Nonlinear Functional Analysis and Applications
Vol. 21, No. 3 (2016), pp. 497-500

http://nfaa.kyungnam.ac.kr/journal-nfaa

THE CONVERGENCE OF FIXED POINT FOR A KIND OF WEAK CONTRACTION

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Abstract. In this paper we will provide a new fixed point convergence theorem. Our theorem includes the known results of [2].

1. Introduction

In this paper, let (X, d) be a metric space and T be a self-map of X. Denote $\Psi = \{$the class of all continuous nondecreasing function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(0) = 0\}$. Now we recall some nonlinear mappings.

A mapping T is said to be contraction if there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \tag{1.1}$$

holds for all $x, y \in X$. T is called φ-weak contraction if there exists $\varphi \in \Psi$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.2}$$

holds for all $x, y \in X$. Let $\varphi(t) = (1 - \alpha)t$, then φ-weak contraction contains contraction as special cases.

The definition of the φ-weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who proved the existence of fixed points in Hilbert spaces. Later, Rhoades [2] in 2001 extended the results of [1] to metric spaces.

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[0]Received January 18, 2016. Revised March 31, 2016.
[0]2010 Mathematics Subject Classification: 47H10, 47H17.
[0]Keywords: Complete metric space, generalized φ-weak contraction, fixed point.
[0]This work was supported by Natural Science Foundation of Hebei Province (No. A2015210098) and Science and Technology Research of Higher Education in Hebei province(No. ZD2015035).

Theorem 1.1. ([2]) *Let (X, d) be a complete metric space, and let T be a φ-weak contraction on X. Then T has a unique fixed point.*

We notice immediately that if T is φ-weak contraction, then T satisfies the following inequality

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)). \tag{1.3}$$

However, the converse is not true in general. See example as follows.

Example 1.2. Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ and let $Tx = \frac{1}{3}x$ for each $x \in X$. Define $\varphi(t) : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{3}{2}t$. Then T satisfies (1.3), but T does not satisfy inequality (1.2). Indeed,

$$\begin{aligned} d(Tx, Ty) =& |\frac{1}{3}x - \frac{1}{3}y| \\ \leq& |x - y| - \frac{3}{2} \cdot \frac{|x - y|}{3} \\ =& d(x, y) - \varphi(d(Tx, Ty)) \end{aligned}$$

and

$$\begin{aligned} d(Tx, Ty) =& |\frac{1}{3}x - \frac{1}{3}y| \\ \geq& |x - y| - \frac{3}{2}|x - y| \\ =& d(x, y) - \varphi(d(x, y)) \end{aligned}$$

hold for all $x, y \in X$. The examples above show that (1.3) properly includes the class of φ-weak contractions.

Example 1.3. Let $X = [0, +\infty)$ be endowed by $d(x, y) = |x - y|$ and let $Tx = \frac{x}{1+x}$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{t^2}{1+t}$. Then

$$\begin{aligned} d(Tx, Ty) =& |\frac{x}{1+x} - \frac{y}{1+y}| = \frac{|x - y|}{(1+x)(1+y)} \\ \leq& \frac{|x - y|}{1 + |x - y|} = |x - y| - \frac{|x - y|^2}{1 + |x - y|} \\ =& d(x, y) - \varphi(d(x, y)) \end{aligned}$$

holds for all $x, y \in X$. So T is a φ-weak contraction. However T is not a contraction. Therefore it is a more significance to study the class of mappings in fixed point theory and applications field.

In this paper, we will research the convergence theorems of fixed point for more generalized φ-weak contractions in complete metric spaces.

2. Preliminaries

Theorem 2.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a generalized φ-weak contraction, i.e., T satisfies the following inequality*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)). \tag{2.1}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be defined by $x_{n+1} = Tx_n$ with $x_n \neq Tx_n$. Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq d(x_n, x_{n-1}) - \varphi(d(x_{n+1}, x_n)) \\ &\leq d(x_n, x_{n-1}), \end{aligned}$$

it implies that $\{d(x_{n+1}, x_n)\}$ is a monotone decreasing bounded sequence. There exists a real number $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r.$$

Since φ is continuous, we have

$$r \leq r - \varphi(r).$$

That is $\varphi(r) \leq 0$, *i.e.*, $r = 0$. So $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Next we prove that $\{x_n\}$ is a Cauchy sequence. Let $c_n = \sup\{d(x_i, x_j) : i, j \geq n\}$. Then $\{c_n\}$ is decreasing. If $\lim_{n\to\infty} c_n = 0$, then we are done. Assume that $\lim_{n\to\infty} c_n = c > 0$. For taking $\varepsilon < \frac{c}{2}$ small enough, there exists N such that

$$d(x_{n+1}, x_n) < \varepsilon, \quad c - \varepsilon < c_n < c + \varepsilon$$

for all $n > N$. By the definition of $\{c_n\}$, there exists n_k, n_l such that

$$d(x_{n_k}, x_{n_l}) > c_n - \varepsilon > c - 2\varepsilon > 0.$$

And we also have

$$d(x_{n_k-1}, y_{n_l-1}) \leq c_{n-1} < c + \varepsilon.$$

Since

$$\begin{aligned} d(x_{n_k}, x_{n_l}) &= d(Tx_{n_k-1}, Tx_{n_l-1}) \\ &\leq d(x_{n_k-1}, x_{n_l-1}) - \varphi(d(x_{n_k}, x_{n_l})), \end{aligned}$$

then

$$c - 2\varepsilon < d(x_{n_k}, x_{n_l}) \leq d(x_{n_k-1}, x_{n_l-1}) < c + \varepsilon.$$

Because of arbitrariness of ε, it follows that

$$d(x_{n_k}, x_{n_l}), d(x_{n_k-1}, x_{n_l-1}) \to c \ \ as \ \ \varepsilon \to 0.$$

Hence, we obtain from above inequality that

$$\varphi(c) \leq 0,$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and so it is convergent in the complete metric space X. Let $\lim_{n\to\infty} x_n = q$.

Finally, we show that q is unique fixed point of T. If $q \neq Tq$, then $d(q, Tq) > 0$. Since

$$\begin{aligned} d(Tq, x_{n+1}) =& d(Tq, Tx_n) \\ \leq& d(q, x_n) - \varphi(d(Tq, Tx_n)) \\ =& d(q, x_n) - \varphi(d(Tq, x_{n+1})), \end{aligned}$$

i.e.,

$$d(Tq, x_{n+1}) + \varphi(d(Tq, x_{n+1})) \leq d(q, x_n).$$

Taking limit as $n \to \infty$ for above inequality,

$$d(Tq, q) + \varphi(d(Tq, q)) \leq 0,$$

which is a contradiction and so $q = Tq$. For uniqueness of fixed point of T. Indeed, if it is not true, then there exists $p \in X$ with $Tp = p \neq q$. Observe that

$$\begin{aligned} d(q, p) =& d(Tq, Tp) \\ \leq& d(q, p) - \varphi(d(Tq, Tp)) \\ =& d(q, p)) - \varphi(d(q, p)), \end{aligned}$$

i.e., $\varphi(d(q, p)) \leq 0$, which is a contradiction. The proof is completed. □

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a φ-weak contraction. Then T has a unique fixed point.*

References

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