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SOME FIXED POINT AND COMMON FIXED POINT THEOREMS ON G-METRIC SPACES

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Abstract. In this paper, we extend, improve and generalize some earlier results in G-metric spaces in [1].

1. Introduction

In 2007, Mustafa and Sims introduced the notion of G-metric and investigated the topology of such spaces. The authors also characterized some fixed point result in the context of G-metric space. A number of authors have published so many fixed point results on the G-metric space in [2]-[24].

2. Preliminaries

Definition 2.1. Let X be a non empty set. A function $G : X \times X \times X \longrightarrow \mathbb{R}_+$ is called a G-metric if the following conditions are satisfied:

(G1) If $x = y = z$, then $G(x, y, z) = 0$;
(G2) $0 < G(x, y, y)$, for any $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for any points $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;

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(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for any $x, y, z, a \in X$.

Then the pair (X, G) is called a G-metric space.

Definition 2.2. Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to+\infty} G(x, x_m, x_n) = 0$, and we say that the sequence $\{x_n\}$ is G-convergent to x and denote it by $x_n \longrightarrow x$.

We have the following useful results.

Proposition 2.3. ([20]) *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x;*
(2) $\lim_{n\to+\infty} G(x_n, x_n, x) = 0$*;*
(3) $\lim_{n\to+\infty} G(x_n, x, x) = 0$.

Definition 2.4. ([20]) Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq k$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 2.5. ([20]) *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *the sequence $\{x_n\}$ is G-Cauchy;*
(2) *for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq k$.*

Definition 2.6. ([20]) A G-metric space (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.7. ([20]) *Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$ it follows that:*

(i) *If $G(x, y, z) = 0$ then $x = y = z$;*
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;
(iii) $G(x, y, y) \leq 2G(y, x, x)$;
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;
(v) $G(x, y, z) \leq \frac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$;
(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Proposition 2.8. ([20]) *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Mustafa [11] extended the well-known Banach Contraction Principle Mapping in the framework of G-metric spaces as follows:

Theorem 2.9. ([11]) *Let (X,G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$:*

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \tag{2.1}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 2.10. ([11]) *Let (X,G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$:*

$$G(Tx, Ty, Ty) \leq kG(x, y, y), \tag{2.2}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Remark 2.11. We notice that the condition (2.1) implies the condition (2.2). The converse is true only if $k \in [0, \frac{1}{2})$. For details see [11].

Lemma 2.12. ([11]) *By the rectangle inequality $(G5)$ together with the symmetry $(G4)$, we have*

$$G(x, y, y) = G(y, y, x) \leq G(y, x, x) + G(x, y, x) = 2G(y, x, x). \tag{2.3}$$

3. Main results

At first we assume that

$$\Psi_1 = \{\psi : [0,\infty) \to [0,\infty) \text{ such that } \psi \text{ is non-decreasing and continuous}\}$$

and

$$\Phi = \{\varphi : [0,\infty) \to [0,\infty) \text{ such that } \varphi \text{ is lower semicontinuous}\},$$

where $\psi(t) = \phi(t) = 0$ if and only if $t = 0$.

Theorem 3.1. *Let (X,G) be a complete G-metric space and $T, S : X \to X$ be mappings satisfying the following condition for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$ holds,*

$$\psi(G(Tx, STx, Ty)) \leq \psi(G(x, Sx, y)) - \phi(G(x, Sx, y)). \tag{3.1}$$

Then T, S have a unique common fixed point.

Proof. From (3.1) and by ψ properties we reach to

$$G(Tx, STx, Ty) \leq G(x, Sx, y). \tag{3.2}$$

Now put $y := x$,

$$G(Tx, STx, Tx) \leq G(x, Sx, x). \tag{3.3}$$

Let $x_0 \in X$ and $x_n := Tx_{n-1}$ so by hypothesis $x := x_{n-1}$ in the relation (3.3) we have

$$G(x_n, Sx_n, x_n) \leq G(x_{n-1}, Sx_{n-1}, x_{n-1}). \tag{3.4}$$

Therefore sequence $\{G(x_n, Sx_n, x_n)\}$ is decreasing to some t. Put $k_n := G(x_n, Sx_n, x_n)$, now by (3.1)

$$\psi(k_n) \leq \psi(k_n) - \phi(k_n), \tag{3.5}$$

if $n \to \infty$ in (3.5) we reach $\phi(t) = 0$ so

$$G(x_n, Sx_n, x_n) \to 0. \tag{3.6}$$

We shall show that the sequences $\{x_n\}$ and $\{Sx_n\}$ are G-Cauchy.

$$\forall \varepsilon > 0,\ \exists N > 0 \text{ such that } \forall m, n\ (n \geq m > N \Rightarrow G(x_m, x_m, x_n) < \varepsilon). \tag{3.7}$$

Let

$$\exists \varepsilon > 0,\ \forall n_k > 0,\ \exists n_k, m_k\ (n_k \geq m_k > k \text{ but } G(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon). \tag{3.8}$$

By (G4) and (G5), we get

$$\begin{aligned} &0 < \varepsilon \leq G(x_{m_k}, x_{m_k}, x_{n_k}) = G(x_{n_k}, x_{m_k}, x_{m_k}), \\ &0 < \varepsilon \leq G(x_{n_k}, Sx_{m_k}, x_{m_k}) + G(Sx_{m_k}, x_{m_k}, x_{m_k}). \end{aligned} \tag{3.9}$$

Now if $k \to \infty$ right hand goes to zero by (3.8) and (3.6), therefore $0 < \varepsilon \leq 0$. We note that $G(x_{n_k}, Sx_{m_k}, x_{m_k}) \to 0$, since by (G4)

$$A_{m_k,n_k} := G(x_{n_k}, Sx_{m_k}, x_{m_k}) = G(x_{m_k}, Sx_{m_k}, x_{n_k}),$$

$\{A_{m_k,n_k}\}_{m_k,n_k}$ is a decreasing and positive sequence. Because from the (3.2) we have

$$\begin{aligned} A_{m_k,n_k} &= G(x_{m_k}, Sx_{m_k}, x_{n_k}) \\ &= G(Tx_{m_k-1}, STx_{m_k-1}, Tx_{n_k-1}) \\ &\leq G(x_{m_k-1}, Sx_{m_k-1}, x_{n_k-1}) \\ &= A_{m_k-1,n_k-1} \end{aligned}$$

that is, $A_{m_k,n_k} \to t$ for some t. Now by (3.1)

$$\psi(A_{m_k,n_k}) \leq \psi(A_{m_k-1,n_k-1}) - \phi(A_{m_k-1,n_k-1}), \tag{3.10}$$

if $k \to \infty$ in (3.10), we reach $\phi(t) = 0$ so $A_{m_k,n_k} \to 0$. Thus $\{x_n\}$ in Cauchy, *i.e.*, there exists $u \in X$ such that $x_n \to u$.

Now we want to prove G-Cauchy of $\{Sx_n\}$. By (G4) and (G5),

$$G(x_n, Sx_n, x_m) = G(Sx_n, x_m, x_n) \leq G(Sx_n, x_n, x_n) + G(x_n, x_m, x_n). \tag{3.11}$$

Now by G-Cauchy of $\{x_n\}$ and (3.6), we get

$$\lim_{n,m\to\infty} G(x_n, Sx_n, x_m) = 0 \quad \text{likewise} \quad \lim_{n,m\to\infty} G(x_m, Sx_m, x_n) = 0. \tag{3.12}$$

On the other hand

$$G(Sx_n, Sx_n, x_m) \leq G(Sx_n, x_n, x_m) + G(x_n, Sx_n, x_m), \quad \text{by (G5)} \tag{3.13}$$

by (3.12) and (3.13) again we find

$$\lim_{n,m\to\infty} G(Sx_n, Sx_n, x_m) = 0. \tag{3.14}$$

And again we have

$$\begin{aligned} G(x_n, Sx_m, x_n) &\leq G(x_n, x_m, x_n) + G(x_m, Sx_m, x_n) \quad \text{by (G5)} \\ &\leq G(x_n, x_m, x_n) + G(x_n, x_m, Sx_m) \quad \text{by (G4)} \end{aligned}$$

$$\begin{aligned} G(x_n, Sx_m, x_n) \leq\, & G(x_n, x_m, x_n) + G(x_n, x_m, Sx_m) \\ & + G(x_m, x_m, Sx_m) \quad \text{by (G5)} \end{aligned} \tag{3.15}$$

the right hand of (3.15) by (3.6), (3.12) and G-Cauchy of $\{x_n\}$ tends to zero, so

$$\lim_{n,m\to\infty} G(x_n, Sx_m, x_n) = 0. \tag{3.16}$$

Also we have

$$G(x_n, Sx_n, Sx_n) \leq 2G(Sx_n, x_n, x_n) \to 0, \quad \text{by (3.6)}. \tag{3.17}$$

For G-Cauchy of $\{Sx_n\}$

$$\begin{aligned} G(Sx_m, Sx_n, Sx_n) &\leq G(Sx_m, x_n, Sx_n) + G(x_n, Sx_n, Sx_n) \quad \text{by (G5)} \\ &= G(Sx_m, Sx_n, x_n) + G(x_n, Sx_n, Sx_n) \quad \text{by (G4)} \end{aligned}$$

$$\begin{aligned} G(Sx_m, Sx_n, Sx_n) \leq\, & G(Sx_m, x_m, x_n) + G(x_m, Sx_n, x_n) \\ & + G(x_n, Sx_n, Sx_n) \quad \text{by (G5)} \end{aligned} \tag{3.18}$$

so by right hand (3.18) tends to zero by (3.17) and (3.12), hence we reach

$$\lim_{n,m\to\infty} G(Sx_m, Sx_n, Sx_n) = 0. \tag{3.19}$$

Namely $\{Sx_n\}$ is G-Cauchy, so $Sx_n \to z$ for some $z \in X$.

Now we want to show that the sequences $\{x_n\}$ and $\{Sx_n\}$ are convergent to a point in $z \in X$. And more z is unique common fixed point T, S. To prove this, if $x_n \to u$ and $Sx_n \to z$ from (3.1) and hypothesis $x := x_{n-1} = y$

$$\psi(G(x_n, Sx_n, x_n)) \leq \psi(G(x_{n-1}, Sx_{n-1}, x_{n-1})) - \phi(G(x_{n-1}, Sx_{n-1}, x_{n-1})),$$

so $G(u, z, u) = 0$ namely $u = z$. In (3.1), let $y = z$ and $x := x_{n-1}$ so

$$\psi(G(x_n, Sx_n, Tz)) \leq \psi(G(x_{n-1}, Sx_{n-1}, z)) - \phi(G(x_{n-1}, Sx_{n-1}, z)). \tag{3.20}$$

Thus

$$\psi(G(z, z, Tz)) \leq \psi(G(z, z, z)) - \phi(G(z, z, z)) = 0 \Rightarrow Tz = z. \tag{3.21}$$

By (3.1) and $x = y = z$,

$$\begin{aligned} &\psi(G(Tz, STz, Tz)) \le \psi(G(z, Sz, z)) - \phi(G(z, Sz, z)) \\ &\Rightarrow \phi(G(z, Sz, z)) = 0 \\ &\Rightarrow Sz = z. \end{aligned} \tag{3.22}$$

Now $z = Tz = Sz$.
For uniqueness, let $u = Tu = Su$, $v = Tv = Sv$ and $u \neq v$. By (3.1) and $x := u$, $y = v$ we have

$$\begin{aligned} &\psi(G(Tu, STu, Tv)) \le \psi(G(u, Su, v)) - \phi(G(u, Su, v)) \\ &\Rightarrow \phi(G(u, Su, v)) = 0 \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &\psi(G(u, u, v)) \le \psi(G(u, u, v)) - \phi(G(u, u, v)) \\ &\Rightarrow \phi(G(u, u, v)) = 0 \\ &\Rightarrow u = v. \end{aligned} \tag{3.24}$$

□

Corollary 3.2. *Let (X, G) be a complete G-metric space and $T, S : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$ holds,*

$$\psi(G(Tx, TSx, Ty)) \le \psi(G(x, Sx, y)) - \phi(G(x, Sx, y)) \tag{3.25}$$

and T and S commute, i.e., $TS = ST$. Then T has a unique fixed point.

The next corollary is Theorem 2.3 in [1].

Corollary 3.3. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$ holds,*

$$\psi(G(Tx, T^2x, Ty)) \le \psi(G(x, Tx, y)) - \phi(G(x, Tx, y)). \tag{3.26}$$

Then T has a unique fixed point.

Corollary 3.4. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$ holds,*

$$\psi(G(Tx, T^m x, Ty)) \le \psi(G(x, T^{m-1}x, y)) - \phi(G(x, T^{m-1}x, y)), \tag{3.27}$$

for some $2 \le m \in \mathbb{N}$. Then T has a unique fixed point.

In Corollary 3.3, we take $\psi(t) = t$ and $\phi(t) = (1 - r)t$ where $0 \le r < 1$, then we deduce the following corollary.

Corollary 3.5. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$ where $0 \leq r < 1$ holds,*

$$G(Tx, T^2x, Ty) \leq rG(x, Tx, y).$$

Then T has a unique fixed point.

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