winkler [26] proved the blow up of solutions of a degenerate parabolic equation, not in divergence form. On the other hand, apart from the literature mentioned above for the system of partial differential equations, that is, regarding the blow up of solutions of parabolic system in particular only few articles appeared; for example, the readers can see, Chen and Wang [4], Chunlai et al. [5], Deng [6], Deng et al. [7], Lei and Zheng [12], Li et al. [13], Li [16], Lu and Wang [17], Payne and Schaefer [18] and Pao [19]. We remark that as for the existence and blow up of solutions of parabolic system with variable exponents is concerned, very few articles appeared in the literature [10]. Shangerganesh and Balachandran [23] considered the predator–prey model in $\mathbb{R}^3$ with mixed boundary conditions on the Lipschitz boundary and prove the existence of solutions by Schauders fixed point theorem and uniqueness of solutions by Gronwalls lemma. Further Shangerganesh et al. proved existence of weak-renormalized solutions to the predator-prey system under the suitable assumptions of no growth conditions and integrable data in [24]. Contrary to the above mentioned works, our paper deals with the existence and blow up properties of strongly coupled nonlinear parabolic system related to biological problems with nonlocal nonlinearities and variable exponents.

We briefly summarize the contents of the paper: In Section 2, we give the definition of blow up of solutions and we prove the existence of solutions of the given parabolic system (1.1). In Section 3, we state and prove the main theorem, that is, the result related to blow up phenomena of solutions of the system (1.1).

2. Existence of solutions

In this section, first we define the blow up of solutions and then we state and prove the global existence of solutions of given parabolic system (1.1) using Banach's contraction principle.

Before establishing our main results, we introduce certain assumptions on the exponents $p(x), q(x) : \Omega \to (1, +\infty)$, and the continuous functions $a_i(x)$, $b_i(x), c_i(x) : \Omega \to \mathbb{R}, (i = 1, 2)$ as follows:

$$\begin{cases} 1 < p^- \leq p(x) \leq p^+ < +\infty, \quad 1 < q^- \leq q(x) \leq q^+ < +\infty, \\ 0 < c_{ia} \leq a_i(x) \leq C_{ia} < +\infty, \quad 0 < c_{ib} \leq b_i(x) \leq C_{ib} < +\infty, \\ 0 < c_{ic} \leq c_i(x) \leq C_{ic} < +\infty, \end{cases} \tag{2.1}$$

where $i = 1, 2$. We introduce the space $X := C(\overline{\Omega}_T) \cap C^{1,2}(\Omega_T)$ and $X_m := \{(u, v) \in C(\overline{\Omega}_T) \cap C^{1,2}(\Omega_T) : \|(u, v)\|_\infty \leq m\}$, where $\Omega_T = \Omega \times (0, T], m > M_0$ is a fixed positive constant and $M_0 = \|(u_0(x), v_0(x))\|_\infty$.

Definition 2.1. Suppose there exists a solution pair (u, v) for the system (1.1) then we say that the pair (u, v) blows up at finite time if there exists an instant $T_f < \infty$ such that

$$\|(u, v)\| \to \infty \quad \text{as} \quad t \to T_f,$$

where $\|(u, v)\| = \sup_{t \in [0,T)} \{\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty\}$.

Theorem 2.2. *Suppose that the exponents $p(x), q(x)$ and the continuous functions $a_i(x)$, $b_i(x)$, $c_i(x), i = 1, 2$ are satisfy the assumption* (2.1)*, then there exists a unique solution pair (u, v) to the system* (1.1)*.*

Proof. We rewrite the given parabolic system (1.1) as the following equivalent integral equations

$$\begin{aligned} u(x,t) &= \int_\Omega G(x,z,t)u_0(z)\mathrm{d}z \\ &\quad + \int_0^t \int_\Omega G(x,z,\tau-s)\left(\int_\Omega u^{p(y)}\left(a_1(y)-b_1(y)u+c_1(y)v\right)\mathrm{d}y\right)\mathrm{d}z\mathrm{d}s, \end{aligned} \tag{2.2}$$

$$\begin{aligned} v(x,t) &= \int_\Omega G(x,z,t)v_0(z)\mathrm{d}z \\ &\quad + \int_0^t \int_\Omega G(x,z,\tau-s)\left(\int_\Omega v^{q(y)}\left(a_2(y)+b_2(y)u-c_2(y)v\right)\mathrm{d}y\right)\mathrm{d}z\mathrm{d}s, \end{aligned} \tag{2.3}$$

where $G(x,z,t)$ is the Green function.

To prove the existence and uniqueness of solutions of (1.1) by using the fixed point argument, we inductively define

$$\begin{cases} u_{n_1}(x,t) &= 0, \\ u_{n_1+1}(x,t) &= \displaystyle\int_\Omega G(x,z,t)u_{n_0}(z)\mathrm{d}z + \int_0^t \int_\Omega G(x,z,\tau-s) \\ & \quad \times \displaystyle\left(\int_\Omega u_{n_1}^{p(y)}\left(a_1(y)-b_1(y)u_{n_1}+c_1(y)v_{n_1}\right)\mathrm{d}y\right)\mathrm{d}z\mathrm{d}s. \end{cases} \tag{2.4}$$

Similarly we also define

$$\begin{cases} v_{n_1}(x,t) &= 0, \\ v_{n_1+1}(x,t) &= \displaystyle\int_\Omega G(x,z,t)v_{n_0}(z)\mathrm{d}z + \int_0^t \int_\Omega G(x,z,\tau-s) \\ & \quad \times \displaystyle\left(\int_\Omega v_{n_1}^{q(y)}\left(a_2(y)+b_2(y)u_{n_1}-c_2(y)v_{n_1}\right)\mathrm{d}y\right)\mathrm{d}z\mathrm{d}s. \end{cases} \tag{2.5}$$

We define the mapping $\Gamma_1 : X_m \times X_m \to X_m \times X_m$ by

$$\begin{aligned} &\Gamma_1(u_{n_1}, v_{n_1}) \\ &= \int_0^t \int_\Omega G(x,z,\tau-s)\left(\int_\Omega u_{n_1}^{p(y)}(a_1(y)-b_1(y)u_{n_1}+c_1(y)v_{n_1})\mathrm{d}y\right)\mathrm{d}z\mathrm{d}s. \end{aligned} \tag{2.6}$$

Now we want to prove that Γ_1 is contraction in X_m. We first note that, see [20], for any $x \in \Omega$ fixed, we have

$$u^{p(x)} - v^{p(x)} = p(x)w^{p(x)-1}(u-v), \tag{2.7}$$

with $w = su + (1-s)v,\ s \in (0,1)$. Although s depends on x, we always have

$$\|p(x)w^{p(x)-1}(u-v)\|_\infty \le p^+(2M)^{p^+-1}\|u-v\|_\infty. \tag{2.8}$$

Now we define $\mu(t)$ as in the following form:

$$\mu(t) = \sup_{x\in\overline{\Omega}, 0\le\tau<t} \int_0^\tau \int_\Omega G(x,z,t-s)\mathrm{d}z\mathrm{d}s. \tag{2.9}$$

Clearly it is easy to understand that from the definition of $\mu(t)$, $\mu(t) \to 0$ when $t \to 0^+$. Now

$$\begin{aligned}
&\Gamma_1(u_{n_1}, v_{n_1}) - \Gamma_1(u_{n_2}, v_{n_2}) \\
&= \int_0^t \int_\Omega G(x,z,\tau-s) \left[\int_\Omega a_1(y)(u_{n_1}^{p(y)} - u_{n_2}^{p(y)})\mathrm{d}y \right. \\
&\quad \left. - \int_\Omega b_1(y) \left(u_{n_1}^{p(y)+1} - u_{n_2}^{p(y)+1} \right) \mathrm{d}y + \int_\Omega c_1(y)(u_{n_1}^{p(y)} v_{n_1} - u_{n_2}^{p(y)} v_{n_2})\mathrm{d}y \right] \mathrm{d}z\mathrm{d}s.
\end{aligned}$$

By using (2.1) and (2.7)-(2.9), we have to prove that Γ_1 is contraction in X_m, that is, there exists a $k<1$ such that

$$\|\Gamma_1(u_{n_1}, v_{n_1}) - \Gamma_1(u_{n_2}, v_{n_2})\|_\infty \le k\left(\|u_{n_1} - u_{n_2}\|_\infty + \|v_{n_1} - v_{n_2}\|_\infty\right),$$

for every $(u_{n_i}, v_{n_i}) \in X_m \times X_m$. Now

$$\begin{aligned}
&\|\Gamma_1(u_{n_1}, v_{n_1}) - \Gamma_1(u_{n_2}, v_{n_2})\|_\infty \\
&= \left\| \int_0^t \int_\Omega G(x,z,\tau-s) \left[\int_\Omega a_1(y)(u_{n_1}^{p(y)} - u_{n_2}^{p(y)})\mathrm{d}y \right. \right. \\
&\quad - \int_\Omega b_1(y)(u_{n_1}^{p(y)+1} - u_{n_2}^{p(y)+1})\mathrm{d}y + \int_\Omega c_1(y) u_{n_1}^{p(y)}(v_{n_1} - v_{n_2})\mathrm{d}y \\
&\quad \left. \left. + \int_\Omega c_1(y) v_{n_2}(u_{n_1}^{p(y)} - u_{n_2}^{p(y)})\mathrm{d}y \right] \mathrm{d}z\mathrm{d}s \right\|_\infty \\
&\le \mu(t) \Big[C\ \|p^+\|_\infty (2m)^{p^+-1} \|u_{n_1} - u_{n_2}\|_\infty |\Omega|_n \\
&\quad + C\|p^+ + 1\|_\infty (2m)^{p^+} \|u_{n_1} - u_{n_2}\|_\infty |\Omega|_n \\
&\quad + Cm|\Omega|_n \|v_{n_1} - v_{n_2}\|_\infty + Cm\|p^+\|_\infty (2m)^{p^+-1} |\Omega|_n \|u_{n_1} - u_{n_2}\|_\infty \Big] \\
&\le \mu(t) \Big[C|\Omega|_n \|p^+\|_\infty (2m)^{p^+-1} + C\|p^+ + 1\|_\infty (2m)^{p^+} |\Omega|_n \\
&\quad + Cm\|p^+\|_\infty (2m)^{p^+-1} |\Omega|_n \Big] \|u_{n_1} - u_{n_2}\|_\infty \\
&\quad + \mu(t)\left[Cm|\Omega|_n\right] \|v_{n_1} - v_{n_2}\|_\infty \\
&\le \mu(t)\alpha_1 \|u_{n_1} - u_{n_2}\|_\infty + \mu(t)\alpha_2 \|v_{n_1} - v_{n_2}\|_\infty,
\end{aligned}$$

where the constants α_1 and α_2 depend only on the given data $|\Omega|_n$. Therefore

$$\|\Gamma_1(u_{n_1}, v_{n_1}) - \Gamma_1(u_{n_2}, v_{n_2})\|_\infty \le \mu(t)\alpha \|u_{n_1} - u_{n_2}\|_\infty + \mu(t)\beta \|v_{n_1} - v_{n_2}\|_\infty.$$

Then, for sufficiently small $\mu(t), 0 \leq t \leq \delta$ shows that Γ_1 is contraction on X_m. Similarly define Γ_2 by

$$\begin{aligned}&\Gamma_2(u_{n_1}, v_{n_1})\\&= \int_0^t \int_\Omega G(x,z,\tau - s)\left(\int_\Omega v_{n_1}^{q(y)}\left(a_2(y) + b_2(y)u_{n_1} - c_2(y)v_{n_1}\right)\mathrm{d}y\right)\mathrm{d}z\mathrm{d}s\end{aligned}$$

and to prove that Γ_2 is a contraction, proceeding as in the previous argument, we get

$$\begin{aligned}&\|\Gamma_2(u_{n_1}, v_{n_1}) - \Gamma_2(u_{n_2}, v_{n_2})\|_\infty\\&\leq \mu(t)\beta_1\|v_{n_1} - v_{n_2}\|_\infty + \mu(t)\beta_2\|u_{n_1} - u_{n_2}\|_\infty,\end{aligned} \tag{2.10}$$

where the constants β_1 and β_2 depend only on the given data $|\Omega|_n$. Then, for sufficiently small $\mu(t), 0 \leq t \leq \delta$, Γ_2 is contraction on X_m. Therefore, by using Banach contraction principle, there exists a unique fixed point of Γ_1 and Γ_2 in X_m. Hence the given parabolic system (1.1) has a unique solution in X_m. □

3. Blow up of solutions

In this section, we establish the blow up of solutions of the given parabolic system(1.1) using some technical lemma and eigenfunction argument.

Lemma 3.1. ([20]) *Let $y(t)$ be the solution of $y'(t) \geq cy^r(t)$, $y(0) > 0$, where $r > 1$ and $c > 0$. Then $y(t)$ can not be globally defined and*

$$y(t) \geq \left(y(0)^{1-r} - \frac{r-1}{c}t\right)^{-1/(r-1)}.$$

Theorem 3.2. *Suppose there exists a solution pair (u, v) for the given parabolic system (1.1) and the assumptions (2.1) hold true. Then, for sufficiently large initial data (u_0, v_0), there exists a finite time $T_f > 0$ such that*

$$\sup_{0\leq t\leq T_f}\left(\|u(x,t)\|_{L^\infty(\Omega)} + \|v(x,t)\|_{L^\infty(\Omega)}\right) = +\infty.$$

Proof. Let λ_1 be the first eigenvalue of the Laplacian in Ω with zero Dirichlet boundary conditions, $-\Delta\phi = \lambda_1\phi$ and let ϕ be the corresponding eigenfunction. Choose the eigenfunction ϕ positive on Ω and also assume that $\int_\Omega \phi(x)\,\mathrm{d}x = 1$. Now we introduce the auxilary functions $\eta(t)$ and $\zeta(t)$ such that

$$\eta(t) = \int_\Omega u(x,t)\phi(x)\mathrm{d}x \quad \text{and} \quad \zeta(t) = \int_\Omega v(x,t)\phi(x)\mathrm{d}x. \tag{3.1}$$

Then

$$\eta'(t) = \int_\Omega u_t(x,t)\phi(x)\mathrm{d}x,$$
$$= -\lambda_1\eta(t) + \int_\Omega \phi(x)\left(\int_\Omega u^{p(y)}\left(a_1(y) - b_1(y)u + c_1(y)v\right)\mathrm{d}y\right)\mathrm{d}x. \quad (3.2)$$

For each $t > 0$, we divide the domain Ω into two sets, that is,

$$\Omega\{<1\} = \{x \in \Omega : (u(x,t), v(x,t)) < 1\} \quad (3.3)$$

and

$$\Omega\{\geq 1\} = \{x \in \Omega : (u(x,t), v(x,t)) \geq 1\}. \quad (3.4)$$

Now (3.4) follows that

$$\eta'(t) \geq -\lambda_1\eta(t) + \int_\Omega \phi(x)a_1(x)\mathrm{d}x \int_\Omega u^{p(y)}\mathrm{d}y - \int_\Omega b_1(x)\phi(x)\mathrm{d}x \int_\Omega u^{p(y)+1}\mathrm{d}y$$
$$+ \int_\Omega c_1(x)\phi(x)\mathrm{d}x \int_\Omega u^{p(y)}v\mathrm{d}y. \quad (3.5)$$

By using equations (3.4) and (2.1) we obtain

$$\eta'(t) \geq -\lambda_1\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\int_\Omega u^{p(y)}\phi(y)\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_\Omega u^{p(y)+1}\phi(y)\mathrm{d}y$$
$$+ \frac{c_{1c}}{\|\phi\|_\infty}\int_\Omega u^{p(y)}v\phi(y)\mathrm{d}y$$
$$\geq -\lambda_1\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)}\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)+1}\phi(y)\mathrm{d}y$$
$$+ \frac{c_{1c}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)}v\phi(y)\mathrm{d}y + \frac{c_{1a}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)}\mathrm{d}y$$
$$- \frac{C_{1b}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)+1}\phi(y)\mathrm{d}y + \frac{c_{1c}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)}v\phi(y)\mathrm{d}y$$
$$\geq -\lambda_1\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)}\phi)(y)\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)+1}\phi(y)\mathrm{d}y$$
$$+ \frac{c_{1c}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)}v\phi(y)\mathrm{d}y$$
$$\geq -\lambda_1\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)}\phi)(y)\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)+1}\phi(y)\mathrm{d}y$$

$$
\begin{aligned}
&+\frac{c_{1c}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} u^{p(y)} v\phi(y)\mathrm{d}y + \Bigg[\frac{c_{1a}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)}\mathrm{d}y \\
&-\frac{C_{1b}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)+1}\phi(y)\mathrm{d}y + \frac{c_{1c}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)} v\phi(y)\mathrm{d}y\Bigg] \\
&-\Bigg[\frac{c_{1a}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)}\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)+1}\phi(y)\mathrm{d}y \\
&+\frac{c_{1c}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)} v\phi(y)\mathrm{d}z\Bigg] \\
\geq &-\lambda_1\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\int_\Omega u^{p(y)}\phi)(y)\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_\Omega u^{p(y)+1}\phi(y)\mathrm{d}y \\
&+\frac{c_{1c}}{\|\phi\|_\infty}\int_\Omega u^{p(y)} v\phi(y)\mathrm{d}y - \Bigg[\frac{c_{1a}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)}\mathrm{d}y \\
&-\frac{C_{1b}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)+1}\phi(y)\mathrm{d}y + \frac{c_{1c}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} u^{p(y)} v\phi(y)\mathrm{d}y\Bigg]. \qquad (3.6)
\end{aligned}
$$

Now we using the value of u and v in the domain $\Omega\{<1\}$, we obtain

$$
\begin{aligned}
\eta'(t) \geq &-\lambda_1\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\int_\Omega u^{p(y)}\phi)(y)\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_\Omega u^{p(y)+1}\phi(y)\mathrm{d}y \\
&+\frac{c_{1c}}{\|\phi\|_\infty}\int_\Omega u^{p(y)} v\phi(y)\mathrm{d}y - \Bigg[\frac{c_{1a}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} \phi(y)\mathrm{d}y \\
&-\frac{C_{1b}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} \phi(y)\mathrm{d}y + \frac{c_{1c}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} \phi(y)\mathrm{d}y\Bigg]. \qquad (3.7)
\end{aligned}
$$

We replace smaller domain $\Omega\{<1\}$ by larger domain Ω we obtain

$$
\begin{aligned}
\eta'(t) \geq &-\lambda_1\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\int_\Omega u^{p(y)}\phi)(y)\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_\Omega u^{p(y)+1}\phi(y)\mathrm{d}y \\
&+\frac{c_{1c}}{\|\phi\|_\infty}\int_\Omega u^{p(y)} v\phi(y)\mathrm{d}y - \left[\frac{c_{1a}}{\|\phi\|_\infty} - \frac{C_{1b}}{\|\phi\|_\infty} + \frac{c_{1c}}{\|\phi\|_\infty}\right]. \qquad (3.8)
\end{aligned}
$$

Here we may choose the constant $\Gamma_1 = \frac{c_{1a}}{\|\phi\|_\infty} - \frac{C_{1b}}{\|\phi\|_\infty} + \frac{c_{1c}}{\|\phi\|_\infty}$, then

$$
\begin{aligned}
\eta'(t) \geq &-\lambda_1\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\int_\Omega u^{p(y)}\phi)(y)\mathrm{d}y - \frac{C_{1b}}{\|\phi\|_\infty}\int_\Omega u^{p(y)+1}\phi(y)\mathrm{d}y \\
&+\frac{c_{1c}}{\|\phi\|_\infty}\int_\Omega u^{p(y)} v\phi(y)\mathrm{d}y - \Gamma_1. \qquad (3.9)
\end{aligned}
$$

Using the assumptions on the exponents $p(x)$ and Jensen's inequality in (3.9), we can get

$$\eta'(t) \geq -\lambda_1 \eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\eta^{p^-}(t) - \frac{C_{1b}}{\|\phi\|_\infty}\eta^{p^+ +1}(t) - \Gamma_1. \tag{3.10}$$

We now will estimate (3.10). Consider

$$-\frac{\lambda_1}{2}\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\eta^{p^-}(t) - \frac{\Gamma_1}{2}. \tag{3.11}$$

Thus

$$-\frac{\lambda_1}{2}\eta(t) + \frac{c_{1a}}{\|\phi\|_\infty}\eta^{p^-}(t) - \frac{\Gamma_1}{2} \geq c_1/2\eta^{p^-}(t), \tag{3.12}$$

where $c_1 = \frac{c_{1a}}{\|\phi\|_\infty}$. Next consider

$$-\frac{\lambda_1}{2}\eta(t) - \frac{C_{1b}}{\|\phi\|_\infty}\eta^{p^+ +1}(t) - \frac{\Gamma_1}{2}. \tag{3.13}$$

Thus

$$\begin{aligned} & -\frac{\lambda_1}{2}\eta(t) - \frac{C_{1b}}{\|\phi\|_\infty}\eta^{p^+ +1}(t) - \frac{\Gamma_1}{2} \\ & = -\frac{\lambda_1}{2}\eta(t) + c_2\eta^{p^+ +1}(t) - 2c_2\eta^{p^+ +1}(t) - \frac{\Gamma_1}{2} \\ & \geq \frac{c_2}{2}\eta^{p^+ +1}(t) + 2c_2\eta^{p^+ +1}(t) \geq \frac{5c_2}{2}\eta^{p^+ +1}(t), \end{aligned} \tag{3.14}$$

where $c_2 = \frac{C_{1b}}{\|\phi\|_\infty}$. By using (3.12) and (3.14), we obtain

$$\eta'(t) \geq \frac{C_1}{2}\eta^{p^-}(t) + \frac{5c_2}{2}\eta^{p^+ +1}(t). \tag{3.15}$$

If we choose $\min\{c_1, 5c_2\} = \delta_1/2$ and $\min\{p^-, p^+ + 1\} = p$, Eqn.(3.10) becomes

$$\eta'(t) \geq \frac{\delta_1}{2}\eta^p(t). \tag{3.16}$$

The solution of (3.16) is

$$\eta(t) \geq \left(\eta^{1-p}(0) - \frac{\delta_2}{2}(p-1)t\right)^{\frac{-1}{p-1}}. \tag{3.17}$$

Now by using (3.1) we get

$$\begin{aligned}
\zeta'(t) =& \int_\Omega v_t(x,t)\phi(x)\mathrm{d}x \\
=& -\lambda_1\zeta(t) + \int_\Omega \phi(x)\left(\int_\Omega v^{q(y)}\left(a_2(x)+b_2(x)u - c_2(x)v\right)\mathrm{d}y\right)\mathrm{d}x \\
\geq& -\lambda_1\zeta(t) + \int_\Omega \phi(x)a_2(x)\mathrm{d}x\int_\Omega v^{q(y)}\mathrm{d}y + \int_\Omega b_2(x)\phi(x)\mathrm{d}x\int_\Omega uv^{q(y)}\mathrm{d}y \\
&- \int_\Omega c_2(x)\phi(x)\mathrm{d}x\int_\Omega v^{q(y)}v\mathrm{d}y. \qquad (3.18)
\end{aligned}$$

Now by using (3.4) we can get

$$\begin{aligned}
\zeta'(t) \geq& -\lambda_1\zeta(t) + \frac{c_{2a}}{\|\phi\|_\infty}\int_\Omega v^{q(y)}\phi(y)\mathrm{d}y + \frac{c_{2b}}{\|\phi\|_\infty}\int_\Omega uv^{q(y)}\phi(y)\mathrm{d}y \\
&- \frac{C_{2c}}{\|\phi\|_\infty}\int_\Omega v^{q(y)+1}\phi(y)\mathrm{d}y \\
\geq& -\lambda_1\zeta(t) + \frac{c_{2a}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} v^{q(y)}\phi(y)\mathrm{d}y + \frac{c_{2b}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} uv^{q(y)}\phi(y)\mathrm{d}y \\
&- \frac{C_{2c}}{\|\phi\|_\infty}\int_{\Omega\{\geq 1\}} v^{q(y)+1}\phi(y)\mathrm{d}y + \left[\frac{c_{2a}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} v^{q(y)}\phi(y)\mathrm{d}y\right. \\
&\left. + \frac{c_{2b}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} uv^{q(y)}\phi(y)\mathrm{d}y - \frac{C_{2c}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} v^{q(y)+1}\phi(y)\mathrm{d}y\right] \\
&- \left[\frac{c_{2a}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} v^{q(y)}\phi(y)\mathrm{d}y + \frac{c_{2b}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} uv^{q(y)}\phi(y)\mathrm{d}y\right. \\
&\left. - \frac{C_{2c}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} v^{q(y)+1}\phi(y)\mathrm{d}y\right]. \qquad (3.19)
\end{aligned}$$

Now we using the value of u and v in the domain $\Omega\{<1\}$

$$\begin{aligned}
\zeta'(t) \geq& -\lambda_1\zeta(t) + \frac{c_{2a}}{\|\phi\|_\infty}\int_\Omega v^{q(y)}\phi(y)\mathrm{d}y + \frac{c_{2b}}{\|\phi\|_\infty}\int_\Omega uv^{q(y)}\phi(y)\mathrm{d}y \\
&- \frac{C_{2c}}{\|\phi\|_\infty}\int_\Omega v^{q(y)+1}\phi(y)\mathrm{d}y - \left[\frac{c_{2a}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} \phi(y)\mathrm{d}y\right. \\
&\left. + \frac{c_{2b}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} \phi(y)\mathrm{d}y - \frac{C_{2c}}{\|\phi\|_\infty}\int_{\Omega\{<1\}} \phi(y)\mathrm{d}y\right]. \qquad (3.20)
\end{aligned}$$

By replacing smaller domain $\Omega\{<1\}$ by larger domain Ω, then

$$\zeta'(t) \geq -\lambda_1\zeta(t) + \frac{c_{2a}}{\|\phi\|_\infty}\int_\Omega v^{q(y)}\phi(y)\mathrm{d}y + \frac{c_{2b}}{\|\phi\|_\infty}\int_\Omega uv^{q(y)}\phi(y)\mathrm{d}y$$
$$- \frac{C_{2c}}{\|\phi\|_\infty}\int_\Omega v^{q(y)+1}\phi(y)\mathrm{d}y - \left[\frac{c_{2a}}{\|\phi\|_\infty} + \frac{c_{2b}}{\|\phi\|_\infty} - \frac{C_{2c}}{\|\phi\|_\infty}\right]. \tag{3.21}$$

Here we may choose the constants $\Gamma_2 = \frac{c_{2a}}{\|\phi\|_\infty} + \frac{c_{2b}}{\|\phi\|_\infty} - \frac{C_{2c}}{\|\phi\|_\infty}$, it follows from (3.21) that

$$\zeta'(t) \geq -\lambda_1\zeta(t) + \frac{c_{2a}}{\|\phi\|_\infty}\int_\Omega v^{q(y)}\phi(y)\mathrm{d}y + \frac{c_{2b}}{\|\phi\|_\infty}\int_\Omega uv^{q(y)}\phi(y)\mathrm{d}y$$
$$- \frac{C_{2c}}{\|\phi\|_\infty}\int_\Omega v^{q(y)+1}\phi(y)\mathrm{d}y - \Gamma_2. \tag{3.22}$$

Using the assumptions on the exponents $p(x)$ and Jensen's inequality in (3.22)

$$\zeta'(t) \geq -\lambda_1\zeta(t) + \frac{c_{2a}}{\|\phi\|_\infty}\zeta^{q^-}(t) - \frac{C_{2c}}{\|\phi\|_\infty}\zeta^{q^+ +1}(t) - \Gamma_2. \tag{3.23}$$

By performing simple calculation as in the above for $\zeta(t)$, we get

$$\zeta'(t) \geq \frac{\delta_2}{2}\zeta^q(t). \tag{3.24}$$

The solution of (3.24) as

$$\zeta(t) \geq \left(\zeta^{1-q}(0) - \frac{\delta_2}{2}(q-1)t\right)^{\frac{-1}{q-1}}. \tag{3.25}$$

According to the definition of $(\eta(t), \zeta(t))$ and for sufficiently large initial data $\eta(0)$ and $\zeta(0)$, the result follows from Lemma 3.1, since

$$\eta(t) = \int_\Omega u(x,t)\phi(x)\mathrm{d}x \leq \|u(\cdot,t)\|_\infty \int_\Omega \phi(x)\mathrm{d}x = \|u(\cdot,t)\|_\infty \tag{3.26}$$

and

$$\zeta(t) = \int_\Omega v(x,t)\phi(x)\mathrm{d}x \leq \|v(\cdot,t)\|_\infty \int_\Omega \phi(x)\mathrm{d}x = \|v(\cdot,t)\|_\infty. \tag{3.27}$$

The solutions of the given parabolic system (1.1) blow up and hence the solutions satisfy

$$\sup_{0\leq t\leq T_f}\left(\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty\right) = +\infty. \tag{3.28}$$

This completes the proof. □

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