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ALMOST JORDAN QUARTIC HOMOMORPHISMS AND JORDAN QUARTIC DERIVATIONS ON FUZZY BANACH ALGEBRAS

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Abstract. In this paper, we establish the generalized Hyers–Ulam stability of Jordan quartic homomorphisms and Jordan quartic derivations associate to the following quartic functional equation

$$\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right) f\left(\sum_{i=1, i\neq i_1,\ldots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{n} x_i\right)$$
$$= 2^{n-2}\sum_{1\le i<j\le n}(f(x_i+x_j)+f(x_i-x_j)) - 2^{n-5}(n-2)\sum_{i=1}^{n} f(2x_i)$$

($n \in \mathbb{N}$, $n \geq 3$) in fuzzy Banach algebras.

1. Introduction and preliminaries

Definition 1.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $a, b \in \mathbb{R}$:

(N_1) $N(x,a) = 0$ for $a \leq 0$;
(N_2) $x = 0$ if and only if $N(x,a) = 1$ for all $a > 0$;
(N_3) $N(ax,b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;

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(N_4) $N(x+y,a+b) \geq \min\{N(x,a),N(y,b)\}$;
(N_5) $N(x,.)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x,a)=1$;
(N_6) for $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,a)$ as the truth value of the statement "the norm of x is less than or equal to the real number a" (see [1]).

Example 1.2. Let $(X,\|.\|)$ be a normed linear space. Then

$$N(x,a)=\begin{cases} \frac{a}{a+\|x\|}, & a>0, \quad x\in X; \\ 0, & a\leq 0, \quad x\in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.3. Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty} N(x_n-x,a)=1$ for all $a>0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N-\lim_{n\to\infty}\ x_n=x$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon>0$ and each a there exists n_0 such that for all $n\geq n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,a)>1-\epsilon$. It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 1.4. Let X be an algebra and (X,N) be complete fuzzy normed space, the pair (X,N) is said to be a fuzzy Banach algebra if for every $x,y\in X$, $a,b\in\mathbb{R}$,

$$N(xy,ab)\geq \max\{N(x,a),N(y,b)\}. \tag{1.1}$$

Definition 1.5. Let (X,N) be a fuzzy Banach algebra and $\{x_n\},\{y_n\}$ be convergent sequences in (X,N) such that $N-\lim_{n\to\infty}\ x_n=x$ and $N-\lim_{n\to\infty}\ y_n=y$. Then

$$\begin{aligned} N(x_ny_n-xy,2t) &\geq \min\{N((x_n-x)y_n,t),N(x(y_n-y),t)\} \\ &\geq \min\{N(x_n-x,t),N(y_n-y,t)\} \end{aligned}$$

for all $t>0$. Therefore, $N-\lim_{n\to\infty}\ x_ny_n=xy$.

The generalized Hyers–Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [7, 12, 14] and [19]. The stability of different functional equations on Banach algebras has been recently studied in [2], [3]–[6], [8]–[10] and [16]–[17].

In this paper, we deal with the stability problem of the following generalized quartic functional equation

$$\begin{aligned}
&\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)f\left(\sum_{i=1,i\neq i_1,\ldots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\right)\\
&+f\left(\sum_{i=1}^{n}x_i\right)\\
&=2^{n-2}\sum_{1\leq i<j\leq n}(f(x_i+x_j)+f(x_i-x_j))-2^{n-5}(n-2)\sum_{i=1}^{n}f(2x_i). \qquad (1.2)
\end{aligned}$$

Definition 1.6. Suppose A and B are two (fuzzy) Banach algebras. We say that a mapping $h : A \to B$ is a Jordan quartic homomorphism if

$$h(a^2) = h(a)^2$$

for all $a \in A$ and h satisfies (1.2).

Definition 1.7. Suppose A is a (fuzzy) Banach algebra. We say that a mapping $d : A \to A$ is a Jordan derivation if

$$d(a^2) = a^4 d(a) + d(a)a^4$$

for all $a \in A$ and h satisfies (1.2).

We investigate the fuzzy stability of Jordan quartic homomorphisms and Jordan quartic derivations related to quartic functional equation (1.2) in fuzzy Banach algebras.

2. Fuzzy stability of Jordan homomorphisms

In this section, using direct method, we investigate the fuzzy stability of Jordan homomorphisms of functional equation (1.2) in fuzzy Banach algebras. Throughout this section, we assume that (A, N) and (B, N) are two fuzzy Banach algebras and (C, N') be a fuzzy normed space. Moreover, we assume that $N(x, .)$ is a left continuous function on $\mathbb{R}$. For convenience, we define the difference operator D_f for a given mapping f:

$$\begin{aligned}&D_f(x_1,..,x_n)\\&=\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)f\left(\sum_{i=1,i\neq i_1,\dots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\right)\\&\quad+f\left(\sum_{i=1}^{n}x_i\right)-2^{n-2}\sum_{1\leq i<j\leq n}(f(x_i+x_j)+f(x_i-x_j))+2^{n-5}(n-2)\sum_{i=1}^{n}f(2x_i).\end{aligned}$$

We will use the following well-known lemma:

Lemma 2.1. *A mapping $f : X \to Y$ satisfies* (1.2) *if and only if the mapping $f : X \to Y$ is quartic.*

Theorem 2.2. *Suppose (A, N) and (B, N) are two fuzzy Banach algebras and (C, N') is a fuzzy normed space. Let $\varphi : A^n \to C$ be a function such that for some $0 < |r| < 16$,*

$$N'(\varphi(2a_1,\cdots,2a_n),t)\geq N'(r\varphi(a_1,\cdots,a_n),t) \tag{2.1}$$

for all $a_1, a_2, \cdots, a_n \in A$ and all $t > 0$. If $f : A \to B$ is a mapping such that

$$N(\Omega_f(a_1,a_2,\cdots,a_n),t)\geq N'(\varphi(a_1,a_2,\cdots,a_n),t) \tag{2.2}$$

and

$$N(f(a^2)-f(a)^2,s)\geq N'(\varphi(a,0,\cdots,0),s) \tag{2.3}$$

for all $a_1, a_2, \cdots, a_n \in A$ and all $t, s > 0$. Then there exists a unique Jordan quartic homomorphism $h : A \to B$ such that

$$N(f(a)-h(a),t)\geq N'\left(\varphi(a,0,\cdots,0),\frac{2^{n+3}(n-2)t}{16-|r|}\right), \tag{2.4}$$

where $a \in A$ and $t > 0$.

Proof. It follows from (2.1) that

$$N'\left(\varphi\left(2^ja_1,2^ja_2,\cdots,2^ja_n\right),t\right)\geq N'\left(\varphi(a_1,a_2,\cdots,a_n),\frac{t}{|r|^j}\right). \tag{2.5}$$

So

$$N'\left(\varphi\left(2^ja_1,2^ja_2,\cdots,2^ja_n\right),|r|^jt\right)\geq N'\left(\varphi(a_1,a_2,\cdots,a_n),t\right)$$

for all $x, y, z \in X$ and all $t > 0$. Putting $a_1 = a$ and $a_2 = \cdots = a_n = 0$ in (2.2), we get

$$\begin{aligned} &N\left(\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)f(a)+f(a))\right.\\ &\qquad\left.-2^{n-2}\sum_{j=2}^{n}2f(a+2^{n-5}(n-2)f(2a),t\right)\\ &\geq N'(\varphi(a,0,\cdots,0),t), \end{aligned} \tag{2.6}$$

for all $a \in A$. That is

$$\begin{aligned} &N\left(\left(1+\sum_{\ell=1}^{n-1}\binom{n-1}{\ell}\right)f(a)-2^{n-1}(n-1)f(a)+2^{n-5}(n-2)f(2a),t\right)\\ &\geq N'(\varphi(a,0,\cdots,0),t), \end{aligned} \tag{2.7}$$

for all $a \in A$. So by using the equation

$$1+\sum_{\ell=1}^{n-1}\binom{n-1}{\ell}=\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}=2^{n-1},$$

gives

$$N\left(f(a)-\frac{1}{2^4}f(2a),\frac{t}{2^{n-1}(n-2)}\right)\geq N'(\varphi(a,0,\cdots,0),t). \tag{2.8}$$

Replacing a by $2^j a$ in (2.8), we have

$$\begin{aligned} &N\left(\frac{f\left(2^{j+1}a\right)}{2^{4j+4}}-\frac{f\left(2^{j}a\right)}{2^{4j}},\frac{t}{2^{n+4j-1}(n-2)}\right)\\ &\geq N'\left(\varphi\left(2^j a,0,\cdots,0\right),t\right)\geq N'\left(\varphi\left(a,0,\cdots,0\right),\frac{t}{|r|^j}\right), \end{aligned} \tag{2.9}$$

for all $a \in A$, all $t > 0$ and any integer $j \geq 0$. So

$$\begin{aligned} &N\left(f(a)-\frac{f\left(2^{m}a\right)}{2^{4m}},\sum_{j=0}^{m-1}\frac{|r|^j t}{2^{n+4j-1}(n-2)}\right)\\ &=N\left(\sum_{j=0}^{m-1}\left\{\frac{f\left(2^{j+1}a\right)}{2^{4j+4}}-\frac{f\left(2^{j}a\right)}{2^{4j}}\right\},\sum_{j=0}^{m-1}\frac{|r|^j t}{2^{n+4j-1}(n-2)}\right)\\ &\geq\min_{0\leq j\leq m-1}\left\{N\left(\frac{f\left(2^{j+1}a\right)}{2^{4j+4}}-\frac{f\left(2^{j}a\right)}{2^{4j}},\frac{|r|^j t}{2^{n+4j-1}(n-2)}\right)\right\}\\ &\geq N'(\varphi(a,0,\cdots,0),t) \end{aligned} \tag{2.10}$$

which yields

$$\begin{aligned}
&N\left(\frac{f\left(2^{m+p}a\right)}{2^{4m+4p}}-\frac{f\left(2^{p}a\right)}{2^{4p}},\sum_{j=0}^{m-1}\frac{|r|^{j}t}{2^{n+4j+4p-1}(n-2)}\right)\\
&\geq N'\left(\varphi\left(2^{p}a,0,\cdots,0\right),t\right)\\
&\geq N'\left(\varphi\left(a,0,\cdots,0\right),\frac{t}{|r|^{p}}\right),
\end{aligned}\tag{2.11}$$

for all $a\in A$, $t>0$ and any integers $n>0$, $p\geq 0$. So

$$N\left(\frac{f\left(2^{m+p}a\right)}{2^{4m+4p}}-\frac{f\left(2^{p}a\right)}{2^{4p}},\sum_{j=0}^{m-1}\frac{|r|^{j+p}t}{2^{n+4j+4p-1}(n-2)}\right)\geq N'\left(\varphi\left(a,0,\cdots,0\right),t\right),$$

for all $x\in X$, $t>0$ and any integers $n>0$, $p\geq 0$. Hence one obtains

$$\begin{aligned}
&N\left(\frac{f\left(2^{m+p}a\right)}{2^{4m+4p}}-\frac{f\left(2^{p}a\right)}{2^{4p}},t\right)\\
&\geq N'\left(\varphi(a,0,\cdots,0),\frac{t}{\frac{|r|^{p}}{2^{n+4p-1}(n-2)}\sum_{j=0}^{m-1}\frac{|r|^{j}}{2^{4j}}}\right),
\end{aligned}\tag{2.12}$$

for all $x\in X$, $t>0$ and any integers $n>0$, $p\geq 0$. Since, the series $\sum_{j=0}^{\infty}\frac{|r|^{j}}{2^{4j}}$ is convergent series, we see by taking the limit $m\to\infty$ in the last inequality that a sequence $\left\{\frac{f(2^{m}a)}{2^{4m}}\right\}_{m\geq 1}$ is a Cauchy sequence in the fuzzy Banach space (B,N) and so it converges in B. It follows that the mapping $h:A\to B$ defined by $h(x):=N-\lim_{n\to\infty}\frac{f(2^{m}a)}{2^{4m}}$ is well defined for all $a\in A$. It means that

$$\lim_{n\to\infty}N\left(h(x)-\frac{f\left(2^{m}a\right)}{2^{4m}},t\right)=1,\tag{2.13}$$

for all $a\in A$ and all $t>0$. In addition, it follows from (2.12) that

$$N\left(f(x)-\frac{f\left(2^{m}a\right)}{2^{4m}},t\right)\geq N'\left(\varphi(a,0,\cdots,0),\frac{t}{\frac{1}{2^{n-1}(n-2)}\sum_{j=0}^{m-1}\frac{|r|^{j}}{2^{4j}}}\right)$$

for all $a \in A$ and all $t > 0$. So

$$
\begin{aligned}
& N(f(a)-h(a),t) \\
& \geq \min\left\{N\left(f(a)-\frac{f\left(2^m a\right)}{2^{4m}},(1-\epsilon)t\right), N\left(h(a)-\frac{f\left(2^m a\right)}{2^{4m}},\epsilon t\right)\right\} \\
& \geq N'\left(\varphi(a,0,\cdots,0), \frac{2^{n-1}(n-2)t}{\sum_{j=0}^{m-1}\frac{|r|^j}{2^{4j}}}\right) \\
& \geq N'\left(\varphi(a,0,\cdots,0), \frac{2^{n+3}(n-2)\epsilon t}{16-|r|}\right)
\end{aligned}
$$

for sufficiently large n and for all $a \in A$, $t > 0$ and ϵ with $0 < \epsilon < 1$. Since ϵ is arbitrary and N' is left continuous, we obtain

$$
N(f(a)-h(a),t) \geq N'\left(\varphi(a,0,\cdots,0), \frac{2^{n+3}(n-2)t}{16-|r|}\right)
$$

for all $x \in X$ and $t > 0$. It follows from (2.2) that

$$
\begin{aligned}
N\left(\frac{D_f(2^m x_1, 2^m x_2, \cdots, 2^m x_n)}{2^{4m}}, t\right) & \geq N'\left(\varphi\left(2^m a_1, 2^m a_2, \cdots, 2^m a_n\right), 2^{4m} t\right) \\
& \geq N'\left(\varphi(a_1, a_2, \cdots, a_n), \frac{2^{4m} t}{|r|^m}\right)
\end{aligned}
$$

for all $a_1, a_2, \cdots, a_n \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since

$$
\lim_{m\to\infty} N'\left(\varphi(a_1, a_2, \cdots, a_n), \frac{2^{4m} t}{|r|^m}\right) = 1
$$

and so

$$
N\left(\frac{D_f(2^m x_1, 2^m x_2, \cdots, 2^m x_n)}{2^{4m}}, t\right) \quad \to \quad 1 \quad \text{when} \quad m \to +\infty
$$

for all $x, y, z \in X$ and all $t > 0$. Therefore, we obtain in view of (2.13)

$$
\begin{aligned}
& N\left(D_h(x_1, x_2, \cdots, x_n), t\right) \\
& \geq \min\left\{N\left(D_h(x_1, x_2, \cdots, x_n) - \frac{D_f(2^m x_1, 2^m x_2, \cdots, 2^m x_n)}{2^{4m}}, \frac{t}{2}\right),\right. \\
& \left. N\left(\frac{D_f(2^m x_1, 2^m x_2, \cdots, 2^m x_n)}{2^{4m}}, \frac{t}{2}\right)\right\} \\
& = N\left(\frac{D_f(2^m x_1, 2^m x_2, \cdots, 2^m x_n)}{2^{4m}}, \frac{t}{2}\right) \\
& \geq N'\left(\varphi(a_1, a_2, \cdots, a_n), \frac{2^{4m-1} t}{|r|^m}\right) \quad \to \quad 1 \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

which implies

$$\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)h\left(\sum_{i=1,i\neq i_1,\ldots,i_{n-k+1}}^{n}a_i-\sum_{r=1}^{n-k+1}a_{i_r}\right)$$
$$+h\left(\sum_{i=1}^{n}x_i\right)$$
$$=2^{n-2}\sum_{1\leq i<j\leq n}(h(a_i+a_j)+h(a_i-a_j))-2^{n-5}(n-2)\sum_{i=1}^{n}h(2a_i)$$

for all $a_1, a_2, \cdots, a_n \in X$. Thus $h : A \to B$ is a mapping satisfying the equation (1.2) and the inequality (2.22).

To prove the uniqueness, let there is another mapping $h' : A \to B$ which satisfies the inequality (2.22). Then, for all $a \in A$, we have

$$\begin{aligned}
&N(h(a)-h'(a),t)\\
&=N\left(\frac{h(2^m a)}{2^{4m}}-\frac{h'(2^m a)}{2^{4m}},t\right)\\
&\geq \min\left\{N\left(\frac{h(2^m a)}{2^{4m}}-\frac{f(2^m a)}{2^{4m}},\frac{t}{2}\right),N\left(\frac{f(2^m a)}{2^{4m}}-\frac{h'(2^m a)}{2^{4m}},\frac{t}{2}\right)\right\}\\
&\geq N'\left(\varphi(2^m a,0,\cdots,0),\frac{2^{4m+n+2}(n-2)t}{16-|r|}\right)\\
&\geq N'\left(\varphi(a,0,\cdots,0),\frac{2^{4m+n+2}(n-2)t}{|r|^m(16-|r|)}\right) \quad\to\quad 1 \quad \text{as} \quad m\to+\infty
\end{aligned}$$

for all $t > 0$. Therefore, $h(a) = h'(a)$ for all $a \in A$. Now we only need to show that $h(a^2) = h(a)^2$ for all $a \in A$. It follows from (2.1) that

$$\begin{aligned}
N(f(2^m a)-h(2^m a),t) &\geq N'\left(\varphi(2^m a,0,\cdots,0),\frac{2^{n+3}(n-2)t}{16-|r|}\right)\\
&\geq N'\left(r^m\varphi(a,0,\cdots,0),\frac{2^{n+3}(n-2)t}{16-|r|}\right)
\end{aligned} \tag{2.14}$$

for all $a \in A$ and all $t > 0$. Thus

$$N(2^{-4m}f(2^m a)-2^{-4m}h(2^m a),2^{-4m}t)\geq N'\left(\varphi(a,0,\cdots,0),\frac{2^{n+3}(n-2)t}{|r|^m(16-|r|)}\right)$$

for all $a \in A$ and all $t > 0$. Since h is quartic, then it is easy to see that

$$N(2^{-4m}f(2^m a)-h(a),t)\geq N'\left(\varphi(a,0,\cdots,a),\frac{2^{n+3}(n-2)t}{|r|^m(16-|r|)}\right) \tag{2.15}$$

for all $a \in A$ and all $t > 0$. Letting n to infinity in (2.15) and using (N_5), we see that

$$h(a) = N - \lim_{m\to\infty} 2^{-4m} f(2^m a) \tag{2.16}$$

for all $a \in A$. Similarly, we obtain

$$h(a^2) = N - \lim_{m\to\infty} 2^{-8m} f(2^{2m} a^2) \tag{2.17}$$

for all $a \in A$. Using inequality (2.3), we get

$$\begin{aligned} N(f(2^{2m}a^2) - f(2^m a)^2, s) &\geq N'(\varphi(2^m a, 0, \cdots, 0), s) \\ &\geq N'(r^m \varphi(a, 0, \cdots, 0), s) \end{aligned}$$

for all $a \in A$ and all $s > 0$. Thus

$$N\Big(2^{-8m}[f(2^{2m}a^2) - f(2^m a)^2], s\Big) \geq N'\Big(\varphi(a, 0, \cdots, 0), \frac{2^{8m}s}{|r|^m}\Big) \tag{2.18}$$

for all $a \in A$ and all $s > 0$. Letting n to infinity in (2.18) and using (N_5), we see that

$$N - \lim_{m\to\infty} 2^{-8m}[f(2^{2m}a^2) - f(2^m a)^2] = 0. \tag{2.19}$$

Applying (2.16), (2.17) and (2.19), we have

$$\begin{aligned} h(a^2) &= N - \lim_{m\to\infty} 2^{-8m} f(2^{2m}a^2) \\ &= N - \lim_{m\to\infty} 2^{-8m}[f(2^{2m}a^2) - f(2^{2m}a^2) + f(2^m a)^2] \\ &= N - \lim_{m\to\infty} 2^{-8m} f(2^m a)^2 \\ &= \Big\{N - \lim_{m\to\infty} 2^{-4m} f(2^m a)\Big\}^2 = h(a)^2 \end{aligned}$$

for all $a \in A$. This completes the proof. □

Corollary 2.3. *Suppose (A, N) and (B, N) are two fuzzy Banach algebras and (C, N') is a fuzzy normed space. Also p be a positive integer with $p < 4$, then if $f : A \to B$ is a mapping such that*

$$N(\Omega_f(a_1, a_2, \cdots, a_n), t) \geq N'\left(\sum_{i=1}^{n} \|a_i\|^p, t\right) \tag{2.20}$$

and

$$N(f(a^2) - f(a)^2, s) \geq N'(\|a\|^p, s) \tag{2.21}$$

for all $a, a_1, a_2, \cdots, a_n \in A$ and all $t, s > 0$. Then there exists a unique Jordan quartic homomorphism $h : A \to B$ such that

$$N(f(a) - h(a), t) \geq N'\left(\|a\|^p, \frac{2^{n+3}(n-2)t}{15}\right), \tag{2.22}$$

where $a \in A$ and $t > 0$.

Proof. Let $\varphi(a_1, a_2, \cdots, a_n) = \sum_{i=1}^{n} \|a_i\|^p$ and $|r| = 1$. Applying Theorem 2.2, we get the desired results. $\square$

Remark 2.4. If in (2.20) we replace $\sum_{i=1}^{n} \|a_i\|^p$ by $\prod_{i=1}^{n} \|a_i\|^p$, then we get the superstability, *i.e.*, in the above corollary $f = h$.

3. Fuzzy stability of Jordan derivations

In this section, we prove the stability of Jordan quartic derivations on fuzzy Banach algebras.

Theorem 3.1. *Let (A, N) be a fuzzy Banach algebra and (B, N') be a fuzzy normed space. Let $\varphi : A^n \to B$ be a function such that for some $0 < |r| < 16$,*

$$N'(\varphi(2a_1, \cdots, 2a_n), t) \geq N'(r\varphi(a_1, \cdots, a_n), t) \tag{3.1}$$

for all $a_1, \cdots, a_n \in A$ and all $t > 0$. Suppose that $f : A \to A$ is a function such that

$$N(D_f(a_1, a_2, \cdots, a_n), t) \geq N'(\varphi(a_1, \cdots, a_n), t) \tag{3.2}$$

and

$$N(f(a^2) - a^4 f(a) - f(a)a^4, s) \geq N'(\varphi(a, 0, \cdots, 0), s) \tag{3.3}$$

for all $a \in A$ and all $t, s > 0$. Then there exists a unique Jordan quartic derivation $d : A \to A$ such that

$$N(f(a) - d(a), t) \geq N'\left(\varphi(a, 0, \cdots, 0), \frac{2^{n+3}(n-2)t}{16 - |r|}\right), \tag{3.4}$$

where $a \in A$ and $t > 0$.

Proof. Proceeding as in the proof of Theorem 2.2, we find that the mapping $d : A \to B$ defined by $d(a) := N - \lim_{m\to\infty} \frac{f(2^m a)}{2^{4m}}$ is a quartic function satisfying (3.4). Now we only need to show that d satisfies

$$d(a^2) = a^4 d(a) + d(a)a^4$$

for all $a \in A$. The inequalities (3.1) and (3.4) imply that

$$\begin{aligned} N(f(2^m a) - d(2^m a), t) &\geq N'\left(\varphi(2^m a, 0, \cdots, 0), \frac{2^{n+3}(n-2)t}{16 - |r|}\right) \\ &\geq N'\left(r^m \varphi(a, 0, \cdots, 0), \frac{2^{n+3}(n-2)t}{16 - |r|}\right), \end{aligned} \tag{3.5}$$

for all $a \in A$ and all $t > 0$. Thus

$$N\Big(2^{-4m} f(2^m a) - 2^{-4m} d(2^m a), 2^{-4m} t\Big) \geq N'\left(\varphi(a, 0, \cdots, 0), \frac{2^{n+3}(n-2)t}{|r|^m(16 - |r|)}\right)$$

for all $a \in A$ and all $t > 0$. By the additivity of d it is easy to see that

$$N(2^{-4m} f(2^m a) - d(a), t) \geq N'\left(\varphi(a, 0, \cdots, 0), \frac{2^{n+4m+3}(n-2)t}{|r|^m(16-|r|)}\right) \tag{3.6}$$

for all $a \in A$ and all $t > 0$. Letting m to infinity in (3.6) and using (N_5), we get

$$d(a) = N - \lim_{m\to\infty} 2^{-4m} f(2^m a) \tag{3.7}$$

for all $a \in A$. Similarly, we get

$$d(a^2) = N - \lim_{m\to\infty} 2^{-8m} f(2^{2m} a^2) \tag{3.8}$$

for all $a \in A$. Using (3.1) and (3.3), we get

$$\begin{aligned} &N\Big(f(2^{2m}a^2) - (2^{4m}a^4) f(2^m a) - f(2^m a)(2^{4m}a^4), s\Big) \\ &\geq N'\big(\varphi(2^m a, 0, \cdots, 0), s\big) \geq N'\Big(\varphi(a, 0, \cdots, 0), \frac{s}{r^m}\Big), \end{aligned} \tag{3.9}$$

for all $a \in A$ and all $s > 0$. Let $g : A \times A \to A$ be a function defined by $g(a, a) = f(a^2) - a^4 f(a) - f(a)a^4$ for all $a \in A$. Hence, (3.9) implies that

$$N - \lim_{m\to\infty} 2^{-4m} g(2^m a, 2^m a) = 0$$

and

$$N - \lim_{m\to\infty} 2^{-8m} g(2^m a, 2^m a) = 0 \tag{3.10}$$

for all $a \in A$. Since (A, N) is a fuzzy Banach algebra, applying (3.7), (3.8) and (3.10), we get

$$\begin{aligned} d(a^2) &= N - \lim_{m\to\infty} 2^{-8m} f(2^{2m} a^2) \\ &= N - \lim_{m\to\infty} [a^4 2^{-4m} f(2^m a) + 2^{-4m} f(2^m a) a^4 + 2^{-8m} g(2^m a, 2^m a)] \\ &= a^4(N - \lim_{m\to\infty} 2^{-4m} f(2^m a)) + (N - \lim_{m\to\infty} 2^{-4m} f(2^m a)) a^4 \\ &\quad + N - \lim_{m\to\infty} 2^{-8m} g(2^m a, 2^m a) \\ &= a^4 d(a) + d(a) a^4. \end{aligned}$$

for all $a \in A$.

To prove the uniqueness property of d, assume that d' is another Jordan derivation satisfying (3.4). Since both d and d' are additive, we get from (3.1) and (3.4) that

$$\begin{aligned}
&N(d(a)-d'(a),t)\\
&=N(d(2^m a)-d'(2^m a),2^{4m}t)\\
&=N\left([d(2^m a)-f(2^m a)]+[f(2^m a)-d'(2^m a)],2^{4m}t\right)\\
&\geq \min\left\{N\left(d(2^m a)-f(2^m a),\frac{2^{4m}t}{2}\right),N\left(f(2^m a)-d'(2^m a),\frac{2^{4m}t}{2}\right)\right\}\\
&\geq N'\left(\varphi(a,0,\cdots,0),\frac{2^{n+4m+2}(n-2)t}{|r|^m(16-|r|)}\right)\quad\to\quad 1\quad\text{as}\quad m\to+\infty,
\end{aligned}$$

for all $a\in A$ and all $t>0$. Letting m to infinity in the above inequality, we get $N(d(a)-d'(a),t)=1$ for all $a\in A$ and all $t>0$. Hence $d(a)=d'(a)$ for all $a\in A$. □

Corollary 3.2. *Suppose (A,N) and (B,N) are two fuzzy Banach algebras and (C,N') is a fuzzy normed space. Also p be a positive integer with $p<4$, then if $f:A\to B$ is a mapping such that*

$$N(\Omega_f(a_1,a_2,\cdots,a_n),t)\geq N'\left(\sum_{i=1}^{n}\|a_i\|^p,t\right) \tag{3.11}$$

and

$$N(f(a^2)-f(a)^2,s)\geq N'(\|a\|^p,s) \tag{3.12}$$

for all $a,a_1,a_2,\cdots,a_n\in A$ and all $t,s>0$. Then there exists a unique Jordan quartic homomorphism $h:A\to B$ such that

$$N(f(a)-h(a),t)\geq N'\left(\|a\|^p,2^n(n-2)t\right), \tag{3.13}$$

where $a\in A$ and $t>0$.

Proof. Let $\varphi(a_1,a_2,\cdots,a_n)=\sum_{i=1}^{n}\|a_i\|^p$ and $|r|=8$. Applying Theorem 2.2, we get the desired results. □

Remark 3.3. If in (3.11) we replace $\sum_{i=1}^{n}\|a_i\|^p$ by $\prod_{i=1}^{n}\|a_i\|^p$, then we get the superstability, *i.e.*, in the above corollary $f=h$.

4. Conclusion

We establish the generalized Hyers–Ulam stability of Jordan quartic homomorphisms and Jordan quartic derivations on fuzzy Banach algebras. We show that every approximately Jordan quartic homomorphism (Jordan quartic derivation) is near to an exact Jordan quartic homomorphism (Jordan quartic derivation).

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