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RESULTS ON THE SEMI-LOCAL CONVERGENCE OF ITERATIVE METHODS WITH APPLICATIONS IN k-MULTIVARIATE FRACTIONAL CALCULUS

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Abstract. We provide new semi-local convergence results for general iterative methods in order to approximate a solution of a nonlinear operator equation. Moreover, applications are suggested in many areas including k-multivariate fractional calculus, where k is a positive integer.

1. Introduction

Many problems are special cases of the equation

$$M(x) = 0, \tag{1.1}$$

where $M : \Omega \to B_2$ is a continuous operator, B_1, B_2 are Banach spaces and $\Omega \subseteq B_1$. These problems are reduced to (1.1) using Mathematical Modelling.

Then, it is very important to find solutions x^* of equation (1.1). However, the solutions x^* can rarely be obtained in closed form. That is why we use mostly iterative methods to approximate such solutions [1], [8-20].

Let $\mathcal{L}(B_1, B_2)$ stand for space of bounded linear operators from B_1 into B_2. Let also $A(\cdot) : \Omega \to \mathcal{L}(B_1, B_1)$ be a continuous operator. Set

$$F = LM, \tag{1.2}$$

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where $L \in \mathcal{L}(B_2, B_1)$. We shall approximate x^* using a sequence $\{x_n\}$ generated by the fixed point scheme:

$$\begin{aligned} & x_{n+1} := x_n + y_n, \;\; A(x_n) y_n + F(x_n) = 0 \\ & \Leftrightarrow y_n = Q(y_n) := (I - A(x_n)) y_n - F(x_n), \end{aligned} \tag{1.3}$$

where $x_0 \in \Omega$. The sequence $\{x_n\}$ defined by

$$x_{n+1} = Q(x_n) = Q^{(n+1)}(x_0) \tag{1.4}$$

exists. In case of convergence we write:

$$Q^{\infty}(x_0) := \lim_{n\to\infty} (Q^n(x_0)) = \lim_{n\to\infty} x_n. \tag{1.5}$$

Many methods in the literature can be considered special cases of method (1.3). We can choose A to be: $A(x) = F'(x)$ (Newton's method), $A(x) = F'(x_0)$ (Modified Newton's method), $A(x) = [x, g(x); F]$, $g : \Omega \to B_1$ (Steffensen's method). Many other choices for A can be found in [1-21] and the references there in. Therefore, it is important to study the convergence of method (1.3) under generalized conditions. In particular, we present the semi-local convergence of method (1.3) using only continuity assumptions on operator F and for a so general operator A as to allow applications to k-multivariate fractional calculus and other areas.

The rest of the paper is organized as follows: Section 2 contains the semi-local convergence of method (1.3). In the concluding Section 3, we suggest some applications to k-multivariate fractional calculus.

2. Convergence

Let $B(w,\xi)$, $\overline{B}(w,\xi)$ stand, respectively for the open and closed balls in B_1 with center $w \in B_1$ and of radius $\xi > 0$.

We present the semi-local convergence of method (1.3) in this section.

Theorem 2.1. *Let $F : \Omega \subset B_1 \to B_2$, $A(\cdot) : \Omega \to \mathcal{L}(B_1, B_1)$ and $x_0 \in \Omega$ be as defined in the Introduction. Suppose there exist $\delta_0 \in (0,1)$, $\delta_1 \in (0,1)$, $\eta \geq 0$ such that for each $x, y \in \Omega$*

$$\delta := \delta_0 + \delta_1 < 1, \tag{2.1}$$

$$\|F(x_0)\| \leq \eta, \tag{2.2}$$

$$\|I - A(x)\| \leq \delta_0, \tag{2.3}$$

$$\|F(y) - F(x) - A(x)(y - x)\| \leq \delta_1 \|y - x\| \tag{2.4}$$

and

$$\overline{B}(x_0, \delta) \subseteq \Omega, \tag{2.5}$$

where

$$\rho = \frac{\eta}{1-\delta}. \tag{2.6}$$

Then, sequence $\{x_n\}$ *generated for* $x_0 \in \Omega$ *by*

$$x_{n+1} = x_n + Q_n^{\infty}(0), \quad Q_n(y) := (I - A(x_n))\, y - F(x_n) \tag{2.7}$$

is well defined in $B(x_0, \rho)$*, remains in* $\overline{B}(x_0, \rho)$ *for each* $n = 0, 1, 2, ...$ *and converges to* x^* *which is the only solution of equation* $F(x) = 0$ *in* $\overline{B}(x_0, \rho)$*. Moreover, an apriori error estimate is given by the sequence* $\{\rho_n\}$ *defined by*

$$\rho_0 := \rho, \quad \rho_n = T_n^{\infty}(0), \quad T_n(t) = \delta_0 + \delta_1 \rho_{n-1} \tag{2.8}$$

for each $n = 1, 2, ...$ *and satisfying*

$$\lim_{n\to\infty} \rho_n = 0. \tag{2.9}$$

Furthermore, an aposteriori error estimate is given by the sequence $\{\sigma_n\}$ *defined by*

$$\sigma_n := H_n^{\infty}(0), \; H_n(t) = \delta t + \delta_1 p_{n-1}, \tag{2.10}$$

$$q_n := \|x_n - x_0\| \le \rho - \rho_n \le \rho, \tag{2.11}$$

where

$$p_{n-1} := \|x_n - x_{n-1}\| \quad \text{for each} \quad n = 1, 2, ... \tag{2.12}$$

Proof. We shall show using mathematical induction the following assertion is true:

(A_n) $x_n \in X$ and $\rho_n \ge 0$ are well defined and such that

$$\rho_n + p_{n-1} \le \rho_{n-1}. \tag{2.13}$$

By the definition of ρ, (2.3)-(2.6) we have that there exists $r \le \rho$ (Lemma 1.4 [8, p. 3]) such that

$$\delta_0 r + \|F(x_0)\| = r$$

and

$$\delta_0^k r \le \delta_0^k \rho \to 0 \quad \text{as} \quad k \to \infty.$$

That is (Lemma 1.5 [8, p. 4]) x_1 is well defined and $p_0 \le r$.

We need the estimate:

$$T_1(\rho - r) = \delta_0(\rho - r) + \delta_1 \rho_0 = \delta_0 \rho - \delta_0 r + \delta_1 \rho = G_0(\rho) - r = \rho - r.$$

That is (Lemma 1.4 [8, p. 3]) ρ_1 exists and satisfies

$$\rho_1 + p_0 \le \rho - r + r = \rho = \rho_0.$$

Hence (I_0) is true. Suppose that for each $k = 1, 2, ..., n$, assertion (I_k) is true. We must show: x_{k+1} exists and find a bound r for p_k. Indeed, we have in turn that

$$\begin{aligned}\delta_0 \rho_k + \delta_1(\rho_{k-1} - \rho_k) &= \delta_0 \rho_k + \delta_1 \rho_{k-1} - \delta_1 \rho_k \\ &= T_k(\rho_k) - \delta_1 \rho_k \le \rho_k.\end{aligned}$$

That is there exists $r \leq \rho_k$ such that

$$r = \delta_0 r + \delta_1 (\rho_{k-1} - \rho_k) \quad \text{and} \quad (\delta_0 + \delta_1)^i r \to 0 \tag{2.14}$$

as $i \to \infty$.

The induction hypothesis gives that

$$q_k \leq \sum_{m=0}^{k-1} p_m \leq \sum_{m=0}^{k-1} (\rho_m - \rho_{m+1}) = \rho - \rho_k \leq \rho,$$

so $x_k \in \overline{B}(x_0, \rho) \subseteq \Omega$ and x_1 satisfies $\|I - A(x_1)\| \leq \delta_0$ (by (2.3)).

Using the induction hypothesis, (1.3) and (2.4), we get

$$\begin{aligned} \|F(x_k)\| &= \|F(x_k) - F(x_{k-1}) - A(x_{k-1})(x_k - x_{k-1})\| \\ &\leq \delta_1 p_{k-1} \leq \delta_1 (\rho_{k-1} - \rho_k) \end{aligned} \tag{2.15}$$

leading together with (2.14) to:

$$\delta_0 r + \|F(x_k)\| \leq r,$$

which implies x_{k+1} exists and $p_k \leq r \leq \rho_k$. It follows from the definition of ρ_{k+1} that

$$T_{k+1}(\rho_k - r) = T_k(\rho_k) - r = \rho_k - r,$$

so ρ_{k+1} exists and satisfies

$$\rho_{k+1} + p_k \leq \rho_k - r + r = \rho_k$$

so the induction for (I_n) is completed.

Let $j \geq k$. Then, we obtain in turn that

$$\|x_{j+k} - x_k\| \leq \sum_{i=k}^{j} p_i \leq \sum_{i=k}^{j} (\rho_j - \rho_{j+1}) = \rho_k - \rho_{j+k} \leq \rho_k. \tag{2.16}$$

We also obtain using induction that

$$\rho_{k+1} = T_{k+1}(\rho_{k+1}) \leq T_{k+1}(\rho_k) \leq \delta \rho_k \leq \ldots \leq \delta^{k+1} \rho. \tag{2.17}$$

Hence, by (2.1) and (2.17) $\lim_{k\to\infty} \rho_k = 0$, so $\{x_k\}$ is a complete sequence in a Banach space X and as such it converges to some x^*. By letting $j \to \infty$ in (2.16), we conclude that $x^* \in \overline{B}(x_k, \rho_k)$. Moreover, by letting $k \to \infty$ in (2.15) and using the continuity of F we get that $F(x^*) = 0$. Notice that

$$H_k(\rho_k) \leq T_k(\rho_k) \leq \rho_k,$$

so the apriori bound exists. That is σ_k is smaller in general than ρ_k. Clearly, the conditions of the theorem are satisfied for x_k replacing x_0 (by (2.16)). Hence, by (2.8) $x^* \in \overline{B}(x_n, \sigma_n)$, which completes the proof for the aposteriori bound. □

Remark 2.2. (a) It follows from the proof of Theorem 2.1 that the conclusions hold, if $A(\cdot)$ is replaced by a more general continuous operator $A:\Omega\to B_1$.

(b) In the next section some applications are suggested for special choices of the "A" operators with $\gamma_0 := \delta_0$ and $\gamma_1 := \delta_1$.

3. Applications to k-multivariate fractional calculus

Our presented earlier semi-local convergence results, see Theorem 2.1, apply in the next three multivariate fractional settings given that the following inequalities are fulfilled:

$$\|1-A(x)\|_{\infty}\leq\gamma_0\in(0,1) \tag{3.1}$$

and

$$\left\|(F(y)-F(x))\overrightarrow{i}-A(x)(y-x)\right\|\leq\gamma_1\|y-x\|, \tag{3.2}$$

where $\gamma_0,\gamma_1\in(0,1)$, furthermore

$$\gamma=\gamma_0+\gamma_1\in(0,1), \tag{3.3}$$

for all $x,y\in\prod_{i=1}^{N}[a_i^*,b_i^*]$, where $a_i<a_i^*<b_i^*<b_i$, $i=1,...,N$.

Above $\overrightarrow{i}$ is the unit vector in $\mathbb{R}^N$, $N\in\mathbb{N}$, $\left\|\overrightarrow{i}\right\|=1$, and $\|\cdot\|$ is a norm in $\mathbb{R}^k$.

The specific functions $A(x)$, $F(x)$ will be described next.

(I) Consider the k-left multidimensional Riemann-Liouville fractional integral of order $\alpha=(\alpha_1,...,\alpha_N)$, $k=(k_1,...,k_N)$, $(\alpha_i>0,\ k_i>0,\ i=1,...,N)$:

$$\begin{aligned}&\left({}_kI_{a+}^{\alpha}f\right)(x)\\&=\frac{1}{\prod_{i=1}^{N}k_i\Gamma_{k_i}(\alpha_i)}\int_{a_1}^{x_1}...\int_{a_N}^{x_N}\prod_{i=1}^{N}(x_i-t_i)^{\frac{\alpha_i}{k_i}-1}f(t_1,...,t_N)\,dt_1...dt_N,\end{aligned} \tag{3.4}$$

where $\Gamma_{k_i}(\alpha_i)$ is the k_i-gamma function given by $\Gamma_{k_i}(\alpha_i)=\int_0^{\infty}t^{\alpha_i-1}e^{-\frac{t^{k_i}}{k_i}}dt$, $i=1,...,N$ (it holds ([21]) $\Gamma_{k_i}(\alpha_i+k_i)=\alpha_i\Gamma_{k_i}(\alpha_i)$, $\Gamma(\alpha_i)=\lim_{k_i\to1}\Gamma_{k_i}(\alpha_i)$, where Γ is the gamma function, and ${}_kI_{a+}^{0}f:=f$), $f\in L_{\infty}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $a=(a_1,...,a_N)$ and $x=(x_1,...,x_N)\in\prod_{i=1}^{N}[a_i,b_i]$.

By [6], we get that $\left({}_kI^{\alpha}_{a+}f\right)$ is a continuous function on $\prod_{i=1}^{N}[a_i,b_i]$. Furthermore, by [6], we get that ${}_kI^{\alpha}_{a+}$ is a bounded linear operator which is a positive operator.

We notice the following

$$\left|{}_kI^{\alpha}_{a+}f(x)\right| \le \frac{1}{\prod_{i=1}^{N}k_i\Gamma_{k_i}(\alpha_i)}\left(\int_{a_1}^{x_1}\dots\int_{a_N}^{x_N}\prod_{i=1}^{N}(x_i-t_i)^{\frac{\alpha_i}{k_i}-1}dt_1\dots dt_N\right)\|f\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]} \tag{3.5}$$

$$= \frac{1}{\prod_{i=1}^{N}k_i\Gamma_{k_i}(\alpha_i)}\prod_{i=1}^{N}\frac{(x_i-a_i)^{\frac{\alpha_i}{k_i}}}{\left(\frac{\alpha_i}{k_i}\right)}\|f\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]} = \left(\prod_{i=1}^{N}\frac{(x_i-a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i+k_i)}\right)\|f\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]}. \tag{3.6}$$

That is, it holds

$$\left|{}_kI^{\alpha}_{a+}f(x)\right| \le \left(\prod_{i=1}^{N}\frac{(x_i-a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i+k_i)}\right)\|f\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]} \le \left(\prod_{i=1}^{N}\frac{(b_i-a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i+k_i)}\right)\|f\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]}. \tag{3.7}$$

We get that

$${}_kI^{\alpha}_{a+}f(a)=0. \tag{3.8}$$

In particular, $\left({}_kI^{\alpha}_{a+}f\right)$ is continuous on $\prod_{i=1}^{N}[a_i^*,b_i^*]$. Thus there exist $x_1,x_2\in\prod_{i=1}^{N}[a_i^*,b_i^*]$ such that

$$\begin{aligned}\left({}_kI^{\alpha}_{a+}f\right)(x_1)&=\min\left({}_kI^{\alpha}_{a+}f\right)(x),\\ \left({}_kI^{\alpha}_{a+}f\right)(x_2)&=\max\left({}_kI^{\alpha}_{a+}f\right)(x),\end{aligned} \tag{3.9}$$

over all $x\in\prod_{i=1}^{N}[a_i^*,b_i^*]$.

We assume that

$$\left({}_kI^{\alpha}_{a+}f\right)(x_1)>0. \tag{3.10}$$

Hence

$$\left\| {}_{k}I_{a+}^{\alpha} f \right\|_{\infty, \prod_{i=1}^{N} [a_i^*, b_i^*]} = \left({}_{k}I_{a+}^{\alpha} f \right)(x_2) > 0. \tag{3.11}$$

Here we define

$$Jf(x) = mf(x), \quad 0 < m < \frac{1}{2}, \tag{3.12}$$

for any $x \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Therefore the equation

$$Jf(x) = 0, \quad x \in \prod_{i=1}^{N} [a_i^*, b_i^*] \tag{3.13}$$

has the same solutions as the equation

$$F(x) := \frac{Jf(x)}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)} = 0, \quad x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.14}$$

Notice that

$$\begin{aligned} {}_{k}I_{a+}^{\alpha} \left(\frac{f}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)} \right)(x) &= \frac{\left({}_{k}I_{a+}^{\alpha} f \right)(x)}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)} \\ &\leq \frac{1}{2} < 1, \quad x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \end{aligned} \tag{3.15}$$

Call

$$A(x) := \frac{\left({}_{k}I_{a+}^{\alpha} f \right)(x)}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)}, \quad \forall x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.16}$$

We notice that

$$0 < \frac{\left({}_{k}I_{a+}^{\alpha} f \right)(x_1)}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)} \leq A(x) \leq \frac{1}{2}, \quad \forall x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.17}$$

Hence the first condition (3.1) is fulfilled by

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{\left({}_{k}I_{a+}^{\alpha} f \right)(x_1)}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)} =: \gamma_0, \quad \forall x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.18}$$

So that $\|1 - A(x)\|_{\infty} \leq \gamma_0$, where $\|\cdot\|_{\infty}$ is over $\prod_{i=1}^{N} [a_i^*, b_i^*]$. Clearly $\gamma_0 \in (0,1)$.

Next we assume that $\frac{f(x)}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)}$ is a contraction, that is

$$\left| \frac{f(x)}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)} - \frac{f(y)}{2\left({}_{k}I_{a+}^{\alpha} f \right)(x_2)} \right| \leq \theta \|x - y\|, \tag{3.19}$$

for all $x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]$, $0 < \theta < 1$. Hence

$$\left| \frac{mf(x)}{2\left({}_kI_{a+}^{\alpha} f\right)(x_2)} - \frac{mf(y)}{2\left({}_kI_{a+}^{\alpha} f\right)(x_2)} \right| \leq m\theta \|x - y\| \leq \frac{\theta}{2} \|x - y\|, \tag{3.20}$$

for all $x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Set $\lambda = \frac{\theta}{2}$, it is $0 < \lambda < \frac{1}{2}$. We have that

$$|F(x) - F(y)| \leq \lambda \|x - y\|, \tag{3.21}$$

for all $x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]$.

Equivalently, we have

$$|Jf(x) - Jf(y)| \leq 2\lambda \left({}_kI_{a+}^{\alpha} f\right)(x_2) \|x - y\|, \text{ for all } x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.22}$$

We observe that

$$\begin{aligned} &\left\| (F(y) - F(x)) \overrightarrow{i} - A(x)(y - x) \right\| \\ &\leq |F(y) - F(x)| + |A(x)| \|y - x\| \\ &\leq \lambda \|y - x\| + |A(x)| \|y - x\| = (\lambda + |A(x)|) \|y - x\| \\ &=: (\psi_1), \ \ \forall x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \end{aligned} \tag{3.23}$$

By (3.7), we have that

$$\left| \left({}_kI_{a+}^{\alpha} f\right)(x) \right| \leq \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)} \right) \|f\|_\infty, \ \ \forall x \in \prod_{i=1}^{N} [a_i^*, b_i^*], \tag{3.24}$$

where $\|\cdot\|_\infty$ now is over $\prod_{i=1}^{N} [a_i, b_i]$.

Hence

$$\begin{aligned} |A(x)| &= \frac{\left| \left({}_kI_{a+}^{\alpha} f\right)(x) \right|}{2\left({}_kI_{a+}^{\alpha} f\right)(x_2)} \\ &\leq \frac{1}{2\left({}_kI_{a+}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)} \right) \|f\|_\infty < \infty, \end{aligned} \tag{3.25}$$

for all $x \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Therefore we get

$$(\psi_1) \leq \left(\lambda + \frac{1}{2 \left({}_k I_{a+}^{\alpha} f \right) (x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i} (\alpha_i + k_i)} \right) \|f\|_{\infty} \right) \|y - x\|, \qquad (3.26)$$

for all $x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Call

$$0 < \gamma_1 := \lambda + \frac{1}{2 \left({}_k I_{a+}^{\alpha} f \right) (x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i} (\alpha_i + k_i)} \right) \|f\|_{\infty}, \qquad (3.27)$$

and by choosing $(b_i - a_i)$ small enough, $i = 1, ..., N$, we can make $\gamma_1 \in (0, 1)$, fulfilling (3.2).

Next we call and we need that

$$\begin{aligned} 0 < \gamma &:= \gamma_0 + \gamma_1 \\ &= \left(1 - \frac{\left({}_k I_{a+}^{\alpha} f \right) (x_1)}{2 \left({}_k I_{a+}^{\alpha} f \right) (x_2)} \right) \\ &\quad + \left(\lambda + \frac{1}{2 \left({}_k I_{a+}^{\alpha} f \right) (x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i} (\alpha_i + k_i)} \right) \|f\|_{\infty} \right) \\ &< 1, \end{aligned} \qquad (3.28)$$

equivalently,

$$\lambda + \frac{1}{2 \left({}_k I_{a+}^{\alpha} f \right) (x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i} (\alpha_i + k_i)} \right) \|f\|_{\infty} < \frac{\left({}_k I_{a+}^{\alpha} f \right) (x_1)}{2 \left({}_k I_{a+}^{\alpha} f \right) (x_2)}, \qquad (3.29)$$

equivalently,

$$2\lambda \left({}_k I_{a+}^{\alpha} f \right) (x_2) + \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i} (\alpha_i + k_i)} \right) \|f\|_{\infty} < \left({}_k I_{a+}^{\alpha} f \right) (x_1), \qquad (3.30)$$

which is possible for small λ and small $(b_i - a_i)$, all $i = 1, ..., N$. That is $\gamma \in (0, 1)$, fulfilling (3.3). So our numerical method converges and solves (3.13).

(II) Consider the k-right multidimensional Riemann-Liouville fractional integral of order $\alpha = (\alpha_1, ..., \alpha_N)$, $k = (k_1, ..., k_N)$, $(\alpha_i > 0,\ k_i > 0,\ i = 1, ..., N)$:

$$\begin{aligned} &\left({}_k I_{b-}^{\alpha} f \right) (x) \\ &= \frac{1}{\prod_{i=1}^{N} k_i \Gamma_{k_i} (\alpha_i)} \int_{x_1}^{b_1} ... \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\frac{\alpha_i}{k_i} - 1} f(t_1, ..., t_N)\, dt_1 ... dt_N, \end{aligned} \qquad (3.31)$$

we set

$$ {}_{k}I_{b-}^{0} f = f, \tag{3.32} $$

where $f \in L_{\infty}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $b=(b_1,...,b_N)$ and $x=(x_1,...,x_N)\in\prod_{i=1}^{N}[a_i,b_i]$.

By [7], we get that ${}_{k}I_{b-}^{\alpha} f$ is a continuous function on $\prod_{i=1}^{N}[a_i,b_i]$. Furthermore by [7] we get that ${}_{k}I_{b-}^{\alpha}$ is a bounded linear operator, which is a positive operator.

We notice the following

$$ \begin{aligned} \left|{}_{k}I_{b-}^{\alpha} f(x)\right| &\leq \frac{1}{\prod_{i=1}^{N} k_i \Gamma_{k_i}(\alpha_i)} \prod_{i=1}^{N} \frac{(b_i - x_i)^{\frac{\alpha_i}{k_i}}}{\left(\frac{\alpha_i}{k_i}\right)} \|f\|_{\infty} \\ &= \left(\prod_{i=1}^{N} \frac{(b_i - x_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)}\right) \|f\|_{\infty}. \end{aligned} \tag{3.33} $$

That is it holds

$$ \begin{aligned} \left|{}_{k}I_{b-}^{\alpha} f(x)\right| &\leq \left(\prod_{i=1}^{N} \frac{(b_i - x_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)}\right) \|f\|_{\infty} \\ &\leq \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)}\right) \|f\|_{\infty}. \end{aligned} \tag{3.34} $$

We get that

$$ {}_{k}I_{b-}^{\alpha} f(b) = 0. \tag{3.35} $$

In particular, $\left({}_{k}I_{b-}^{\alpha} f\right)$ is continuous on $\prod_{i=1}^{N}[a_i^*,b_i^*]$. Thus there exist $x_1, x_2 \in \prod_{i=1}^{N}[a_i^*,b_i^*]$ such that

$$ \begin{aligned} \left({}_{k}I_{b-}^{\alpha} f\right)(x_1) &= \min \left({}_{k}I_{b-}^{\alpha} f\right)(x), \\ \left({}_{k}I_{b-}^{\alpha} f\right)(x_2) &= \max \left({}_{k}I_{b-}^{\alpha} f\right)(x), \end{aligned} \tag{3.36} $$

over all $x \in \prod_{i=1}^{N}[a_i^*,b_i^*]$.

We assume that

$$ \left({}_{k}I_{b-}^{\alpha} f\right)(x_1) > 0. \tag{3.37} $$

Hence

$$ \left\|{}_{k}I_{b-}^{\alpha} f\right\|_{\infty, \prod_{i=1}^{N}[a_i^*,b_i^*]} = \left({}_{k}I_{b-}^{\alpha} f\right)(x_2) > 0. \tag{3.38} $$

Here we define

$$Jf(x) = mf(x), \ \ 0 < m < \frac{1}{2}, \tag{3.39}$$

for any $x \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Therefore the equation

$$Jf(x) = 0, \ \ x \in \prod_{i=1}^{N} [a_i^*, b_i^*], \tag{3.40}$$

has the same solutions as the equation

$$F(x) := \frac{Jf(x)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)} = 0, \quad x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.41}$$

Notice that

$${}_kI_{b-}^{\alpha}\left(\frac{f}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)}\right)(x) = \frac{\left({}_kI_{b-}^{\alpha} f\right)(x)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)} \leq \frac{1}{2} < 1, \quad x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.42}$$

Call

$$A(x) := \frac{\left({}_kI_{b-}^{\alpha} f\right)(x)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)}, \ \ \forall x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.43}$$

We notice that

$$0 < \frac{\left({}_kI_{b-}^{\alpha} f\right)(x_1)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)} \leq A(x) \leq \frac{1}{2}, \ \ \forall x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.44}$$

Hence the first condition (3.1) is fulfilled by

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{\left({}_kI_{b-}^{\alpha} f\right)(x_1)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)} =: \gamma_0, \ \ \forall x \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.45}$$

So that $\|1 - A(x)\|_{\infty} \leq \gamma_0$, where $\|\cdot\|_{\infty}$ is over $\prod_{i=1}^{N} [a_i^*, b_i^*]$. Clearly $\gamma_0 \in (0,1)$.

Next we assume that $\frac{f(x)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)}$ is a contraction, that is

$$\left|\frac{f(x)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)} - \frac{f(y)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)}\right| \leq \theta \|x - y\|, \tag{3.46}$$

for all $x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]$, $0 < \theta < 1$. Hence

$$\left|\frac{mf(x)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)} - \frac{mf(y)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)}\right| \leq m\theta \|x - y\| \leq \frac{\theta}{2} \|x - y\|, \tag{3.47}$$

for all $x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Set $\lambda = \frac{\theta}{2}$, it is $0 < \lambda < \frac{1}{2}$. We have that

$$|F(x) - F(y)| \leq \lambda \|x - y\|, \tag{3.48}$$

all $x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda \left({}_kI_{b-}^{\alpha} f\right)(x_2) \|x - y\|, \ \ \forall\, x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \tag{3.49}$$

We observe that

$$\begin{aligned} &\left\| (F(y) - F(x)) \overrightarrow{i} - A(x)(y - x) \right\| \\ &\leq |F(y) - F(x)| + |A(x)| \|y - x\| \\ &\leq \lambda \|y - x\| + |A(x)| \|y - x\| \\ &= (\lambda + |A(x)|) \|y - x\| =: (\psi_2), \ \ \forall\, x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]. \end{aligned} \tag{3.50}$$

By (3.34), we have that

$$\left| \left({}_kI_{b-}^{\alpha} f\right)(x) \right| \leq \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)} \right) \|f\|_\infty, \ \ \forall\, x \in \prod_{i=1}^{N} [a_i^*, b_i^*], \tag{3.51}$$

where $\|\cdot\|_\infty$ now is over $\prod_{i=1}^{N} [a_i, b_i]$. Hence

$$\begin{aligned} |A(x)| &= \frac{\left| \left({}_kI_{b-}^{\alpha} f\right)(x) \right|}{2 \left({}_kI_{b-}^{\alpha} f\right)(x_2)} \\ &\leq \frac{1}{2 \left({}_kI_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)} \right) \|f\|_\infty < \infty, \end{aligned} \tag{3.52}$$

for all $x \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Therefore we get

$$(\psi_2) \leq \left(\lambda + \frac{1}{2 \left({}_kI_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)} \right) \|f\|_\infty \right) \|y - x\|, \tag{3.53}$$

for all $x, y \in \prod_{i=1}^{N} [a_i^*, b_i^*]$. Call

$$0 < \gamma_1 := \lambda + \frac{1}{2 \left({}_kI_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)} \right) \|f\|_\infty, \tag{3.54}$$

and by choosing $(b_i - a_i)$ small enough, $i = 1, ..., N$, we can make $\gamma_1 \in (0,1)$, fulfilling (3.2).

Next we call and we need that

$$\begin{aligned} 0 < \gamma &:= \gamma_0 + \gamma_1 \\ &= \left(1 - \frac{\left({}_kI_{b-}^{\alpha} f\right)(x_1)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)}\right) \\ &\quad + \left(\lambda + \frac{1}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)}\right) \|f\|_\infty \right) < 1, \end{aligned} \tag{3.55}$$

equivalently,

$$\lambda + \frac{1}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)}\right) \|f\|_\infty < \frac{\left({}_kI_{b-}^{\alpha} f\right)(x_1)}{2\left({}_kI_{b-}^{\alpha} f\right)(x_2)}, \tag{3.56}$$

equivalently,

$$2\lambda \left({}_kI_{b-}^{\alpha} f\right)(x_2) + \left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\frac{\alpha_i}{k_i}}}{\Gamma_{k_i}(\alpha_i + k_i)}\right) \|f\|_\infty < \left({}_kI_{b-}^{\alpha} f\right)(x_1), \tag{3.57}$$

which is possible for small λ and small $(b_i - a_i)$, all $i = 1, ..., N$. That is $\gamma \in (0,1)$, fulfilling (3.3). So our numerical method converges and solves (3.40).

(III) Here we deal with the following multivariate mixed fractional derivative:

let $\alpha = (\alpha_1, ..., \alpha_N)$, where $0 < \alpha_i < 1$, $i = 1, ..., N$; $f \in C^N\left(\prod_{i=1}^{N}[0, b_i]\right)$; $b_i > 0$, $i = 1, ..., N$,

$$\begin{aligned} \left({}^{CF}D_*^{\alpha} f\right)(t) = &\frac{1}{\prod_{i=1}^{N}(1-\alpha_i)} \cdot \int_0^{t_1} ... \int_0^{t_N} \prod_{i=1}^{N} \exp\left(-\frac{\alpha_i}{1-\alpha_i}(t_i - s_i)\right) \\ &\times \frac{\partial^N f(s_1, ..., s_N)}{\partial s_1 ... \partial s_N} ds_1 ... ds_N, \end{aligned} \tag{3.58}$$

for all $0 \le t_i \le b_i$, $i = 1, ..., N$ with $t = (t_1, ..., t_N)$. When $N = 1$, the univariate case is known as the M. Caputo-Fabrizio fractional derivative, see [18]. Call

$$\gamma_i := \frac{\alpha_i}{1 - \alpha_i} > 0, \tag{3.59}$$

i.e.,

$$\left({}^{CF}D_{*}^{\alpha}f\right)(t)=\frac{1}{\prod_{i=1}^{N}(1-\alpha_i)}\cdot\int_0^{t_1}...\int_0^{t_N}\prod_{i=1}^{N}e^{-\gamma_i(t_i-s_i)}\times\frac{\partial^N f(s_1,...,s_N)}{\partial s_1...\partial s_N}ds_1...ds_N, \tag{3.60}$$

for all $0\leq t_i\leq b_i$, $i=1,...,N$. We notice that

$$\left|\left({}^{CF}D_{*}^{\alpha}f\right)(t)\right| \leq \frac{1}{\prod_{i=1}^{N}(1-\alpha_i)}\cdot\left(\int_0^{t_1}...\int_0^{t_N}\prod_{i=1}^{N}e^{-\gamma_i(t_i-s_i)}ds_1...ds_N\right)\left\|\frac{\partial^N f}{\partial x_1...\partial x_N}\right\|_\infty \tag{3.61}$$

$$=\prod_{i=1}^{N}\left(\frac{1}{1-\alpha_i}\int_0^{t_i}e^{-\gamma_i(t_i-s_i)}ds_i\right)\left\|\frac{\partial^N f}{\partial x_1...\partial x_N}\right\|_\infty =\left(\prod_{i=1}^{N}\frac{e^{-\gamma_i t_i}}{\alpha_i}\left(e^{\gamma_i t_i}-1\right)\right)\left\|\frac{\partial^N f}{\partial x_1...\partial x_N}\right\|_\infty =\left(\prod_{i=1}^{N}\frac{1}{\alpha_i}\left(1-e^{-\gamma_i t_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1...\partial x_N}\right\|_\infty \leq\left(\prod_{i=1}^{N}\left(\frac{1-e^{-\gamma_i b_i}}{\alpha_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1...\partial x_N}\right\|_\infty. \tag{3.62}$$

That is

$$\left({}^{CF}D_{*}^{\alpha}f\right)(0,...,0)=0 \tag{3.63}$$

and

$$\left|\left({}^{CF}D_{*}^{\alpha}f\right)(t)\right|\leq\left(\prod_{i=1}^{N}\left(\frac{1-e^{-\gamma_i b_i}}{\alpha_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1...\partial x_N}\right\|_\infty. \tag{3.64}$$

Notice here that $1-e^{-\gamma_i t_i}$, $t_i\geq 0$ is an increasing function, $i=1,...,N$. Thus the smaller the t_i, the smaller it is $1-e^{-\gamma_i t_i}$, $i=1,...,N$. We can rewrite

$$\left({}^{CF}D_{*}^{\alpha}f\right)(t) =\prod_{i=1}^{N}\left(\frac{e^{-\gamma_i t_i}}{1-\alpha_i}\right)\int_0^{t_1}...\int_0^{t_N}e^{\sum_{i=1}^{N}\gamma_i s_i}\frac{\partial^N f(s_1,...,s_N)}{\partial s_1...\partial s_N}ds_1...ds_N \tag{3.65}$$

$$=\prod_{i=1}^{N}\left(\frac{e^{-\gamma_i t_i}}{1-\alpha_i}\right)\int_0^{b_1}...\int_0^{b_N}\chi_{\prod_{i=1}^{N}[0,t_i]}(s_1,...,s_N)\,e^{\sum_{i=1}^{N}\gamma_i s_i}\times\frac{\partial^N f(s_1,...,s_N)}{\partial s_1...\partial s_N}ds_1...ds_N, \tag{3.66}$$

where χ stands for the characteristic function. Let $t_n \to t$, as $n \to \infty$, then

$$\chi_{\prod_{i=1}^{N}[0,t_{in}]}(s_1,...,s_N) \to \chi_{\prod_{i=1}^{N}[0,t_i]}(s_1,...,s_N), \text{a.e, as } n \to \infty,$$

where $t_n = (t_{1n},...,t_{Nn})$. Hence we have

$$\chi_{\prod_{i=1}^{N}[0,t_{in}]}(s_1,...,s_N)\, e^{\sum_{i=1}^{N}\gamma_i s_i} \frac{\partial^N f(s_1,...,s_N)}{\partial s_1 ... \partial s_N}$$
$$\to \chi_{\prod_{i=1}^{N}[0,t_i]}(s_1,...,s_N)\, e^{\sum_{i=1}^{N}\gamma_i s_i} \frac{\partial^N f(s_1,...,s_N)}{\partial s_1 ... \partial s_N}, \quad \text{a.e.,}$$

in $(s_1,...,s_N) \in \prod_{i=1}^{N}[0,b_i]$. Furthermore, it holds

$$\begin{aligned} &\chi_{\prod_{i=1}^{N}[0,t_{iN}]}(s_1,...,s_N)\, e^{\sum_{i=1}^{N}\gamma_i s_i} \left| \frac{\partial^N f(s_1,...,s_N)}{\partial s_1 ... \partial s_N} \right| \\ &\leq e^{\sum_{i=1}^{N}\gamma_i b_i} \left\| \frac{\partial^N f}{\partial x_1 ... \partial x_N} \right\|_{\infty}. \end{aligned} \tag{3.67}$$

Thus, by dominated convergence theorem we get

$$\left({}^{CF}D_*^{\alpha} f\right)(t_n) \to \left({}^{CF}D_*^{\alpha} f\right)(t) \quad \text{as } n \to \infty,$$

proving continuity of $\left({}^{CF}D_*^{\alpha} f\right)(t)$, $t \in \prod_{i=1}^{N}[0,b_i]$. In particular, $\left({}^{CF}D_*^{\alpha} f\right)(t)$ is continuous, for all $t \in \prod_{i=1}^{N}[a_i,b_i]$, where $0 < a_i < b_i$, $i = 1,...,N$. Therefore there exist $x_1, x_2 \in \prod_{i=1}^{N}[a_i,b_i]$ such that

$${}^{CF}D_*^{\alpha} f(x_1) = \min\ {}^{CF}D_*^{\alpha} f(x) \tag{3.68}$$

and

$${}^{CF}D_*^{\alpha} f(x_2) = \max\ {}^{CF}D_*^{\alpha} f(x), \quad \text{for } x \in \prod_{i=1}^{N}[a_i,b_i]. \tag{3.69}$$

We assume that

$${}^{CF}D_*^{\alpha} f(x_1) > 0 \tag{3.70}$$

$(i.e., {}^{CF}D_*^{\alpha} f(x) > 0, \quad \forall\, x \in \prod_{i=1}^{N}[a_i,b_i])$. Furthermore

$$\left\| {}^{CF}D_*^{\alpha} fG \right\|_{\infty,[a,b]} = {}^{CF}D_*^{\alpha} f(x_2). \tag{3.71}$$

Here we define

$$Jf(x) = mf(x), \quad 0 < m < \frac{1}{2}, \quad \forall\, x \in \prod_{i=1}^{N}[a_i,b_i]. \tag{3.72}$$

The equation

$$Jf(x) = 0, \quad x \in \prod_{i=1}^{N}[a_i,b_i] \tag{3.73}$$

has the same set of solutions as the equation

$$F(x) := \frac{Jf(x)}{{}^{CF}D_*^{\alpha} f(x_2)} = 0, \quad x \in \prod_{i=1}^{N} [a_i, b_i]. \tag{3.74}$$

Notice that

$${}^{CF}D_*^{\alpha}\left(\frac{f(x)}{2^{CF}D_*^{\alpha} f(x_2)}\right) = \frac{{}^{CF}D_*^{\alpha} f(x)}{2^{CF}D_*^{\alpha} f(x_2)} \leq \frac{1}{2} < 1, \quad \forall x \in \prod_{i=1}^{N} [a_i, b_i]. \tag{3.75}$$

We call

$$A(x) := \frac{{}^{CF}D_*^{\alpha} f(x)}{2^{CF}D_*^{\alpha} f(x_2)}, \quad \forall x \in \prod_{i=1}^{N} [a_i, b_i]. \tag{3.76}$$

We notice that

$$0 < \frac{{}^{CF}D_*^{\alpha} f(x_1)}{2^{CF}D_*^{\alpha} f(x_2)} \leq A(x) \leq \frac{1}{2}. \tag{3.77}$$

Furthermore it holds

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{{}^{CF}D_*^{\alpha} f(x_1)}{2^{CF}D_*^{\alpha} f(x_2)} =: \gamma_0, \quad \forall x \in \prod_{i=1}^{N} [a_i, b_i]. \tag{3.78}$$

Clearly $\gamma_0 \in (0,1)$. We have proved that

$$\|1 - A(x)\|_{\infty} \leq \gamma_0 \in (0,1), \quad \forall x \in \prod_{i=1}^{N} [a_i, b_i], \tag{3.79}$$

see (3.1) fulfilled.

Next we assume that $F(x)$ is a contraction over $\prod_{i=1}^{N} [a_i, b_i]$, *i.e.*,

$$|F(x) - F(y)| \leq \lambda \|x - y\|, \quad \forall x, y \in \prod_{i=1}^{N} [a_i, b_i] \tag{3.80}$$

and $0 < \lambda < \frac{1}{2}$. Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda \left({}^{CF}D_*^{\alpha} f(x_2)\right) \|x - y\|, \quad \forall x, y \in [a, b]. \tag{3.81}$$

We observe that

$$\begin{aligned} &\left\| (F(y) - F(x)) \overrightarrow{i} - A(x)(y - x) \right\| \\ &\leq |F(y) - F(x)| + |A(x)| \|y - x\| \\ &\leq \lambda \|y - x\| + |A(x)| \|y - x\| \\ &= (\lambda + |A(x)|) \|y - x\| =: (\xi), \quad \forall x, y \in \prod_{i=1}^{N} [a_i, b_i], \end{aligned} \tag{3.82}$$

where $\overrightarrow{i}$ the unit vector in $\mathbb{R}^N$. Here we have (3.64) valid on $\prod_{i=1}^N [a_i, b_i]$. Hence, for all $x \in \prod_{i=1}^N [a_i, b_i]$, we get that

$$|A(x)| = \frac{\left|{}^{CF}D_*^\alpha f(x)\right|}{2\left({}^{CF}D_*^\alpha f\right)(x_2)} \leq \frac{\left(\prod_{i=1}^N \left(\frac{1-e^{-\gamma_i b_i}}{\alpha_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1 ... \partial x_N}\right\|_\infty}{2\alpha\left({}^{CF}D_*^\alpha f\right)(x_2)} < \infty. \quad (3.83)$$

Consequently we observe

$$(\xi) \leq \left(\lambda + \frac{\left(\prod_{i=1}^N \left(\frac{1-e^{-\gamma_i b_i}}{\alpha_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1 ... \partial x_N}\right\|_\infty}{2\alpha\left({}^{CF}D_*^\alpha f\right)(x_2)}\right)\|y - x\|, \quad (3.84)$$

for all $x, y \in \prod_{i=1}^N [a_i, b_i]$. Call

$$0 < \gamma_1 := \lambda + \frac{\left(\prod_{i=1}^N \left(\frac{1-e^{-\gamma_i b_i}}{\alpha_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1 ... \partial x_N}\right\|_\infty}{2\alpha\left({}^{CF}D_*^\alpha f\right)(x_2)}, \quad (3.85)$$

choosing b_i small enough, $i = 1, ..., N$, we can make $\gamma_1 \in (0, 1)$. We have proved (3.2) over $\prod_{i=1}^N [a_i, b_i]$.

Next we call and need

$$\begin{aligned} 0 < \gamma &:= \gamma_0 + \gamma_1 \\ &= 1 - \frac{{}^{CF}D_*^\alpha f(x_1)}{2\,{}^{CF}D_*^\alpha f(x_2)} + \lambda + \frac{\left(\prod_{i=1}^N \left(\frac{1-e^{-\gamma_i b_i}}{\alpha_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1 ... \partial x_N}\right\|_\infty}{2\alpha\left({}^{CF}D_*^\alpha f\right)(x_2)} \\ &< 1, \end{aligned} \quad (3.86)$$

equivalently,

$$\lambda + \frac{\left(\prod_{i=1}^N \left(\frac{1-e^{-\gamma_i b_i}}{\alpha_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1 ... \partial x_N}\right\|_\infty}{2\alpha\left({}^{CF}D_*^\alpha f\right)(x_2)} < \frac{{}^{CF}D_*^\alpha f(x_1)}{2\,{}^{CF}D_*^\alpha f(x_2)}, \quad (3.87)$$

equivalently,

$$2\lambda\,{}^{CF}D_*^\alpha f(x_2) + \left(\prod_{i=1}^N \left(\frac{1-e^{-\gamma_i b_i}}{\alpha_i}\right)\right)\left\|\frac{\partial^N f}{\partial x_1 ... \partial x_N}\right\|_\infty < {}^{CF}D_*^\alpha f(x_1), \quad (3.88)$$

which is possible for small λ, b_i, $i = 1, ..., N$. We have proved that

$$\gamma = \gamma_0 + \gamma_1 \in (0, 1), \quad (3.89)$$

fulfilling (3.3). Hence equation (3.73) can be solved with our presented numerical methods. Consequently, our presented Numerical methods here, Theorem 2.1, apply to solve

$$f(x) = 0. \quad (3.90)$$

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