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CONVERGENCE TO COMMON FIXED POINTS FOR A FINITE FAMILY OF STRICTLY ASYMPTOTICALLY PSEUDO-CONTRACTIVE MAPPINGS IN THE INTERMEDIATE SENSE

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Abstract. In this paper, we establish a weak convergence theorem for an explicit iteration process for a finite family of asymptotically k-strictly pseudo-contractive mappings in the intermediate sense which is not necessarily Lipschitzian in the framework of Hilbert spaces. We also construct (CQ) method for this explicit iteration process which generates a strongly convergent sequence. The results presented in this paper generalize and extend the corresponding results of [1, 5, 7, 9, 12, 14, 15, 17, 18] and many others.

1. Introduction and Preliminaries

Throughout this paper, let H be a real Hilbert space with the scalar product and norm denoted by the symbols $\langle .,. \rangle$ and $\| . \|$ respectively. Let C be a closed convex subset of H, we denote by $P_C(.)$ the metric projection from H onto C. It is known that $z = P_C(x)$ is equivalent to $\langle z - y, x - z \rangle \geq 0$ for every $y \in C$. A point $x \in C$ is a fixed point of T provided that $Tx = x$. Denote by $F(T)$ the set of fixed point of T, that is, $F(T) = \{x \in C : Tx = x\}$. It is known that $F(T)$ is closed and convex. Let T be a (possibly) nonlinear mapping from C into C. We now consider the following classes:

T is contractive, i.e., there exists a constant $k < 1$ such that

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$$\|Tx - Ty\| \quad \leq \quad k\,\|x-y\|\,, \tag{1.1}$$

for all $x, y \in C$.

T is nonexpansive, i.e.,

$$\|Tx - Ty\| \quad \leq \quad \|x-y\|\,, \tag{1.2}$$

for all $x, y \in C$.

T is uniformly L-Lipschitzian, i.e., if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad L\,\|x-y\|\,, \tag{1.3}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

T is pseudo-contractive, i.e.,

$$\langle Tx - Ty, j(x-y)\rangle \quad \leq \quad \|x-y\|^2\,, \tag{1.4}$$

for all $x, y \in C$.

T is asymptotically nonexpansive [3], i.e., if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n\,\|x-y\|\,, \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] as a generalization of the class of nonexpansive mappings. T is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\,\|T^n x - T^n y\| - \|x-y\|\,\big) \quad \leq \quad 0. \tag{1.6}$$

Observe that if we define

$$G_n = \max\left\{0, \sup_{x,y\in C} \big(\,\|T^n x - T^n y\| - \|x-y\|\,\big)\right\}, \tag{1.7}$$

then $G_n \to 0$ as $n \to \infty$. It follows that (1.7) is reduced to

$$\|T^n x - T^n y\| \quad \leq \quad \|x-y\| + G_n, \tag{1.8}$$

for all $x, y \in C$ and $n \geq 1$.

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [2]. It is known [6] that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Recall that T is said to be a k-strictly pseudocontraction if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\,\|(I-T)x - (I-T)y\|^2, \tag{1.9}$$

for all $x, y \in C$.

T is said to an asymptotically k-strictly pseudocontraction with sequence $\{\gamma_n\}$ if there exists a sequence $\{\gamma_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\begin{aligned}\|T^n x - T^n y\|^2 \leq\ & (1+\gamma_n)\,\|x-y\|^2 \\ & +k\,\|(x - T^n x) - (y - T^n y)\|^2,\end{aligned} \tag{1.10}$$

for some $k \in [0,1)$ for all $x, y \in C$ and $n \geq 1$.

Remark 1.1. ([11]) If T is k-strictly asymptotically pseudo-contractive mapping, then it is uniformly L-Lipschitzian with $L = \sup\{(a_n + \sqrt{k})/(1+\sqrt{k}) : n \in N\}$ where $\{a_n\}$ is a sequence in $[1,\infty)$ with $a_n \to 1$ as $n \to \infty$, but the converse does not hold.

The class of asymptotically k-strictly pseudocontraction was introduced by Qihou [7] in 1996. Kim and Xu [5] studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically k-strictly pseudocontraction with sequence $\{\gamma_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup\{(k + \sqrt{1+(1-k)\gamma_n})/(1+k) : n \in N\}$.

Recently, Sahu et al. [15] introduced a class of new mappings: asymptotically k-strictly pseudocontractive mappings in the intermediate sense. Recall that T is said to be an asymptotically k-strictly pseudocontraction in the intermediate sense with sequence $\{\gamma_n\}$ if there exists a sequence $\{\gamma_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ and a constant $k \in [0,1)$ such that

$$\begin{aligned}\limsup_{n\to\infty} \sup_{x,y\in C} \Big(& \|T^n x - T^n y\|^2 - (1+\gamma_n)\,\|x-y\|^2 \\ & -k\,\|(I-T^n)x - (I-T^n)y\|^2 \Big) \leq 0.\end{aligned} \tag{1.11}$$

Throughout this paper, we assume that

$$c_n = \max\Big\{0, \sup_{x,y\in C}\Big(\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 \\ -k\|(I-T^n)x-(I-T^n)y\|^2\Big)\Big\}. \tag{1.12}$$

It follows that $c_n \to 0$ as $n \to \infty$ and (1.11) is reduced to the relation

$$\begin{aligned} \|T^n x - T^n y\|^2 \;\leq\; & (1+\gamma_n)\|x-y\|^2 \\ & +k\|(I-T^n)x-(I-T^n)y\|^2 + c_n, \end{aligned} \tag{1.13}$$

for all $x, y \in C$ and $n \geq 1$.

Remark 1.2. ([15]) (1) T is not necessarily uniformly L-Lipschitzian (see Lemma 2.6 of [15]).

(2) When $c_n = 0$ for all $n \in \mathbb{N}$ in (1.13) then T is an asymptotically k-strictly pseudocontractive mapping with sequence $\{\gamma_n\}$.

They obtained a weak convergence theorem of modified Mann iterative processes for the class of mappings which is not necessarily Lipschitzian. Moreover, a strong convergence theorem was also established in a real Hilbert space by hybrid projection method; see [15] for more details.

In 2001, Xu and Ori [18] have introduced the following implicit iteration process for common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in Hilbert spaces:

$$x_n = t_n x_{n-1} + (1-t_n)T_n x_n, \quad n \geq 1 \tag{1.14}$$

where $T_n = T_{n \bmod N}$. (Here the mod N function takes values in $\{1, 2, \dots, N\}$). And they proved the weak convergence of the process (1.14).

In 2003, Sun [16] modified the implicit iteration process of Xu and Ori [18] and applied the modified averaging iteration process for the approximation of fixed points of asymptotically quasi-nonexpansive mappings. Sun introduced the following implicit iteration process for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings $\{T_i\}_{i=1}^N$ in Banach spaces:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_n, \quad n \geq 1 \tag{1.15}$$

where $n = (k-1)N + i$, $i \in I = \{1, 2, \dots, N\}$.

Very recently, Acedo and Xu [1] still in the framework of Hilbert spaces introduced the following cyclic algorithm.

Let C be a closed convex subset of a Hilbert space H and let $\{T_i\}_{i=0}^{N-1}$ be N k-strict pseudo-contractions on C such that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0,1)$. The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_0 x_0, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1 x_1, \\ &\vdots \\ x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1}, \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_0 x_N, \\ &\vdots \end{aligned}$$

In general, $\{x_{n+1}\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]} x_n, \tag{1.16}$$

where $T_{[n]} = T_i$ with $i = n \ (mod\ N)$, $0 \leq i \leq N-1$. They also proved a weak convergence theorem for k-strict pseudo-contractions in Hilbert spaces by cyclic algorithm (1.16). More precisely, they obtained the following theorem:

Theorem AX. ([1]) *Let C be a closed convex subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $0 \leq i \leq N-1$, $T_i\colon C \to C$ be a k_i-strict pseudo-contraction for some $0 \leq k_i < 1$. Let $k = \max\{k_i : 1 \leq i \leq N\}$. Assume the common fixed point the set $\bigcap_{i=0}^{N-1} F(T_i)$ of $\{T_i\}_{i=0}^{N-1}$ is nonempty. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the cyclic algorithm (1.16). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k + \epsilon < \alpha_n < 1 - \epsilon$ for all n and for some $\epsilon \in (0,1)$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Motivated by Xu and Ori [18], Acedo and Xu [1] and some others we introduce and study the following:

Let C be a closed convex subset of a Hilbert space H and let $\{T_i\}_{i=0}^{N-1}$ be N asymptotically k-strictly pseudo-contractive mappings in the intermediate sense on C such that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in $(0,1)$. The explicit iteration scheme generates a sequence $\{x_n\}_{n=0}^{\infty}$

in the following way:

$$
\begin{aligned}
x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_0 x_0, \\
x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1 x_1, \\
&\vdots \\
x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1}, \\
x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_0^2 x_0, \\
&\vdots \\
x_{2N} &= \alpha_{2N-1} x_{2N-1} + (1-\alpha_{2N-1})T_{N-1}^2 x_{2N-1}, \\
x_{2N+1} &= \alpha_{2N} x_{2N} + (1-\alpha_{2N})T_0^3 x_0, \\
&\vdots
\end{aligned}
$$

In general, $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}^s x_n, \tag{1.17}$$

where $T_{[n]}^s = T_{n\,(mod\ N)}^s = T_i^s$ with $n = (s-1)N + i$ and $i \in I = \{0, 1, \dots, N-1\}$.

The aim of this paper is to establish weak convergence theorem to approximating a common fixed point of $\{T_i\}_{i=0}^{N-1}$. The results presented in the paper extend and generalize some recent results of [1, 5, 7, 9, 12, 14, 15, 17, 18].

In order to prove our main results, we need the following lemma:

Lemma 1.3. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let $T_i\colon C \to C$ be asymptotically k_i-strictly pseudocontractive mappings in the intermediate sense for $i = 0, 1, \dots, N-1$ with a sequence $\{\gamma_{n_i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \gamma_{n_i} < \infty$ and for some $0 \le k_i < 1$, then there exists a constant $k \in [0, 1)$ and sequences $\{\gamma_n\}, \{c_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty} c_n = 0$ such that for any $x, y \in C$ and for each $i = 0, 1, \dots, N-1$ and each $n \ge 0$, the following holds:*

$$
\begin{aligned}
\|T_i^n x - T_i^n y\| \le\ & (1+\gamma_n)\|x-y\|^2 \\
& + k\,\|(I-T_i^n)x - (I-T_i^n)y\|^2 + c_n.
\end{aligned} \tag{1.18}
$$

Proof. Since for each $i = 0, 1, \dots, N-1$, T_i is asymptotically k_i-strictly pseudocontractive mapping in the intermediate sense, where $k_i \in [0, 1)$ and $\{\gamma_{n_i}\}, \{c_{n_i}\} \subset [0, \infty)$ with $\lim_{n\to\infty} \gamma_{n_i} = 0$ and $\lim_{n\to\infty} c_{n_i} = 0$. Taking $\gamma_n = \max\{\gamma_{n_i}, i = 0, 1, \dots, N-1\}$, $c_n = \max\{c_{n_i}, i = 0, 1, \dots, N-1\}$ and

$k = \max\{k_i, i = 0, 1, \ldots, N-1\}$, hence, for each $i = 0, 1, \ldots, N-1$, we have from (1.13)

$$\begin{aligned} \|T_i^n x - T_i^n y\| &\leq (1+\gamma_{n_i})\|x-y\|^2 \\ &\quad + k_i \|(x - T_i^n x) - (y - T_i^n y)\|^2 + c_{n_i}, \\ &\leq (1+\gamma_n)\|x-y\|^2 \\ &\quad + k\|(x - T_i^n x) - (y - T_i^n y)\|^2 + c_n. \end{aligned} \tag{1.19}$$

The conclusion (1.18) is proved. This completes the proof of Lemma 1.3. □

We use following notation:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

2. $\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

In the sequel, we will need the following lemmas.

Lemma 1.4. *Let H be a real Hilbert space. There holds the following identities:*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \quad \forall\, x, y \in H.$

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,$

$\forall\, t \in [0,1], \forall\, x, y \in H.$

(iii) If $\{x_n\}$ be a sequence in H weakly converges to z, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2 \quad \forall y \in H.$$

Lemma 1.5. *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$\{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Lemma 1.6. *Let K be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in K$. Then $z = P_K x$ if and only if there holds the relation*

$$\langle x-z, y-z\rangle \leq 0 \quad \forall y \in K,$$

where P_K is the nearest point projection from H onto K, that is, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \le \|x - y\| \quad \forall x \in K.$$

Lemma 1.7. *([15]) Let C be a nonempty subset of a Hilbert space H and $T\colon C \to C$ a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - T x_n\| \to 0$ as $n \to \infty$.*

Lemma 1.8. *([8]) Let K be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - u\| \;=\; \|u - q\| \quad \forall n. \tag{1.20}$$

Then $x_n \to q$.

Lemma 1.9. *([13, 15]) Let $\{a_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le r_n a_n + \beta_n, \quad n \ge 1.$$

If $r_n \ge 1$, $\sum_{n=1}^{\infty}(r_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.10. *(Proposition 3.1, Demiclosed principle [15]) Let C be a nonempty closed convex subset of a Hilbert space H and $T\colon C \to C$ a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{n\to\infty}\|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.*

Lemma 1.11. *(Lemma 2.2, [15]) Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space X. If $\omega_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$.*

Example 1.12. ([15]) Let $X = \mathbb{R}$ be a normed linear space and $C = [0, 1]$. For each $x \in C$, we define

$$T(x) = \begin{cases} \text{kx}, & \text{if } x \in [0, 1/2], \\ 0, & \text{if } x \in (1/2, 1], \end{cases}$$

where $0 < k < 1$. Then $T\colon C \to C$ is discontinuous at $x = 1/2$ and hence T is not Lipschitzian. Set $C_1 := [1, 1/2]$ and $C_2 := (1/2, 1]$. Hence

$$|T^n x - T^n y| = k^n |x - y| \le |x - y|$$

for all $x, y \in C_1$ and $n \in \mathbb{N}$ and

$$|T^n x - T^n y| = 0 \leq |x - y|$$

for all $x, y \in C_2$ and $n \in \mathbb{N}$.

For $x \in C_1$ and $y \in C_2$, we have

$$\begin{aligned} |T^n x - T^n y| &= |k^n x - 0| = |k^n(x-y) + k^n y| \\ &\leq k^n|x-y| + k^n|y| \\ &\leq |x-y| + k^n, \quad \text{for all} \quad n \in \mathbb{N}. \end{aligned}$$

Thus,

$$\begin{aligned} |T^n x - T^n y|^2 &\leq (|x-y| + k^n)^2 \\ &\leq |x-y|^2 + k|x - T^n x - (y - T^n y)|^2 + k^n K, \end{aligned}$$

for all $x, y \in C$, $n \in \mathbb{N}$ and for some $K > 0$. Therefore, T is an asymptotically k-strictly pseudocontractive mapping in the intermediate sense.

Example 1.13. Let $X = \ell_2 = \{\bar{x} = \{x_i\}_{i=1}^{\infty} : x_i \in C, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, and let $\bar{B} = \{\bar{x} \in \ell_2 : \|x\| \leq 1\}$. Define $T\colon \bar{B} \to \ell_2$ by

$$T\bar{x} = (0, x_1^2, a_2 x_2, a_3 x_3, \dots),$$

where $\{a_j\}_{j=1}^{\infty}$ is a real sequence satisfying: $a_2 > 0$, $0 < a_j < 1$, $j \neq 2$, and $\prod_{j=2}^{\infty} a_j = 1/2$. Then

$$\begin{aligned} \|T^n\bar{x} - T^n\bar{y}\|^2 &\leq 2\Big(\prod_{j=2}^{n} a_j\Big) \|\bar{x} - \bar{y}\|^2 \\ &\leq 2\Big(\prod_{j=2}^{n} a_j\Big) \|\bar{x} - \bar{y}\|^2 + k\,\|(I - T^n)\bar{x} - (I - T^n)\bar{y}\|^2 \\ &\quad + k^n Q \end{aligned}$$

for all $k \in (0,1)$, $n \geq 2$, $\bar{x}, \bar{y} \in X$ and for some $Q > 0$. Since $\lim_{n\to\infty} 2\Big(\prod_{j=2}^{n} a_j\Big) = 1$, it follows that T is an asymptotically k-strictly pseudocontractive mapping in the intermediate sense.

2. Weak convergence of the explicit iteration process

Theorem 2.1. *Let C be a closed convex subset of a Hilbert space H. Let $N \geq 1$ be an integer. Let for each $0 \leq i \leq N-1$, $T_i\colon C \to C$ be N asymptotically k_i-strictly pseudo-contraction mappings in the intermediate sense with sequence*

$\{\gamma_{n_i}\}$ *for some* $0 \leq k_i < 1$. *Let* $k = \max\{k_i : 0 \leq i \leq N-1\}$ *and* $\gamma_n = \max\{\gamma_{n_i} : 0 \leq i \leq N-1\}$. *Assume that*

$$F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset.$$

Given $x_0 \in C$, *let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by an explicit iteration scheme (1.17)with the restriction* $\sum_{n=0}^{\infty} \gamma_n < \infty$. *Assume that the control sequence* $\{\alpha_n\}$ *is chosen so that* $k + \delta < \alpha_n < 1 - \delta$ *for all* n *and for some* $\delta \in (0,1)$ *and* $\sum_{n=0}^{\infty}(1-\alpha_n)c_n < \infty$. *Then* $\{x_n\}$ *converges weakly to a common fixed point of the family* $\{T_i\}_{i=0}^{N-1}$.

Proof. Let $p \in F = \bigcap_{i=0}^{N-1} F(T_i)$. It follows from (1.17) and Lemma 1.4(ii) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\alpha_n x_n + (1-\alpha_n)T_{[n]}^s x_n - p\right\|^2 \\
&= \left\|\alpha_n(x_n - p) + (1-\alpha_n)(T_{[n]}^s x_n - p)\right\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\|T_{[n]}^s x_n - p\right\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\left\|x_n - T_{[n]}^s x_n\right\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\Big[(1+\gamma_n)\|x_n - p\|^2 \\
&\quad +k\left\|x_n - T_i^k x_n\right\|^2 + c_n\Big] - \alpha_n(1-\alpha_n)\left\|x_n - T_{[n]}^s x_n\right\|^2 \\
&\leq \Big[\alpha_n(1+\gamma_n) + (1-\alpha_n)(1+\gamma_n)\Big]\|x_n - p\|^2 + (1-\alpha_n)c_n \\
&\quad -(\alpha_n - k)(1-\alpha_n)\left\|x_n - T_{[n]}^s x_n\right\|^2 \\
&= (1+\gamma_n)\|x_n - p\|^2 - (\alpha_n - k)(1-\alpha_n)\left\|x_n - T_{[n]}^s x_n\right\|^2 \\
&\quad +(1-\alpha_n)c_n \\
&\leq (1+\gamma_n)\|x_n - p\|^2 - \delta^2\left\|x_n - T_{[n]}^s x_n\right\|^2 \\
&\quad +(1-\alpha_n)c_n \qquad\qquad (2.1) \\
&\leq (1+\gamma_n)\|x_n - p\|^2 + (1-\alpha_n)c_n
\end{aligned}$$

Since by assumptions $\sum_{n=0}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty}(1-\alpha_n)c_n < \infty$, it follows from Lemma 1.9, we know that

$$\lim_{n\to\infty} \|x_n - p\| \quad \text{exists.} \tag{2.2}$$

Suppose $\lim_{n\to\infty} \|x_n - p\| = r$ for some $r > 0$. It is easy to see from (2.1) that

$$\delta^2 \left\| x_n - T_{[n]}^s x_n \right\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\alpha_n)c_n, \quad (2.3)$$

which implies that $\lim_{n\to\infty} \left\| x_n - T_{[n]}^s x_n \right\| = 0$. Observe that

$$\|x_{n+1} - x_n\| = (1-\alpha_n) \left\| x_n - T_{[n]}^s x_n \right\| \to 0 \ \text{ as } \ n \to \infty. \quad (2.4)$$

Since $\|x_{n+1} - x_n\| \to 0$, $\left\| x_n - T_{[n]}^s x_n \right\| \to 0$ as $n \to \infty$ and T is uniformly continuous, we obtain from Lemma 1.7 that $\left\| x_n - T_{[n]} x_n \right\| \to 0$ as $n \to \infty$.

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. Note that T is uniformly continuous and $\left\| x_n - T_{[n]} x_n \right\| \to 0$, we see that $\left\| x_n - T_{[n]}^m x_n \right\| \to 0$ for all $m \in \mathbb{N}$. By Lemma 1.10, we obtain $x \in F(T_i)$ for all $i = 0, 1, \ldots, N-1$ and hence

$$x \in F = \bigcap_{i=0}^{N-1} F(T_i).$$

To complete the proof, it suffices to show that $\omega_w(\{x_n\})$ consists of exactly one point, namely, x. Suppose there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $z \neq x$. As in the case of x, we must have

$$z \in F = \bigcap_{i=0}^{N-1} F(T_i).$$

It follows from (2.2) that $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - z\|$ exist. Since H satisfies the Opial condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x\| &= \lim_{j\to\infty} \left\|x_{n_j} - x\right\| < \lim_{j\to\infty} \left\|x_{n_j} - z\right\| = \lim_{n\to\infty} \|x_n - z\|, \\
\lim_{n\to\infty} \|x_n - z\| &= \lim_{k\to\infty} \|x_{n_k} - z\| < \lim_{k\to\infty} \|x_{n_k} - x\| = \lim_{n\to\infty} \|x_n - x\|,
\end{aligned}$$

which is a contradiction. Hence $x = z$ so $\omega_w(\{x_n\})$ is a singleton. Thus, $\{x_n\}$ converges weakly to x by Lemma 1.11. □

We remark that Theorem 2.1 is more general than the results studied in Reich [14], Marino and Xu [9], Acedo and Xu [1], Xu and Ori [18] and Sahu at el. [15].

3. The CQ method for the explicit iteration process

It is the purpose of this paper to modify iteration process (1.17) by hybrid method as follows: chosen arbitrary $u = x_0 \in C$ and

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) T^s_{[n]} x_n, \\ C_n &= \Big\{ z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\quad + (1-\alpha_n)(k-\alpha_n) \Big\| x_n - T^s_{[n]} x_n \Big\|^2 + \theta_n \Big\}, \\ Q_n &= \big\{ z \in C : \langle x_n - z, u - x_n \rangle \ge 0 \big\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(u), \end{aligned} \tag{3.1}$$

where $n = (s-1)N + i$, $i \in I = \{0, 1, \ldots, N-1\}$,

$$\theta_n = \gamma_n \Delta_n^2 + (1-\alpha_n) c_n \ \to 0 \ (n \to \infty),$$

and

$$\Delta_n = \sup \left\{ \|x_n - z\| : z \in F = \bigcap_{i=0}^{N-1} F(T_i) \right\} < \infty.$$

Now, we establish a strong convergence theorem of newly proposed (CQ) algorithm (3.1) for a finite family of asymptotically k-strictly pseudo-contractive mappings in the intermediate sense in the framework of Hilbert spaces.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let $N \ge 1$ be an integer. Let for each $0 \le i \le N-1$, $T_i \colon C \to C$ be N asymptotically k_i-strictly pseudo-contraction mappings in the intermediate sense with sequence $\{\gamma_{n_i}\}$ for some $0 \le k_i < 1$. Let $k = \max\{k_i : 0 \le i \le N-1\}$ and $\gamma_n = \max\{\gamma_{n_i} : 0 \le i \le N-1\}$. Assume that*

$$F = \bigcap_{i=0}^{N-1} F(T_i) \ne \emptyset.$$

Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an explicit iteration scheme (3.1) with the restriction $\sum_{n=0}^{\infty} \gamma_n < \infty$. Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1$ and $\sum_{n=0}^{\infty} (1-\alpha_n) c_n < \infty$. Then $\{x_n\}$ converges strongly to $P_F(u)$.

Proof. We break the proof into the following six steps:

Step 1. C_n is convex.

Indeed, the defining inequality in C_n is equivalent to the inequality

$$\langle 2(x_n - y_n), v\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \theta_n,$$

it follows from Lemma 1.5 that C_n is convex.

Step 2. $F \subset C_n$.

Let $p \in F$. From (3.1), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|\alpha_n(x_n - p) + (1-\alpha_n)(T^s_{[n]}x_n - p)\right\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\|T^s_{[n]}x_n - p\right\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\left\|x_n - T^s_{[n]}x_n\right\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\Big[(1+\gamma_n)\|x_n - p\|^2 \\
&\quad +k\left\|x_n - T^s_{[n]}x_n\right\|^2 + c_n\Big] - \alpha_n(1-\alpha_n)\left\|x_n - T^s_{[n]}x_n\right\|^2 \\
&\leq \Big[\alpha_n(1+\gamma_n) + (1-\alpha_n)(1+\gamma_n)\Big]\|x_n - p\|^2 + (1-\alpha_n)c_n \\
&\quad -(\alpha_n - k)(1-\alpha_n)\left\|x_n - T^s_{[n]}x_n\right\|^2 \\
&\leq (1+\gamma_n)\|x_n - p\|^2 + (k-\alpha_n)(1-\alpha_n)\left\|x_n - T^s_{[n]}x_n\right\|^2 \\
&\quad +(1-\alpha_n)c_n \\
&\leq \|x_n - p\|^2 + (k-\alpha_n)(1-\alpha_n)\left\|x_n - T^s_{[n]}x_n\right\|^2 \\
&\quad +\gamma_n\Delta_n^2 + (1-\alpha_n)c_n. \qquad (3.2)
\end{aligned}$$

Hence $p \in C_n$. Thus, $F \subset C_n$.

Step 3. $F \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$.

It suffices to show that $F \subset Q_n$. We prove this by induction.

For $n = 1$, we have $F \subset C = Q_1$. Assume that $F \subset Q_n$. Since x_{n+1} is the projection of u onto $C_n \cap Q_n$. It follows that

$$\langle x_{n+1} - z, u - x_{n+1}\rangle \geq 0$$

for all $z \in C_n \cap Q_n$.

As $F \subset C_n \cap Q_n$, the last inequality holds, in particular for all $z \in F$. By the definition of Q_{n+1},

$$Q_{n+1} = \{z \in C : \langle x_{n+1} - z, u - x_{n+1} \rangle \geq 0\},$$

it follows that $F \subset Q_{n+1}$. By the principle of mathematical induction, we have $F \subset Q_n$ for all $n \in \mathbb{N}$.

Step 4. $\|x_n - x_{n+1}\| \to 0$.

By the definition of Q_n, we have

$$x_n = P_{Q_n}(u)$$

and

$$\|u - x_n\| \leq \|u - y\| \quad \text{for all} \quad y \in F \subset Q_n.$$

Note that boundedness of F implies that $\{\|x_n - u\|\}$ is bounded. Since $x_n = P_{Q_n}(u)$ which together with the fact that $x_{n+1} \in C_n \cap Q_n \subseteq Q_n$ implies that

$$\|u - x_n\| \leq \|u - x_{n+1}\|.$$

Thus, $\{\|x_n - u\|\}$ is increasing. Since $\{\|x_n - u\|\}$ is bounded, we obtain that $\lim_{n\to\infty} \|x_n - u\|$ exists.

Observe that $x_n = P_{Q_n}(u)$ and $x_{n+1} \in Q_n$ which imply that

$$\langle x_{n+1} - x_n, x_n - u \rangle \geq 0.$$

Using Lemma 1.4(i), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - u) - (x_n - u)\|^2 \\ &= \|x_{n+1} - u\|^2 - \|x_n - u\|^2 - 2\langle x_{n+1} - x_n, x_n - u \rangle \\ &\leq \|x_{n+1} - u\|^2 - \|x_n - u\|^2 \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Step 5. $\left\|x_n - T_{[n]} x_n\right\| \to 0$.

Since $x_{n+1} \in C_n$, we get

$$\begin{aligned} \|x_{n+1} - y_n\|^2 &\leq \|x_{n+1} - x_n\|^2 \\ &\quad + (k - \alpha_n)(1 - \alpha_n) \left\|x_n - T^s_{[n]} x_n\right\|^2 + \theta_n. \end{aligned} \tag{3.3}$$

Moreover, since $y_n = \alpha_n x_n + (1-\alpha_n)T^s_{[n]}x_n$, we deduce that

$$\begin{aligned} \|x_{n+1} - y_n\|^2 &= \alpha_n \|x_{n+1} - x_n\|^2 \\ &\quad + (1-\alpha_n)\left\|x_{n+1} - T^s_{[n]}x_n\right\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\left\|x_n - T^s_{[n]}x_n\right\|^2. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) to get

$$\begin{aligned} (1-\alpha_n)\left\|x_{n+1} - T^s_{[n]}x_n\right\|^2 &\le (1-\alpha_n)\|x_{n+1} - x_n\|^2 \\ &\quad + k(1-\alpha_n)\left\|x_n - T^s_{[n]}x_n\right\|^2 + \theta_n. \end{aligned}$$

Since $\limsup_{n\to\infty} \alpha_n < 1$, the last inequality becomes,

$$\begin{aligned} \left\|x_{n+1} - T^s_{[n]}x_n\right\|^2 &\le \|x_{n+1} - x_n\|^2 + k\left\|x_n - T^s_{[n]}x_n\right\|^2 \\ &\quad + \frac{\theta_n}{1-\tau}, \end{aligned} \tag{3.5}$$

for some positive number $\tau > 0$, such that $\alpha_n \le \tau < 1$.

But on the other hand, we compute

$$\begin{aligned} \left\|x_{n+1} - T^s_{[n]}x_n\right\|^2 &= \|x_{n+1} - x_n\|^2 + 2\langle x_{n+1} - x_n, x_n - T^s_{[n]}x_n\rangle \\ &\quad + \left\|x_n - T^s_{[n]}x_n\right\|^2. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we get

$$(1-k)\left\|x_n - T^s_{[n]}x_n\right\|^2 \le \frac{\theta_n}{1-\tau} - 2\langle x_{n+1} - x_n, x_n - T^s_{[n]}x_n\rangle. \tag{3.7}$$

Therefore

$$\begin{aligned} \left\|x_n - T^s_{[n]}x_n\right\|^2 &\le \frac{\theta_n}{(1-\tau)(1-k)} - \frac{2}{1-k}\langle x_{n+1} - x_n, x_n - T^s_{[n]}x_n\rangle \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.8}$$

Now,

$$\begin{aligned} \left\|x_n - T_{[n]}x_n\right\| &\le \left\|x_n - T^s_{[n]}x_n\right\| + \left\|T^s_{[n]}x_n - T_{[n]}x_n\right\| \\ &\le \left\|x_n - T^s_{[n]}x_n\right\| + [(1+\gamma_1)\left\|T^{s-1}_{[n]}x_n - x_n\right\| + c_1] \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.9}$$

Step 6. $x_n \to v \in F$.

Since H is reflexive and $\{x_n\}$ is bounded, we get that $\omega_w(\{x_n\})$ is nonempty. First, we show that $\omega_w(\{x_n\})$ is singleton. Assume that $\{x_{n_j}\}$ is subsequence of $\{x_n\}$ such that $x_{n_j} \to v \in C$. Since $x_n - T_{[n]}x_n \to 0$ by Step 4, it follows from the uniform continuity of T that $x_n - T^m_{[n]}x_n \to 0$ for all $m \in \mathbb{N}$. By Lemma 1.10, $v \in \omega_w(\{x_n\}) \subset F$.

Since $x_{n+1} = P_{c_n \cap Q_n}(u)$, we obtain that

$$\|u - x_{n+1}\| \leq \|u - P_F(u)\| \quad \text{for all} \quad n \in \mathbb{N}.$$

Observe that

$$u - x_{n_j} \rightharpoonup u - v.$$

By the weak lower semicontinuity of norm,

$$\begin{aligned} \|u - P_F(u)\| &\leq \|u - v\| \\ &\leq \liminf_{j\to\infty} \left\|u - x_{n_j}\right\| \\ &\leq \limsup_{j\to\infty} \left\|u - x_{n_j}\right\| \\ &\leq \|u - P_F(u)\|, \end{aligned}$$

which yields

$$\|u - P_F(u)\| = \|u - v\|$$

and

$$\lim_{j\to\infty} \left\|u - x_{n_j}\right\| = \|u - P_F(u)\|. \tag{3.10}$$

Hence $v = P_F(u)$ by the uniqueness of the nearest point projection of u onto F. Thus, $\left\|x_{n_j} - u\right\| \to \|v - u\|$. It shows that $x_{n_j} - u \to v - u$, i.e., $x_{n_j} \to v$. Since $\{x_{n_j}\}$ is an arbitrary weakly convergent subsequence, it follows that $\omega_w(\{x_n\}) = \{v\}$ and hence from Lemma 1.11 we have $x_n \rightharpoonup v$. Now, it is easy to see as (3.10) that $\|x_n - u\| \to \|v - u\|$. Therefore, $x_n \to v$, that is, $\{x_n\}$ converges strongly to $P_F(u)$. This completes the proof. □

Remark 3.2. Theorem 2.1 extends and improves the corresponding result of Reich [14] and Marino and Xu [9] from nonexpansive and strict pseudo-contraction mapping to more general class of finite family of asymptotically k-strictly pseudo-contraction mappings in the intermediate sense and explicit iteration scheme considered in this paper.

Remark 3.3. Theorem 2.1 also extends and improves the corresponding result of Acedo and Xu [1] from k-strictly pseudo-contraction mapping to more general class of asymptotically k-strictly pseudo-contraction mappings in the intermediate sense considered in this paper.

Remark 3.4. Theorem 2.1 also extends and improves the corresponding result of Xu and Ori [18] from nonexpansive mapping to more general class of asymptotically k-strictly pseudo-contraction mappings in the intermediate sense considered in this paper.

Remark 3.5. Theorem 3.1 extends Theorem 3.1 of Thakur [17] to the case of finite family of asymptotically k-strictly pseudo-contractive mappings in the intermediate sense and explicit iteration process considered in this paper.

Remark 3.6. Theorem 3.1 also extends and generalizes Theorem 3.4 of Nakajo and Takahashi [10] and Theorem 2.2 of Kim and Xu [4] from nonexpansive and asymptotically nonexpansive mapping to more general class of finite family of asymptotically k-strictly pseudo-contractive mappings in the intermediate sense and explicit iteration process considered in this paper.

Remark 3.7. Our results also extend the corresponding results of Sahu et al. [15] to the case of explicit iteration process considered in this paper.

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