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LOWER AND UPPER SEMI CONVERGENCE IN NORMED QUASILINEAR SPACES

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Abstract. In this paper concerning with the notion of convergence in normed quasilinear spaces, we show that convergence of a sequence depends directly on partial order relation defined on the normed quasilinear space. Then we introduce the notions of "lower semi convergence" and "upper semi convergence". We note that these new definitions about convergence would be an alternative for not convergent sequences in normed quasilinear spaces.

1. Introduction

Aseev [2] launched a new branch of functional analysis by introducing the concept of quasilinear spaces which is generalization of classical linear spaces. He used a partial order relation to define quasilinear spaces and gave coherent counterparts of some results in linear spaces. Aseev's approach provides a suitable base and necessary tools to proceed algebra and analysis on normed quasilinear spaces as similar to the theory of normed linear spaces. So, Aseev's attempt brings an extended point of view to classical functional analysis and allows us to construct a kind of theory of quasilinear functional analysis, [4, 5, 6, 7, 8, 10].

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This study deals with the notion of convergence of a sequence, which is one of the important concepts in mathematics, in normed quasilinear spaces.

In next section, we give some definitions and auxiliary facts about quasilinear spaces and normed quasilinear spaces. Then we introduce the concepts of lower and upper semi convergence. Also, we obtain some results related to these concepts.

2. Preliminaries and some results on quasilinear spaces and normed quasilinear spaces

Definition 2.1. ([2]) A set X is called *quasilinear space* (*qls,* for short), if a partial order relation "$\preceq$", an algebraic sum operation and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for all elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:

$$x \preceq x, \tag{2.1}$$

$$x \preceq z \quad \text{if } x \preceq y \text{ and } y \preceq z, \tag{2.2}$$

$$x = y \quad \text{if } x \preceq y \text{ and } y \preceq x, \tag{2.3}$$

$$x + y = y + x, \tag{2.4}$$

$$x + (y + z) = (x + y) + z, \tag{2.5}$$

$$\text{there exists an element } \theta \in X \text{ such that } x + \theta = x, \tag{2.6}$$

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x, \tag{2.7}$$

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \tag{2.8}$$

$$1 \cdot x = x, \tag{2.9}$$

$$0 \cdot x = \theta, \tag{2.10}$$

$$(\alpha + \beta) \cdot x \preceq \alpha \cdot x + \beta \cdot x, \tag{2.11}$$

$$x + z \preceq y + v \quad \text{if } x \preceq y \text{ and } z \preceq v, \tag{2.12}$$

$$\alpha \cdot x \preceq \alpha \cdot y \quad \text{if } x \preceq y. \tag{2.13}$$

A qls X with the partial order relation "$\preceq$" is denoted by $(X, \preceq)$.

Every linear space is a qls with the partial order relation "$=$". The most popular example of quasilinear spaces which is not linear space is the set of all closed intervals of real numbers with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and multiplication by a real number λ is defined by

$$\lambda \cdot A = \{\lambda a : a \in A\}.$$

We denote this set by $\Omega_C(\mathbb{R})$. Another one is $\Omega(\mathbb{R})$ which is the set of all nonempty compact subsets of real numbers. In general, $\Omega(E)$ the set of all nonempty, closed and bounded subsets of any normed linear space E and $\Omega_C(E)$ denotes its subset of all nonempty convex sets. Both of them are nonlinear quasilinear spaces with the inclusion relation, a slight modification of addition shaped

$$A+B=\overline{\{a+b : a\in A,\ b\in B\}}$$

and multiplication by a real number λ is defined by $\lambda\cdot A=\{\lambda a : a\in A\}$.

Lemma 2.2. ([2]) *In a qls $(X,\preceq)$ the element θ is minimal, i.e., $x=\theta$ if $x\preceq\theta$.*

An element x' is called *inverse* of $x\in X$ if $x+x'=\theta$. Further, if the inverse element exists, then it is unique. An element x possessing inverse is called *regular*, otherwise is called *singular*. X_r and X_s stand for the sets of all regular and singular elements of X, respectively. We note that the minimality is not only a property of θ but also is shared by the other regular elements ([10]). It will be assumed throughout the paper that $-x=(-1)\cdot x$ and an element x in a qls is regular if and only if $x-x=\theta$ equivalently $x'=-x$.

Suppose that every element x in a qls X has inverse element $x'\in X$. Then the partial order relation in X is determined by equality, the distributivity conditions hold and consequently, X is a linear space ([2]). In a real linear space, equality relation is the only way to define a partial order relation such that the conditions (2.1)-(2.13) hold.

Let X be a qls, $Y\subseteq X$ and Y be a qls with the same partial order relation and the restriction of the operations on X to Y. Then Y is called a *sub-quasilinear spaces or subspace, shortly* of X. The following characterization of subspace in a qls is the same as in linear spaces, and its proof is similar to its counterpart in the classical.

Theorem 2.3. ([10]) *Let X be a qls and $Y\subseteq X$. Then Y is a subspace of X if and only if $\alpha\cdot x+\beta\cdot y\in Y$ for all $x,y\in Y$ and $\alpha,\beta\in\mathbb{R}$.*

Suppose that every element x in Y has inverse element $x'\in Y$. Then the partial order relation on Y is determined by the equality. In this case the distributivity conditions in (2.11) hold on Y and so Y is a *linear subspace* of the qls X.

An element $x\in X$ is said to be *symmetric* if $-x=x$. Let X_d denotes the set of all symmetric elements of X. X_r, X_d and $X_s\cup\{\theta\}$ are subspaces of X and are called regular, symmetric and singular subspaces of X, respectively.

For example, let $X = \Omega_C(\mathbb{R})$ and $Z = \{0\} \cup \{[a,b] : a,b \in \mathbb{R} \text{ and } a < b\}$. Then Z is singular subspace of X. On the other hand, the set of all singletons of real numbers $\{\{a\} : a \in \mathbb{R}\}$ is regular subspace of X ([10]).

Definition 2.4. ([2]) Let $(X, \preceq)$ be a qls. A function $\|.\|_X : X \longrightarrow \mathbb{R}$ is called a *norm* if the following conditions are satisfied:

$$\|x\|_X > 0 \quad \text{if } x \neq \theta, \tag{2.14}$$

$$\|x+y\|_X \leq \|x\|_X + \|y\|_X \,, \tag{2.15}$$

$$\|\alpha \cdot x\|_X = |\alpha| \, \|x\|_X \,, \tag{2.16}$$

$$\|x\|_X \leq \|y\|_X \quad \text{if } x \preceq y,$$

$$\begin{aligned} \text{if for any } \epsilon &> 0 \text{ there exists an element } x_\epsilon \in X \text{ such that} \\ x &\preceq y + x_\epsilon \text{ and } \|x_\epsilon\|_X \leq \epsilon \text{ then } x \preceq y. \end{aligned} \tag{2.17}$$

A qls X with a norm defined on it, is called *normed quasilinear space* (*briefly, normed qls*).

If there is not any confused, we write only $\|x\|$ instead of $\|x\|_X$.

If every element in a qls has inverse then the concept of normed qls coincides with the concept of normed linear space ([2]).

Let $(X, \preceq)$ be a normed qls. *Hausdorff metric* or *norm metric* on X is defined by the equality

$$h_X(x,y) = \inf \{r \geq 0 : x \preceq y + a_1^r, \;\; y \preceq x + a_2^r \quad \text{and} \quad \|a_i^r\| \leq r, \; i = 1,2\} \,.$$

Since $x \preceq y + (x-y)$ and $y \preceq x + (y-x)$, the quantity $h_X(x,y)$ is well defined. It is not hard to see that the function $h_X(x,y)$ satisfies all of the metric axioms. Further, $h_X(x,y) \leq \|x-y\|_X$ for any elements $x, y \in X$. Since $h_X(x,y)$ may not equal to $\|x-y\|_X$ if X is a nonlinear qls, we use the metric instead of the norm to discuss topological properties in normed quasilinear spaces. So $x_n \to x$ if and only if $h_X(x_n,x) \to 0$ in a normed qls. Although $\|x_n - x\|_X \to 0$ always implies $x_n \to x$ in normed quasilinear spaces, $x_n \to x$ may not imply $\|x_n - x\|_X \to 0$.

Let X be a real Banach space. Then X is a complete normed qls with partial order relation given by equality. Conversely, if X is a complete normed qls and every $x \in X$ has inverse element $x' \in X$, then X is a real Banach space. Also the partial order relation on X is equality. In this case $h_x(x,y) = \|x-y\|_X$ ([2]).

Let E be a normed real linear space. Then $\Omega(E)$ and $\Omega_C(E)$ are normed quasilinear spaces with the norm is defined by

$$\|A\|_{\Omega} = \sup_{a\in A} \|a\|_E . \tag{2.18}$$

In this case, the Hausdorff metric is defined as usual:

$$h_{\Omega}(A,B) = \inf\{r \geq 0 : A \subseteq B + S(\theta,r),\ B \subseteq A + S(\theta,r)\},$$

where $S(\theta,r)$ denotes the closed ball of radius r and centered at $\theta \in E$ ([2]).

Lemma 2.5. ([2]) *The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is a continuous function with respect to the Hausdorff metric.*

3. Lower and upper semi convergence in normed quasilinear spaces

The need to discuss convergence in the context of quasilinear functional analysis requires a suitable metric. So, we will use

$$h_X(x,y) = \inf\left\{r \geq 0 : x \preceq y + a_1^r,\ \ y \preceq x + a_2^r \ \text{ and } \ \|a_i^r\| \leq r,\ i = 1,2\right\},$$

as a measure of distance between the elements x and y of the normed qls $(X,\preceq)$. We note that if there is not any confused we write only h instead of h_X.

We need the following proposition before explaining why we introduce concepts of lower and upper semi convergence. The main idea used to define these new notions is based on commentation of this proposition.

Proposition 3.1. *Let $(X,\preceq)$ be a normed qls and h be Hausdorff metric induced by the norm on X. Then the right sides of following propositions about the convergence of a sequence (x_n) in X are equivalent:*

- $x_n \to x \Leftrightarrow$ *"for every $\epsilon > 0$ there exist sequences $\left(a_{i,n}^{\epsilon}\right) \subset X$ and a natural number N=$N(\epsilon)$ such that $x_n \preceq x + a_{1,n}^{\epsilon}$, $x \preceq x_n + a_{2,n}^{\epsilon}$ and $\left\|a_{i,n}^{\epsilon}\right\| \leq \epsilon$ for all $n \geq N$, $i = 1,2$."*
- $x_n \to x \Leftrightarrow$ *"for every $\epsilon > 0$ there exists a natural number N=$N(\epsilon)$ such that $h\left(x_n,x\right) \leq \epsilon$ for all $n \geq N$."*

Proof. Suppose that for every $\epsilon > 0$ there are sequences $\left(a_{i,n}^{\epsilon}\right) \subset X$ and a natural number N=$N(\epsilon)$ such that

$$x_n \preceq x + a_{1,n}^{\epsilon},\ \ x \preceq x_n + a_{2,n}^{\epsilon} \ \text{ and } \ \left\|a_{i,n}^{\epsilon}\right\| \leq \epsilon,\ \ i = 1,2 \tag{3.1}$$

for all $n \geq N$. Since ϵ is one of values r providing (3.1), we have

$$\inf \left\{ r \geq 0 : \ x_n \preceq x + a^r_{1,n}, \ x \preceq x_n + a^r_{2,n} \ \text{ and } \ \left\| a^r_{i,n} \right\| \leq r, \ i = 1, 2 \right\} \leq \epsilon,$$

by definition of infimum. Hence we obtain $h(x_n, x) \leq \epsilon$ for $n \geq N$.

On the other hand, suppose that for every $\epsilon > 0$ there exist sequences $\left(a^\epsilon_{i,n} \right) \subset X$ and a natural number $N{=}N(\epsilon)$ such that

$$\begin{aligned} h(x_n, x) &= \inf \left\{ r \geq 0 : x_n \preceq x + a^r_{1,n}, \ x \preceq x_n + a^r_{2,n}, \ \left\| a^r_{i,n} \right\| \leq r, \ i = 1, 2 \right\} \\ &\leq \epsilon \end{aligned}$$

for all $n \geq N$. Since the infimum of all values r providing the features $x_n \preceq x + a^r_{1,n}$, $x \preceq x_n + a^r_{2,n}$ and $\left\| a^r_{i,n} \right\| \leq r$ is less than or equal to ϵ,

$$x_n \preceq x + a^\epsilon_{1,n}, \ x \preceq x_n + a^\epsilon_{2,n} \ \text{ and } \ \left\| a^\epsilon_{i,n} \right\| \leq \epsilon, \ i = 1, 2$$

hold for this ϵ. □

So, from the Proposition 3.1, $\lim x_n = x$ means that for every $\epsilon > 0$ it can be found sequences $\left(a^\epsilon_{i,n} \right) \subset X$ and $N(\epsilon) \in \mathbb{N}$ such that $x_n \preceq x + a^\epsilon_{1,n}$, $x \preceq x_n + a^\epsilon_{2,n}$ and $\left\| a^\epsilon_{i,n} \right\| \leq \epsilon$ $(i = 1, 2)$ for all $n \geq N$.

With a careful observation, it can be seen that for all $\epsilon > 0$ and $n \geq N$, $x_n \preceq x + a^\epsilon_{1,n}$ and $x \preceq x_n + a^\epsilon_{2,n}$ with $\left\| a^\epsilon_{i,n} \right\| \leq \epsilon$ $(i = 1, 2)$ may not be satisfied simultaneously. Taking into account this situation, in this study we introduce new definitions about convergence of sequences in normed quasilinear spaces. These new definitions about covergence would be an alternative for the not convergent sequences in normed quasilinear spaces.

Definition 3.2. Let (x_n) be a sequence in normed qls $(X, \preceq)$ and $x \in X$. Then, it is called that (x_n) is lower semi convergent to x, if for every $\epsilon > 0$ there exist a sequence $(a^\epsilon_n) \subset X$ and a natural number $N = N(\epsilon)$ such that

$$x \preceq x_n + a^\epsilon_n, \ \ \| a^\epsilon_n \| \leq \epsilon$$

for all $n \geq N$.

It is called that (x_n) is upper semi convergent to y, if for every $\epsilon > 0$ there exist a sequence $(b^\epsilon_n) \subset X$ and a natural number $N = N(\epsilon)$ such that

$$x_n \preceq y + b^\epsilon_n, \ \ \| b^\epsilon_n \| \leq \epsilon$$

for all $n \geq N$.

Otherwise it is called that (x_n) is not lower (upper) semi convergent to x (y).

Also x (y) is said to be the lower (upper) semi limit of the sequence (x_n) and it is written as

$$ls\text{-}lim\ x_n = x\ (us\text{-}lim\ x_n = y).$$

Remark 3.3. The lower or upper semi limit of a sequence in a normed qls $(X, \preceq)$, (if there exists it), is not unique. But all lower (upper) semi limits can be comparable with each other according to the partial order relation on qls. That is, if *ls-lim* $x_n = x_1$ and *ls-lim* $x_n = x_2$, then $x_1 \preceq x_2$ or $x_2 \preceq x_1$. Similarly, if *us-lim* $x_n = y_1$ and *us-lim* $x_n = y_2$, then $y_1 \preceq y_2$ or $y_2 \preceq y_1$.

Proposition 3.4. *Let $(X, \preceq)$ be a qls, (x_n) be a sequence in X and $x, y \in X$.*

(i) *If ls-lim $x_n = x$ and $y \preceq x$, then ls-lim $x_n = y$.*
(ii) *If us-lim $x_n = x$ and $x \preceq y$, then us-lim $x_n = y$.*

Proof. (i) Let *ls-lim* $x_n = x$ and $y \preceq x$. Then for every $\epsilon > 0$ there exist a sequence $(a_n^\epsilon) \subset X$ and a natural number $N = N(\epsilon)$ such that $x \preceq x_n + a_n^\epsilon$, $\|a_n^\epsilon\| \leq \epsilon$ for all $n \geq N$. Since $y \preceq x$, we write $y \preceq x_n + a_n^\epsilon$ for all $n \geq N$. Hence *ls-lim* $x_n = y$.
(ii) Since proof of (ii) is similar to proof of (i), we omit it. □

Theorem 3.5. *Let $(X, \preceq)$ be a normed qls, (x_n) be a sequence in X and $x, y \in X$. If ls-lim $x_n = x$, us-lim $x_n = y$, then $x \preceq y$.*

Proof. Suppose that *ls-lim* $x_n = x$ and *us-lim* $x_n = y$. Then, for every $\epsilon > 0$ there exist sequences (a_n^ϵ), $(b_n^\epsilon) \subset X$ and natural numbers N, N' such that $x \preceq x_n + a_n^\epsilon$, $\|a_n^\epsilon\| \leq \epsilon/2$ for all $n \geq N$ and $x_n \preceq y + b_n^\epsilon$, $\|b_n^\epsilon\| \leq \epsilon/2$ for all $n \geq N'$. From $a_n^\epsilon \preceq a_n^\epsilon$ and $x_n \preceq y + b_n^\epsilon$, we can write $x_n + a_n^\epsilon \preceq y + a_n^\epsilon + b_n^\epsilon$ for all $n \geq N^* = \max\{N, N'\}$. Since $\|a_n^\epsilon + b_n^\epsilon\| \leq \|a_n^\epsilon\| + \|b_n^\epsilon\| \leq \epsilon/2 + \epsilon/2 = \epsilon$, we obtain $x \preceq y$ from (2.17). □

Hence for any sequence (x_n) which has upper and lower semi limits, we say that *ls-lim* $x_n \preceq$ *us-lim* x_n.

A sequence in a normed qls may be convergent to x from below and to y from above. That is, in a normed qls, a sequence can be lower and upper semi convergent. However, this case does not mean that this sequence is convergent. But we have the following result:

Proposition 3.6. *Let $(X, \preceq)$ be a normed qls, (x_n) be a sequence in X and $x \in X$. If the sequence (x_n) is both lower and upper semi convergent to x, then (x_n) is convergent to x.*

Proof. Let $\epsilon > 0$. If *ls-lim* $x_n = x$, then for every $\epsilon > 0$ there exist a sequence $(a_n^\epsilon) \subset X$ and a natural number $N = N(\epsilon)$ such that $x \preceq x_n + a_n^\epsilon$, $\|a_n^\epsilon\| \leq \epsilon$ for all $n \geq N$.

If *us-lim* $x_n = x$, then for every $\epsilon > 0$ there exist a sequence $(b_n^\epsilon) \subset X$ and a natural number $N' = N'(\epsilon)$ such that $x_n \preceq x + b_n^\epsilon$, $\|b_n^\epsilon\| \leq \epsilon$ for all $n \geq N'$.

If we choose $N^* = \max\{N, N'\}$, we can write $x \preceq x_n + a_n^\epsilon$, $\|a_n^\epsilon\| \leq \epsilon$ and $x_n \preceq x + b_n^\epsilon$, $\|b_n^\epsilon\| \leq \epsilon$ for all $n \geq N^*$. This means that $\lim x_n = x$. □

Example 3.7. In $\Omega_C(\mathbb{R})$, consider the sequence $(x_n) = \left(\left[-\frac{1}{n}, 1 + \frac{1}{n}\right]\right)$. Then (x_n) is both lower and upper semi convergent to $x = [0,1] \in \Omega_C(\mathbb{R})$:

Let $\epsilon > 0$, if we choose $N = 1$ and $a_n^\epsilon = \{0\} \in \Omega_C(\mathbb{R})$, we have

$$[0,1] \subseteq \left[-\frac{1}{n}, 1 + \frac{1}{n}\right] + a_n^\epsilon \quad \text{and} \quad \|a_n^\epsilon\| = \|\{0\}\| = 0 \leq \epsilon$$

for all $n \geq N$. Hence *ls-lim* $x_n = x$.

On the other hand, (x_n) is upper semi convergent to $x = [0,1] \in \Omega_C(\mathbb{R})$:

For given every $\epsilon > 0$, there are a sequence (b_n^ϵ) defined by $b_n^\epsilon = \left[-\frac{1}{n}, \frac{1}{n}\right]$ and a natural number

$$N(\epsilon) = \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1$$

such that

$$\|b_n^\epsilon\| = \left\| \left[-\frac{1}{n}, \frac{1}{n}\right] \right\| = \frac{1}{n} < \frac{1}{\left\lfloor \frac{1}{\epsilon} \right\rfloor} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

and

$$\left[-\frac{1}{n}, 1 + \frac{1}{n}\right] \subseteq [0,1] + \left[-\frac{1}{n}, \frac{1}{n}\right] = \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$$

for all $n \geq \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1$, where $\left\lfloor \frac{1}{\epsilon} \right\rfloor$ denotes the integer part of $\frac{1}{\epsilon}$. So we have *us-lim* $x_n = x$. Therefore the sequence (x_n) is convergent to x.

Any sequence in a normed qls can be lower semi convergent without being upper semi convergent, and vice versa. The following example shows this situation:

Example 3.8. In the above example, (x_n) is not upper semi convergent to $y = \left[0, \frac{1}{2}\right] \in \Omega_C(\mathbb{R})$, although the sequence it is lower semi convergent to the element y :

It is obvious that (x_n) is lower semi convergent to y.

Now we show that this sequence is not upper semi convergent to y. To do this, we assume that *us-lim* $x_n = y = \left[0, \frac{1}{2}\right]$. Then, for $\epsilon = \frac{1}{100}$ we can find a sequence $(b_n^\epsilon) \subset \Omega_C(\mathbb{R})$ and an element $N(\epsilon)$ such that

$$\left[-\frac{1}{n}, 1 + \frac{1}{n}\right] = \left[-\frac{1}{N+1000}, 1 + \frac{1}{N+1000}\right] \subseteq [0, \frac{1}{2}] + b_n^\epsilon, \;\; \|b_n^\epsilon\| \leq \epsilon = \frac{1}{100}$$

for $n = N + 1000$. Whereas, it must be $b_n^\epsilon \subseteq \left[-\frac{1}{100}, \frac{1}{100}\right]$ for $\|b_n^\epsilon\| \leq \frac{1}{100}$. In this case, we write

$$\begin{aligned}\left[-\frac{1}{N+1000}, 1 + \frac{1}{N+1000}\right] &\subseteq [0, \frac{1}{2}] + \left[-\frac{1}{100}, \frac{1}{100}\right] \\ &= \left[-\frac{1}{100}, \frac{1}{2} + \frac{1}{100}\right].\end{aligned} \tag{3.2}$$

Since

$$\frac{1}{2} + \frac{1}{100} \leq 1 + \frac{1}{N+1000},$$

the including (3.2) is not possible. This contradiction completes the proof.

Example 3.9. We consider again the sequence (x_n) in Example 3.7. It is not hard to see that it is not lower semi convergent to $z = [0, 2] \in \Omega_C(\mathbb{R})$, although the sequence (x_n) is upper semi convergent to z.

Example 3.10. Consider the sequence (x_n) defined by $x_n = [-n, n]$ in $\Omega_C(\mathbb{R})$. Then (x_n) is not upper semi convergent in $\Omega_C(\mathbb{R})$. But this sequence is lower semi convergent to every elements x of $\Omega_C(\mathbb{R})$ such that $x \subseteq [-1, 1]$:

Let $\epsilon > 0$. If we choose $N = 1$ and $a_n^\epsilon = \{0\} \in \Omega_C(\mathbb{R})$, we have

$$x \subseteq [-1, 1] \subseteq [-n, n] + a_n^\epsilon, \quad \|a_n^\epsilon\| = \|\{0\}\| = 0 \leq \epsilon$$

for all $n \geq N$. Hence, we say that *ls-lim* $x_n = x$ for every elements x such that $x \subseteq [-1, 1] \in \Omega_C(\mathbb{R})$.

On the other hand, since $n \in \mathbb{N}$ is arbitrary, this shows that for given any $\epsilon > 0$, no there exist any elements y and (b_n^ϵ) in $\Omega_C(\mathbb{R})$ and $N = N(\epsilon) \in \mathbb{N}$ such that

$$[-n, n] \subseteq y + b_n^\epsilon \quad \text{and} \quad \|b_n^\epsilon\| \leq \epsilon$$

for all $n \geq N$. Thus the sequence (x_n) is not upper semi convergent to any element of $\Omega_C(\mathbb{R})$.

Example 3.11. Consider the sequence (x_n) defined by

$$x_n = \begin{cases} \left[-\frac{1}{n} + 1, 1 + \frac{1}{n}\right], & \text{if } n \text{ is odd}, \\ \left[-\frac{1}{n} - 1, -1 + \frac{1}{n}\right], & \text{if } n \text{ is even}, \end{cases}$$

in $\Omega_C(\mathbb{R})$. Then (x_n) is upper semi convergent to $[-2, 2]$. But we note that this sequence is not lower semi convergent to any element of $\Omega_C(\mathbb{R})$.

In the following section, we will prove some results that follow immediately from the Definition 3.2.

Theorem 3.12. *Let* $(X, \preceq)$ *be a normed qls,* $(x_n), (y_n)$ *be sequences in* X *and* $x, y \in X$. *Then,*

(i) *If* ls-lim $x_n = x$ *and* $x_n \preceq y_n$ *for all* $n \in \mathbb{N}$*, then* ls-lim $y_n = x$.
(ii) *If* us-lim $x_n = y$ *and* $y_n \preceq x_n$ *for all* $n \in \mathbb{N}$*, then* us-lim $y_n = y$.

Proof. (i) Suppose that ls-lim $x_n = x$ and $x_n \preceq y_n$ for all $n \in \mathbb{N}$. Then for every $\epsilon > 0$ there exist a sequence $(a_n^\epsilon) \subset X$ and a natural number $N = N(\epsilon)$ such that $x \preceq x_n + a_n^\epsilon$, $\|a_n^\epsilon\| \leq \epsilon$ for all $n \geq N$. Since $x_n \preceq y_n$, we can write $x \preceq y_n + a_n^\epsilon$. Thus ls-lim $y_n = x$. One can prove (ii) with a similar way. □

Theorem 3.13. *Let* $(X, \preceq)$ *be a normed qls,* $(x_n), (y_n)$ *be sequences in* X *and* $x, y \in X$. *If* ls-lim $x_n = x$, us-lim $y_n = y$ *and* $x_n \preceq y_n$ *for all* $n \in \mathbb{N}$*, then* $x \preceq y$.

Proof. Assume that ls-lim $x_n = x$ and us-lim $y_n = y$. For every $\epsilon > 0$ there exist sequences $(a_n^\epsilon), (b_n^\epsilon) \subset X$ and natural numbers N, N'such that $x \preceq x_n + a_n^\epsilon$, $\|a_n^\epsilon\| \leq \epsilon/2$ for all $n \geq N$ and $y_n \preceq y + b_n^\epsilon$, $\|b_n^\epsilon\| \leq \epsilon/2$ for all $n \geq N'$. Let $N^* = \max\{N, N'\}$. Because of the fact that $x_n \preceq y_n$ for all $n \in \mathbb{N}$, we get $x \preceq y + a_n^\epsilon + b_n^\epsilon$ for $n \geq N^*$. Since $\|a_n^\epsilon + b_n^\epsilon\| \leq \|a_n^\epsilon\| + \|b_n^\epsilon\| \leq \epsilon/2 + \epsilon/2 = \epsilon$, we obtain $x \preceq y$ from (2.17). □

Now we will give an analog of the Squeeze Theorem in the setting of normed qls.

Theorem 3.14. *Let* $(x_n), (y_n), (z_n)$ *be sequences in normed qls* $(X, \preceq)$ *and* $x \in X$. *If* ls-lim $x_n = x$, us-lim $z_n = x$ *and* $x_n \preceq y_n \preceq z_n$ *for all* $n \in \mathbb{N}$*, then* $\lim y_n = x$.

Proof. Let $(x_n), (y_n), (z_n)$ be as stated above. Then, for every $\epsilon > 0$ there exist sequences $(a_n^\epsilon), (b_n^\epsilon) \subset X$ and natural numbers N, N'such that $x \preceq x_n + a_n^\epsilon$, $\|a_n^\epsilon\| \leq \epsilon$ for all $n \geq N$ and $z_n \preceq x + b_n^\epsilon$, $\|b_n^\epsilon\| \leq \epsilon$ for all $n \geq N'$. Since $x_n \preceq y_n$ and $y_n \preceq z_n$ for all $n \in \mathbb{N}$, we have $x \preceq y_n + a_n^\epsilon$ and $y_n \preceq x + b_n^\epsilon$ for all $n \geq N^* = \max\{N, N'\}$. Consequently, we obtain $\lim y_n = x$. □

Theorem 3.15. *Let* $(X, \preceq)$ *be a normed qls,* $(x_n), (y_n)$ *be sequences in* X *and* $x, y \in X$.

(i) *If* ls-lim $x_n = x$ *and* ls-lim $y_n = y$ *then* ls-lim $(\lambda \cdot x_n + \beta \cdot y_n) = \lambda \cdot x + \beta \cdot y$,
(ii) *If* us-lim $x_n = x$ *and* us-lim $y_n = y$ *then* us-lim $(\lambda \cdot x_n + \beta \cdot y_n) = \lambda \cdot x + \beta \cdot y$,

for $\lambda, \beta \in \mathbb{R}$.

Proof. (i) Since *ls-lim* $x_n = x$ and *ls-lim* $y_n = y$ for every $\epsilon > 0$, there exist sequences (a_n^ϵ), $(b_n^\epsilon) \subset X$ and natural numbers N, N'such that $x \preceq x_n + a_n^\epsilon$, $\|a_n^\epsilon\| \leq \epsilon/(2|\lambda|)$ for all $n \geq N$ and $y \preceq y_n + b_n^\epsilon$, $\|b_n^\epsilon\| \leq \epsilon/(2|\beta|)$ for all $n \geq N'$. Taking into account the axioms (2.8), (2.12), (2.13), (2.15) and (2.16), for $N^* = \max\{N, N'\}$, we get $\lambda \cdot x + \beta \cdot y \preceq \lambda \cdot x_n + \beta \cdot y_n + \lambda \cdot a_n^\epsilon + \beta \cdot b_n^\epsilon$ and $\|\lambda \cdot a_n^\epsilon + \beta \cdot b_n^\epsilon\| \leq |\lambda| \|a_n^\epsilon\| + |\beta| \|b_n^\epsilon\| \leq \epsilon$, for all $n \geq N^*$. Thus *ls-lim* $(\lambda \cdot x_n + \beta \cdot y_n) = \lambda \cdot x + \beta \cdot y$. With the similar argument, (ii) can be proved. □

An immediate consequence of Theorem 3.15 is the following:

Corollary 3.16. *Let* (x_n) *be a sequence in normed qls* $(X, \preceq)$*,* $x, y \in X$ *and* $\lambda \in \mathbb{R}$*. Then,*

(i) *If ls-lim* $x_n = x$ *then ls-lim* $(\lambda \cdot x_n + y) = \lambda \cdot x + y$*,*
(ii) *If us-lim* $x_n = x$ *then us-lim* $(\lambda \cdot x_n + y) = \lambda \cdot x + y$*.*

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