



ON CONSTRUCTION OF FIXED POINT THEORY UNDER IMPLICIT RELATION IN HILBERT SPACES

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Abstract. The aim of this paper is to study existence and uniqueness of common fixed point for a family of self-mappings satisfying implicit relation in a Hilbert space. We also prove well-posedness of a common fixed point problem.

1. INTRODUCTION

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful techniques in the study of nonlinear equations. Different generalizations of the Banach's contraction mapping principle were studied by many authors in metric spaces and Banach spaces, see [1]-[8], [12, 13, 15, 16, 19] and references given therein. These generalizations were made either by using the contractive condition or by imposing some additional conditions on an ambient space. Koparde and Waghmode [11], Pandhare [14], Veerapandi and Anil Kumar [20] investigated the properties of fixed points

⁰Received January 25, 2016. Revised April 1, 2016.

⁰2010 Mathematics Subject Classification: 46C15, 47H10, 58C30.

⁰Keywords: Implicit relation, common fixed point, well-posedness, Hilbert space.

of sequence of mappings under contraction condition in Hilbert spaces. In [2, 3, 4], the authors studied implicit relation in metric spaces.

In spite of the above work, study of contraction condition through implicit relation in Hilbert spaces needs more investigation. In continuation of these results, in this paper, we study common fixed point theorems for a family of self mappings in Hilbert space by applying implicit relation of three dimension and then well-posed problem.

2. PRELIMINARIES

Definition 2.1. A sequence $\{x_n\}$ in a Hilbert space X is said to be asymptotically T -regular if $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.

Definition 2.2. Let X be a Hilbert space and T be a self mapping on X . Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \rightarrow x$ implies that $Tx_n \rightarrow Tx$ as $n \rightarrow \infty$.

Definition 2.3. (Implicit Relation)

Let Φ be the class of real valued continuous functions $\phi : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ non-decreasing in the first argument and satisfying the following condition: for $x, y > 0$,

$$(i) \quad x \leq \phi\left(y, \frac{x+y}{2}, x+y\right),$$

or

$$(ii) \quad x \leq \phi(x, 0, x),$$

there exists a real number $0 < h < 1$ such that $x \leq hy$.

The following are examples of the implicit relation defined above.

Example 2.4. Let $\phi(t_1, t_2, t_3) = t_1 - \alpha \max(t_2, t_3) + (2 + \alpha)t_3$, where $\alpha \geq 0$.

Example 2.5. Let $\phi(t_1, t_2, t_3) = t_1 + a \min\left(t_2, \frac{t_3}{2}\right)$, where $a \geq 2$.

Example 2.6. Let $\phi(t_1, t_2, t_3) = t_1 + c \max(t_2, t_3)$, where $\alpha \geq 1$.

Definition 2.7. Let X be a nonempty set and $S, T : X \rightarrow X$ be two self mappings. An element $x \in X$ is called a common fixed point of S and T , if $x = S(x) = T(x)$.

3. MAIN RESULTS

Saluja [19] studied the concept of implicit relation for obtaining common fixed point for two continuous self mappings in 2-Banach spaces. We formulate it in Hilbert spaces and further we extend the result as follows.

Theorem 3.1. *Let C be a closed subset of a Hilbert space X and $S, T : C \rightarrow C$ be two continuous self mappings such that*

$$\|Sx - Ty\|^2 \leq \phi \left(\|x - y\|^2, \frac{\|x - Sx\|^2 + \|y - Ty\|^2}{2}, \frac{\|x - Ty\|^2 + \|y - Sx\|^2}{2} \right), \quad (3.1)$$

for all $x, y \in C$. Then S and T have a unique common fixed point in C .

Proof. Let $x_0 \in C$ and we choose $x_1, x_2 \in C$ in such a way that $x_1 = Sx_0$ and $x_2 = Tx_1$. In general, we get a sequence in C such that

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, \dots$$

By using (3.1) and parallelogram law, we get

$$\begin{aligned} \|x_{2n+1} - x_{2n}\|^2 &= \|Sx_{2n} - Tx_{2n-1}\|^2 \\ &\leq \phi \left(\|x_{2n} - x_{2n-1}\|^2, \frac{\|x_{2n} - Sx_{2n}\|^2 + \|x_{2n-1} - Tx_{2n-1}\|^2}{2}, \frac{\|x_{2n} - Tx_{2n-1}\|^2 + \|x_{2n-1} - Sx_{2n}\|^2}{2} \right) \\ &\leq \phi \left(\|x_{2n} - x_{2n-1}\|^2, \frac{\|x_{2n} - x_{2n+1}\|^2 + \|x_{2n-1} - x_{2n}\|^2}{2}, \frac{\|x_{2n} - x_{2n}\|^2 + \|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2}{2} \right) \\ &= \phi \left(\|x_{2n} - x_{2n-1}\|^2, \frac{\|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2}{2}, \frac{\|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2}{2} \right). \end{aligned}$$

Hence from Definition 2.3 (i), there exists $0 < h < 1$ such that

$$\|x_{2n+1} - x_{2n}\|^2 \leq h \|x_{2n} - x_{2n-1}\|^2.$$

Proceeding in this way, finally we obtain

$$\|x_{2n+1} - x_{2n}\|^2 \leq h^n \|x_0 - x_1\|^2, \quad n = 1, 2, \dots$$

For any positive integer p , we get

$$\begin{aligned} \|x_n - x_{n+p}\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \cdots + \|x_{n+p-1} - x_{n+p}\| \\ &\leq \left(h^n + h^{n+1} + \cdots + h^{n+p-1} \right) \|x_0 - x_1\| \\ &\leq \frac{h^n}{1-h} \|x_0 - x_1\|. \end{aligned}$$

Since $0 < h < 1$, $\frac{h^n}{1-h} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\|x_n - x_{n+p}\| \rightarrow 0$ as $n \rightarrow \infty$. Since C is closed, there exists an element $u \in C$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. By the continuity of S and T , it is clear that $Su = Tu = u$. Hence u is a common fixed point of S and T .

For the uniqueness, let $v \in C$ be another common fixed point of S and T where $v \neq u$. Then

$$\begin{aligned} \|u - v\|^2 &= \|Su - Tv\|^2 \\ &\leq \phi \left(\|u - v\|^2, \frac{\|u - Su\|^2 + \|v - Tv\|^2}{2}, \frac{\|u - Tv\|^2 + \|v - Su\|^2}{2} \right) \\ &= \phi \left(\|u - v\|^2, 0, \|u - v\|^2 \right). \end{aligned}$$

Now, by the Definition 2.3 (ii), we have

$$\|u - v\|^2 \leq h\|u - v\|^2, \text{ where } 0 < h < 1.$$

Thus $u = v$ which shows that u is a unique common fixed point of S and T . \square

The following corollary is a particular case of Theorem 3.1, when $S = T$.

Corollary 3.2. *Let C be a closed subset of a Hilbert space X and $T : C \rightarrow C$ be a continuous self mapping such that*

$$\begin{aligned} \|Tx - Ty\|^2 \leq \phi \left(\|x - y\|^2, \frac{\|x - Tx\|^2 + \|y - Ty\|^2}{2}, \right. \\ \left. \frac{\|x - Ty\|^2 + \|y - Tx\|^2}{2} \right), \end{aligned} \tag{3.2}$$

for all $x, y \in C$. Then T has a unique fixed point in C .

Proof. The proof follows from the Theorem 3.1, if we take $S = T$. \square

Theorem 3.3. *Let C be a closed subset of a Hilbert space X and $T : C \rightarrow C$ be a self mapping satisfying (3.2). Then T has a unique fixed point in C and*

$\{z_n\}$ is asymptotically T -regular if and only if T is continuous at the fixed point of T .

Proof. Let $v \in C$ be a fixed point of T and $z_n \rightarrow v$ as $n \rightarrow \infty$. Now

$$\begin{aligned} & \|Tz_n - Tv\|^2 \\ & \leq \phi\left(\|z_n - v\|^2, \frac{\|z_n - Tz_n\|^2 + \|v - Tv\|^2}{2}, \frac{\|z_n - Tv\|^2 + \|v - Tz_n\|^2}{2}\right). \end{aligned}$$

Since $Tv = v$ and $\{z_n\}$ is asymptotically T -regular, we get

$$\|Tz_n - Tv\|^2 \leq \phi\left(\|Tz_n - Tv\|^2, 0, \|Tz_n - Tv\|^2\right).$$

By the Definition 2.3 (ii), there exists $0 < h < 1$ such that

$$\|Tz_n - Tv\|^2 \leq h\|Tz_n - Tv\|^2$$

which yields that

$$Tz_n \rightarrow Tv \quad \text{as } n \rightarrow \infty.$$

Hence T is continuous at $v \in X$. Conversely, assume that T is continuous at $v \in X$. Note that, by Corollary 3.2, T has a unique fixed point $v \in C$ and

$$z_n \rightarrow v \Rightarrow Tz_n \rightarrow Tv \quad \text{as } n \rightarrow \infty$$

which shows that

$$\|z_n - Tz_n\|^2 \rightarrow \|v - Tv\|^2 = 0,$$

since $Tv = v$. This completes the proof. □

Now we extend the Theorem 3.1 to the case of pair of mappings S^p and T^q where p and q are some positive integers.

Theorem 3.4. *Let C be a closed subset of a Hilbert space X and $S, T : C \rightarrow C$ be two continuous self mappings such that*

$$\begin{aligned} \|S^p x - T^q y\|^2 \leq \phi\left(\|x - y\|^2, \frac{\|x - S^p x\|^2 + \|y - T^q y\|^2}{2}, \right. \\ \left. \frac{\|x - T^q y\|^2 + \|y - S^p x\|^2}{2}\right), \end{aligned} \tag{3.3}$$

for all $x, y \in X$ where p and q are some positive integers. Then S and T have a unique common fixed point.

Proof. Since S^p and T^q satisfy the conditions of Theorem 3.1, S^p and T^q have a unique common fixed point. Let v be the common fixed point. Now

$$\begin{aligned} S^p v = v & \Rightarrow S(S^p v) = Sv, \\ S^p(Sv) & = Sv. \end{aligned}$$

If $Sv = x_0$ then $S^p(x_0) = x_0$. So, Sv is a fixed point of S^p . Similarly, $T^q(Tv) = Tv$. Now, we have

$$\begin{aligned} \|v - Tv\|^2 &= \|S^p v - T^q(Tv)\|^2 \\ &\leq \phi\left(\|v - Tv\|^2, \frac{\|v - S^p v\|^2 + \|Tv - T^q(Tv)\|^2}{2}, \right. \\ &\quad \left. \frac{\|v - T^q(Tv)\|^2 + \|Tv - S^p v\|^2}{2}\right) \\ &= \phi\left(\|v - Tv\|^2, 0, \|v - Tv\|^2\right). \end{aligned}$$

Hence by the Definition 2.3 (ii), we obtain

$$\|v - Tv\|^2 \leq 0.$$

Thus $v = Tv$ for $v \in C$. Similarly, $v = Sv$.

Finally, for the uniqueness of v , let $w \neq v$ be another common fixed point of S and T . Then clearly w is also a common fixed point of S^p and T^q which implies $w = v$. Hence S and T have a unique common fixed point $v \in X$. \square

Hence we have proved that if x_0 is a unique common fixed point of S^p and T^q , for some positive integers p, q then x_0 is a unique common fixed point of S and T . Further, we extend Theorem 3.1 to the case of family of mappings satisfying the condition (3.1).

Theorem 3.5. *Let C be a closed subset of a Hilbert space X and $\{F_\alpha\}$ be a family of continuous self mappings on C satisfying*

$$\begin{aligned} &\|F_\alpha x - F_\beta y\|^2 \\ &\leq \phi\left(\|x - y\|^2, \frac{\|x - F_\alpha x\|^2 + \|y - F_\beta y\|^2}{2}, \frac{\|x - F_\beta y\|^2 + \|y - F_\alpha x\|^2}{2}\right), \end{aligned}$$

for $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ and $x, y \in C$. Then there exists a unique $v \in C$ satisfying $F_\alpha v = v$ for all $\alpha \in \Lambda$.

Proof. Let us take F_α and F_β in place of S and T respectively in Theorem 3.1, an application of which gives a unique $v \in C$ to satisfy $F_\alpha v = F_\beta v = v$. For any other member F_γ , uniqueness of v gives $F_\gamma v = v$ and this completes the proof. \square

Theorem 3.6. *Let C be a closed subset of a Hilbert space X and $\{F_n\}$ be a sequence of self mappings on C such that $\{F_n\}$ converging pointwise to a self*

mapping F and

$$\begin{aligned} & \|F_n x - F_n y\|^2 \\ & \leq \phi \left(\|x - y\|^2, \frac{\|x - F_n x\|^2 + \|y - F_n y\|^2}{2}, \frac{\|x - F_n y\|^2 + \|y - F_n x\|^2}{2} \right), \end{aligned}$$

for all $x, y \in X$. If F_n has a fixed point v_n and F has a fixed point v . Then the sequence $\{v_n\}$ converges to v .

Proof. Note that $F_n v_n = v_n$ and $Fv = v$. Now consider

$$\begin{aligned} \|v - v_n\|^2 &= \|Fv - F_n v_n\|^2 \\ &\leq \|Fv - F_n v\|^2 + \|F_n v - F_n v_n\|^2 + 2\operatorname{Re}\langle Fv - F_n v, F_n v - F_n v_n \rangle \\ &\leq \|Fv - F_n v\|^2 + \phi \left(\|v - v_n\|^2, \frac{\|v - F_n v\|^2 + \|v_n - F_n v_n\|^2}{2}, \right. \\ &\quad \left. \frac{\|v - F_n v_n\|^2 + \|v_n - F_n v\|^2}{2} \right) + 2\operatorname{Re}\langle Fv - F_n v, F_n v - F_n v_n \rangle. \end{aligned}$$

By the fact that $F_n v \rightarrow Fv$ as $n \rightarrow \infty$, we get

$$\|v - v_n\|^2 \leq \phi \left(\|v - v_n\|^2, 0, \|v - v_n\|^2 \right).$$

Hence, by Definition 2.3 (ii), we obtain

$$\|v - v_n\|^2 \leq 0,$$

which implies that $v_n \rightarrow v$ as $n \rightarrow \infty$. \square

Remark 3.7. Theorem 3.5 generalizes the concept of common fixed point for continuous self mappings under implicit relation in Hilbert spaces. In Theorem 3.6, convergence of sequence of self mappings to another self mapping implies convergence of corresponding sequence of fixed points. Note that, continuity of mappings is not necessary in Theorem 3.6.

4. WELL-POSEDNESS

The study of well-posedness of a fixed point problem has been active area of research and several mathematicians investigated this interesting concept, see [1, 2, 9, 17, 18]. In this section, we prove well-posedness of common fixed point problem of mappings in Theorem 3.1.

Definition 4.1. Let X be a Hilbert space and $f : X \rightarrow X$ be a self mapping. The fixed point problem of f is said to be well-posed if

- (i) f has a unique fixed point $x_0 \in X$,

- (ii) for any sequence $\{x_n\} \subset X$ and $\lim_{n \rightarrow \infty} \|x_n - fx_n\| = 0$, we have

$$\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0.$$

Let $CFP(T, f, X)$ denote a common fixed point problem of self mappings T and f on X and $CF(T, f)$ denote the set of all common fixed points of T and f .

Definition 4.2. $CFP(T, f, X)$ is called well-posed if $CF(T, f)$ is singleton and for any sequence $\{x_n\}$ in X with

$$\tilde{x} \in CF(T, f) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|x_n - fx_n\| = \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$$

implies $\tilde{x} = \lim_{n \rightarrow \infty} x_n$.

Theorem 4.3. Let C be a closed subset of a Hilbert space X , T and f be continuous self mappings on C as in Theorem 3.1. Then the common fixed problem of f and T is well posed.

Proof. From Theorem 3.1, the mappings T and f have a unique common fixed point, say $v \in X$. Let $\{x_n\}$ be a sequence in X and $\lim_{n \rightarrow \infty} \|fx_n - x_n\| = \lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. Without loss of generality, assume that $v \neq x_n$ for any non-negative integer n . Using $fv = Tv = v$, we get

$$\begin{aligned} \|v - x_n\|^2 &= \|Tv - Tx_n + Tx_n - x_n\|^2 \\ &\leq \|Tv - Tx_n\|^2 + \|Tx_n - x_n\|^2 + 2\operatorname{Re}\langle Tv - Tx_n, Tx_n - x_n \rangle \\ &\leq \|Tx_n - x_n\|^2 + \phi\left(\|v - x_n\|^2, \frac{\|v - Tv\|^2 + \|x_n - Tx_n\|^2}{2}, \right. \\ &\quad \left. \frac{\|v - Tx_n\|^2 + \|x_n - Tv\|^2}{2}\right) + 2\operatorname{Re}\langle Tv - Tx_n, Tx_n - x_n \rangle \\ &= \|Tx_n - x_n\|^2 + \phi(\|v - x_n\|^2, 0, \|v - x_n\|^2) \\ &\quad + 2\operatorname{Re}\langle Tv - Tx_n, Tx_n - x_n \rangle. \end{aligned}$$

By the definition of implicit relation and letting $n \rightarrow \infty$, we obtain $\|v - x_n\| \rightarrow 0$. This completes the proof. \square

5. CONCLUSION AND DISCUSSION

In this paper, we investigated the behaviour of a family of self mappings under implicit relation of three dimension and well-posedness of common fixed point problem in Hilbert space. In this connection, we obtained a common

fixed point theorem for the family of self mappings using the property of continuity, since continuity plays a significant role. Further, this work is a generalization of the study made by Koparde and Waghmode [11], Pandhare [14], Veerapandi and Anil Kumar [20] and Saluja [19]. Now, it makes interesting to consider for future investigation

- Does Theorem 3.5 hold for a family of non-continuous mappings?
- Is it possible to get a point of coincidence for two or family of self mappings under implicit relation in Hilbert space?

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