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ZEROS OF CERTAIN POLYNOMIALS

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Abstract. The fundamental theorem of algebra guarantees the existence of exactly n zeros of any nth degree polynomial. But unfortunately, there is no general method for finding these zeros. Hence there is a need of locating at least the regions where some or all the zeros of a polynomial lie. A vast literature is available in this regard. In this paper, we put certain restrictions on the coefficients of a polynomial and specify the regions where some or all its zeros lie. Our results generalize many already well-known results.

1. Introduction

The following Enestrom-Kakeya Theorem [5] is well-known result in the theory of distribution of zeros of a polynomial:

Theorem 1.1. ([5]) *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then all the zeros of $P(z)$ *lie in* $|z| \leq 1$.

In the literature there exist various extensions and generalizations of Theorem 1.1. For a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ whose coefficients satisfy the

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Enestrom-Kakeya condition, Mohammad [6] found a bound for the number of its zeros in $|z| \leq \frac{1}{2}$. In fact he proved the following result:

Theorem 1.2. ([6]) *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$ *is less than or equal to*

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}.$$

Dewan [1] in 1980 generalized Theorem 1.2 to polynomials with complex coefficients and proved the following result:

Theorem 1.3. ([1]) *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a complex polynomial of degree n with* $a_j = \alpha_j + i\beta_j$, $j = 0, 1, 2, \cdots, n$ *where* α_j *and* β_j *are real numbers. If*

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0,$$

then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$ *is less than or equal to*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

Upadhye [8] in 2007 generalized Theorem 1.3 by proving the following result:

Theorem 1.4. ([8]) *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a complex polynomial of degree n with* $a_j = \alpha_j + i\beta_j$, $j = 0, 1, 2, \cdots, n$ *where* α_j *and* β_j *are real numbers. If for some* $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

then the number of zeros of P(z) in $|z| \leq \delta$, $0 < \delta < 1$ *is less than or equal to*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{k(\alpha_n + |\alpha_n|) + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

Gulzar [3] in 2012 generalized Theorem 1.4 as follows:

Theorem 1.5. ([3]) *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a complex polynomial of degree* n *with* $a_j=\alpha_j+i\beta_j$, $j=0,1,2,\cdots,n$ *where* α_j *and* β_j *are real numbers. If for some* $k\geq 1$, $0<\tau\leq 1$

$$k\alpha_n\geq\alpha_{n-1}\geq\cdots\geq\alpha_1\geq\tau\alpha_0,$$

then the number of zeros of $P(z)$ *in* $|z|\leq\delta, 0<\delta<1$ *is less than or equal to*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k\left(\alpha_n+|\alpha_n|\right)+2|\alpha_0|-\tau(\alpha_0+|\alpha_0|)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Irshad *et al.* [4] recently proved the following results:

Theorem 1.6. ([4]) *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree* n *with complex coefficients such that*

$$|a_n|\leq|a_{n-1}|\leq\cdots\leq|a_{k+1}|\leq\lambda|a_k|$$

and

$$\lambda|a_k|\geq|a_{k-1}|\geq\cdots\geq|a_1|\geq|a_0|,$$

where $0\leq k\leq n-1$ *and for some real* β,

$$|arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ \ j=0,1,2,\cdots,n.$$

Then the number of zeros of $P(z)$ *in* $|z|\leq\frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2}\log\left(\frac{2\lambda|a_k|\cos\alpha+2|\lambda-1||a_k|\sin\alpha+|a_n|(\sin\alpha-\cos\alpha+1)}{|a_0|}\right.$$
$$\left.+\frac{2\sin\alpha\sum_{j=0}^{n-1}|a_j|+2|1-\lambda||a_k|}{|a_0|}\right).$$

Theorem 1.7. ([4]) *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree* n *with complex coefficients such that*

$$|a_n|\leq|a_{n-1}|\leq\cdots\leq|a_{k+1}|\leq\lambda|a_k|$$

and

$$\lambda|a_k|\geq|a_{k-1}|\geq\cdots\geq|a_1|\geq|a_0|,$$

where $0\leq k\leq n-1$ *and for some real* β,

$$|arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ \ j=0,1,2,\cdots,n.$$

Then all the zeros of $P(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}-1\right| \le \frac{1}{|a_n|}\Bigg\{2\lambda|a_k|\cos\alpha+2|\lambda-1||a_k|\sin\alpha-|a_{n-1}|(\sin\alpha+\cos\alpha) + \alpha\sum_{j=0}^{n-1}|a_j| - |a_0|(\sin\alpha+\cos\alpha-1)+2|1-\lambda||a_k|\Bigg\}.$$

2. Main results

In this paper we generalize Theorems 1.6 and 1.7 as follows:

Theorem 2.1. *Let $P(z)=\sum_{j=0}^{n-1} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some integer k, $0 \le k \le n-1$, for some $\lambda \ge 1$, $0<\mu\le 1$ and for some real α, β,*

$$|arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \quad j = 0,1,2,\cdots,n,$$

$$|a_n| \le |a_{n-1}| \le \cdots \le |a_{k+1}| \le \lambda|a_k|$$

and

$$\lambda|a_k| \ge |a_{k-1}| \ge \cdots \ge |a_1| \ge \mu|a_0|.$$

Then all the zeros of $P(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}-1\right| \le \frac{1}{|a_n|}\Bigg\{2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha - |a_{n-1}|(\sin\alpha+\cos\alpha)+2\sin\alpha\sum_{j=0}^{n-1}|a_j| - \mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2|\lambda-1||a_k|\Bigg\}.$$

Theorem 2.2. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some integer k, $0 \le k \le n-1$, for some $\lambda \ge 1$, $0<\mu\le 1$ and for some real α, β,*

$$|arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \quad j = 0,1,2,\cdots,n,$$

$$|a_n| \le |a_{n-1}| \le \cdots \le |a_{k+1}| \le \lambda|a_k|$$

and

$$\lambda|a_k| \ge |a_{k-1}| \ge \cdots \ge |a_1| \ge \mu|a_0|.$$

Then the number zeros of $P(z)$ *in* $|z| \leq \frac{R}{c}$, $(c > 0, R > 0, c < R)$ *is less than or equal to*

$$\frac{1}{\log c} \log \frac{1}{|a_0|} R^{n+1} \Big\{ |a_n|(\sin\alpha - \cos\alpha + 1) + 2\lambda|a_k|\cos\alpha + 2(\lambda-1)|a_k|\sin\alpha + \alpha\sum_{j=0}^{n-1}|a_j| + 2|a_0| - \mu|a_0|(\cos\alpha - \sin\alpha + 1)\Big\}, \quad \text{for } R \geq 1$$

and

$$\frac{1}{\log c} \log \frac{1}{|a_0|} \Big\{ |a_0| + R|a_n|(\sin\alpha - \cos\alpha + 1) + 2\lambda|a_k|\cos\alpha + 2(\lambda-1)|a_k|\sin\alpha + \alpha\sum_{j=0}^{n-1}|a_j| + |a_0| - \mu|a_0|(\cos\alpha - \sin\alpha + 1)\Big\}, \quad \text{for } R \leq 1.$$

Remark 2.3. Taking $\mu = 1$ in Theorem 2.1, we get Theorem 1.5.

Taking $c = \frac{1}{\delta}$, $0 < \delta < 1$, $R = 1$ in Theorem 2,2, we get the following result:

Corollary 2.4. *Let* $P(z) = \sum_{j=0}^{n-1} a_j z^j$ *be a polynomial of degree* n *with complex coefficients such that for some integer* k, $0 \leq k \leq n-1$, *for some* $\lambda \geq 1$, $0 < \mu \leq 1$ *and for some reals* α, β,

$$|arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \cdots, n,$$

$$|a_n| \leq |a_{n-1}| \leq \cdots \leq |a_{k+1}| \leq \lambda|a_k|$$

and

$$\lambda|a_k| \geq |a_{k-1}| \geq \cdots \geq |a_1| \geq \mu|a_0|.$$

Then the number of zeros of $P(z)$ *in* $|z| \leq \delta$, $0 < \delta < 1$, *is less than or equal to*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{1}{|a_0|} \Big\{ |a_n|(\sin\alpha - \cos\alpha + 1) + 2\lambda|a_k|\cos\alpha + 2(\lambda-1)|a_k|\sin\alpha + \alpha\sum_{j=0}^{n-1}|a_j| + |a_0| - |a_0|(\cos\alpha - \sin\alpha + 1)\Big\}.$$

Taking $c = 2$ in Theorem 2.2, we get following result:

Corollary 2.5. *Let* $P(z)=\sum\limits_{j=0}^{n-1}a_jz^j$ *be a polynomial of degree* n *with complex coefficients such that for some integer* k, $0\le k\le n-1$, *for some* $\lambda\ge 1$, $0<\mu\le 1$ *and for some reals* α,β,

$$|arga_j-\beta|\le\alpha\le\frac{\pi}{2},\ \ j=0,1,2,\cdots,n,$$

$$|a_n|\le|a_{n-1}|\le\cdots\le|a_{k+1}|\le\lambda|a_k|$$

and

$$\lambda|a_k|\ge|a_{k-1}|\ge\cdots\ge|a_1|\ge\mu|a_0|.$$

Then the number of zeros of $P(z)$ *in* $|z|\le\frac{R}{2}(R>0)$ *is less than or equal to*

$$\frac{1}{\log 2}\log\frac{1}{|a_0|}R^{n+1}\Bigg\{|a_n|(\sin\alpha-\cos\alpha+1)+2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha$$
$$+2\sin\alpha\sum_{j=0}^{n-1}|a_j|+2|a_0|-\mu|a_0|(\cos\alpha-\sin\alpha+1)\Bigg\},\ \textit{for}\ R\ge 1$$

and

$$\frac{1}{\log 2}\log\frac{1}{|a_0|}\Bigg\{|a_0|+R|a_n|(\sin\alpha-\cos\alpha+1)+2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha$$
$$+2\sin\alpha\sum_{j=0}^{n-1}|a_j|+|a_0|-\mu|a_0|(\cos\alpha-\sin\alpha+1)\Bigg\},\ \ \textit{for}\ \ R\le 1.$$

Many other results can be obtained from Theorems 2.1 and 2.2 by taking different values of parameters.

3. Lemmas

For the proofs of the above results we need the following results:

Lemma 3.1. *For any two complex numbers* b_1,b_2 *such that* $|b_1|\ge|b_2|$ *and* $|argb_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=1,2$ *for some reals* α,β, *we have*

$$|b_1-b_2|\le(|b_1|-|b_2|)\cos\alpha+(|b_1|+|b_2|)\sin\alpha.$$

Lemma 3.1 is due to Govil and Rahman [2].

Lemma 3.2. *If* $f(z)$ *is analytic in* $|z|\le R$, *but not identically zero,* $f(0)\ne 0$ *and* $f(a_k)=0$, $k=1,2,\cdots,n$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})d\theta|-\log|f(0)|=\sum_{k=1}^{n}\log\frac{R}{|a_k|}.$$

Lemma 3.2 is the famous Jensen's Theorem(see page 208 of [7]).

Lemma 3.3. *If $f(z)$ is analytic,$|f(z)| \leq M$ in $|z| \leq R$, $f(0) \neq 0$, then the number of zeros of $f(z)$ in $|z| \leq \frac{R}{c}$, $c > 1$ does not exceed $\frac{1}{\log c} \log \frac{M}{|f(0)|}$.*

Lemma 3.3 is simple consequence of Lemma 3.2.

4. Proofs of theorems

Proof of Theorem 2.1. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_{k+1} - a_k) z^{k+1} \\
&\quad + \cdots + (a_2 - a_1) z^2 + (a_1 - \mu a_0) z + (\mu a_0 - a_0) z + a_0 \\
&= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} \\
&\quad + \cdots + (a_{k+1} - \lambda a_k + \lambda a_k - a_k) z^{k+1} \\
&\quad + \cdots + (a_2 - a_1) z^2 + (a_1 - \mu a_0) z + (\mu a_0 - a_0) z + a_0.
\end{aligned}$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1$, $j = 1, 2, .., n$, we have by using the hypothesis and Lemma 3.1,

$$\begin{aligned}
|F(z)| &\geq |a_n z^{n+1} - (a_n - a_{n-1}) z^n| - \Big\{ |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots \\
&\quad + |a_{k+1} - \lambda a_k + \lambda a_k - a_k||z|^{k+1} + |a_k - \lambda a_k + \lambda a_k - a_{k-1}||z|^k \\
&\quad + \cdots + |a_2 - a_1||z|^2 + |a_1 - \mu a_0 + \mu a_0 - a_0||z| + |a_0| \Big\} \\
&\geq |a_n z^{n+1} - (a_n - a_{n-1}) z^n| - \Big\{ |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots \\
&\quad + |a_{k+1} - \lambda a_k||z|^{k+1} + |\lambda a_k - a_k||z|^{k+1} + |a_k - \lambda a_k||z|^k \\
&\quad + |\lambda a_k - a_{k-1}||z|^k + ... + |a_2 - a_1||z|^2 + |a_1 - \mu a_0||z| + |\mu a_0 - a_0||z| + |a_0| \Big\} \\
&= |z|^n |a_n z + a_{n-1} - a_n| - \Big\{ \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_{k+1} - \lambda a_k|}{|z|^{n-k-1}} \\
&\quad + \frac{|\lambda - 1||a_k|}{|z|^{n-k-1}} + \frac{|1 - \lambda||a_k|}{|z|^{n-k}} + \frac{|\lambda a_k - a_{k-1}|}{|z|^{n-k}} + \cdots \\
&\quad + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - \mu a_0|}{|z|^{n-1}} + \frac{|\mu - 1||a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Big\}
\end{aligned}$$

$$
\begin{aligned}
&> |z|^n|a_nz + a_{n-1} - a_n| - \Big\{|a_{n-1} - a_{n-2}| + \cdots + |a_{k+1} - \lambda a_k| \\
&\quad + |\lambda - 1||a_k| + |1 - \lambda||a_k| + |\lambda a_k - a_{k-1}| + \cdots \\
&\quad + |a_2 - a_1| + |a_1 - \mu a_0| + |\mu - 1||a_0| + |a_0|\Big\} \\
&\geq |z|^n|a_nz + a_{n-1} - a_n| - \Big\{(|a_{n-2}| - |a_{n-1}|)\cos\alpha \\
&\quad + (|a_{n-2}| + |a_{n-1}|)\sin\alpha + \cdots + (|\lambda a_k| - |a_{k+1}|)\cos\alpha \\
&\quad + (|\lambda a_k| + |a_{k+1}|)\sin\alpha + (|\lambda a_k| - |a_{k-1}|)\cos\alpha \\
&\quad + (|\lambda a_k| + |a_{k-1}|)\sin\alpha + \cdots + (|a_2| - |a_1|)\cos\alpha \\
&\quad + (|a_2| + |a_1|)\sin\alpha + (|a_1| - \mu|a_0|)\cos\alpha \\
&\quad + (|a_1| + \mu|a_0|)\sin\alpha + (1 - \mu)|a_0| + |a_0| + 2|\lambda - 1||a_k|\Big\} \\
&= |z|^n|a_nz + a_{n-1} - a_n| - \Big\{2\lambda|a_k|\cos\alpha + 2(\lambda - 1)|a_k|\sin\alpha \\
&\quad - |a_{n-1}|(\sin\alpha + \cos\alpha) + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| - |a_0| \\
&\quad - \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2|\lambda - 1||a_k|\Big\} \\
&> 0,
\end{aligned}
$$

if

$$
\begin{aligned}
&|z|^n|a_nz + a_{n-1} - a_n| \\
&> \Big\{2\lambda|a_k|\cos\alpha + 2(\lambda - 1)|a_k|\sin\alpha - |a_{n-1}|(\sin\alpha + \cos\alpha) \\
&\quad + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| - |a_0| - \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2|\lambda - 1||a_k|\Big\}.
\end{aligned}
$$

Thus $F(z)$ does not vanish in

$$
\begin{aligned}
&\left|z + \frac{a_{n-1}}{a_n} - 1\right| \\
&> \frac{1}{|a_n|}\Big\{2\lambda|a_k|\cos\alpha + 2(\lambda - 1)|a_k|\sin\alpha - |a_{n-1}|(\sin\alpha + \cos\alpha) \\
&\quad + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| - |a_0| - \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2|\lambda - 1||a_k|\Big\}.
\end{aligned}
$$

In other words, those zeros of $F(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}&\left|z+\frac{a_{n-1}}{a_n}-1\right|\\&\leq \frac{1}{|a_n|}\Bigg\{2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha-|a_{n-1}|(\sin\alpha+\cos\alpha)\\&\quad+2\sin\alpha\sum_{j=1}^{n-1}|a_j|-|a_0|-\mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2|\lambda-1||a_k|\Bigg\}.\end{aligned}$$

Since those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of $F(z)$ and hence $P(z)$ lie in

$$\begin{aligned}&\left|z+\frac{a_{n-1}}{a_n}-1\right|\\&\leq \frac{1}{|a_n|}\Bigg\{2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha-|a_{n-1}|(\sin\alpha+\cos\alpha)\\&\quad+2\sin\alpha\sum_{j=1}^{n-1}|a_j|-|a_0|-\mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2|\lambda-1||a_k|\Bigg\}.\end{aligned}$$

That completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+...+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+...\\&\quad+(a_{k+1}-a_k)z^{k+1}+...+(a_2-a_1)z^2+(a_1-\mu a_0)z+(\mu a_0-a_0)z+a_0\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+...\\&\quad+(a_{k+1}-\lambda a_k+\lambda a_k-a_k)z^{k+1}+...\\&\quad+(a_2-a_1)z^2+(a_1-\mu a_0)z+(\mu a_0-a_0)z+a_0.\end{aligned}$$

For $|z|\leq R$, we have, by using the hypothesis and Lemma 3.1

$$\begin{aligned}|F(z)|\leq{}&|a_n|R^{n+1}+\big\{(|a_n|-|a_{n-1}|)\cos\alpha+(|a_n|+|a_{n-1}|)\sin\alpha\big\}R^n\\&+\big\{(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha\big\}R^{n-1}+\cdots\\&+\big\{(|a_{k+1}|-\lambda|a_k|)\cos\alpha+(|a_{k+1}|+\lambda|a_k|)\sin\alpha\big\}R^{k+1}\\&+|\lambda-1||a_k|R^{k+1}+|\lambda-1||a_k|R^k\end{aligned}$$

$$+\big\{(\lambda|a_k|-|a_{k-1}|)\cos\alpha+(\lambda|a_k|+|a_{k-1}|)\sin\alpha\big\}R^k+\cdots$$
$$+\big\{(|a_2|-|a_1|)\cos\alpha+(|a_2|+|a_1|)\sin\alpha\big\}R^2$$
$$+\big\{(|a_1|-\mu|a_0|)\cos\alpha+(|a_1|+\mu|a_0|)\sin\alpha\big\}R^2+|1-\mu||a_0|R+|a_0|.$$

This implies for $R\geq 1$ so that $R^j\leq R^{n+1}$, for all $j=0,1,2,...,n$,

$$\begin{aligned}|F(z)|\leq R^{n+1}\Bigg\{&|a_n|+\big\{(|a_{n-1}|-|a_n|)\cos\alpha+(|a_{n-1}|+|a_n|)\sin\alpha\big\}\\&+\big\{(|a_{n-2}|-|a_{n-1}|)\cos\alpha+(|a_{n-2}|+|a_{n-1}|)\sin\alpha\big\}+\cdots\\&+\big\{(\lambda|a_k|-|a_{k+1}|)\cos\alpha+(\lambda|a_k|+|a_{k+1}|)\sin\alpha\big\}+|\lambda-1||a_k|\\&+\big\{(\lambda|a_k|-|a_{k-1}|)\cos\alpha+(\lambda|a_k|+|a_{k-1}|)\sin\alpha\big\}+\cdots\\&+\big\{(|a_2|-|a_1|)\cos\alpha+(|a_2|+|a_1|)\sin\alpha\big\}\\&+\big\{(|a_1|-\mu|a_0|)\cos\alpha+(|a_1|+\mu|a_0|)\sin\alpha\big\}+|\mu-1||a_0|+|a_0|\Bigg\}\\=R^{n+1}\Bigg\{&|a_n|(\sin\alpha-\cos\alpha+1)+2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha\\&+2\sin\alpha\sum_{j=1}^{n-1}|a_j|+2|a_0|-\mu|a_0|(\cos\alpha-\sin\alpha+1)\Bigg\}\end{aligned}$$

and for $R\leq 1$ so that $R^j\leq R$, for all $j=1,2,...,n+1$,

$$\begin{aligned}|F(z)|\leq R\Bigg\{&|a_n|+\big\{(|a_{n-1}|-|a_n|)\cos\alpha+(|a_{n-1}|+|a_n|)\sin\alpha\big\}\\&+\big\{(|a_{n-2}|-|a_{n-1}|)\cos\alpha+(|a_{n-2}|+|a_{n-1}|)\sin\alpha\big\}+\cdots\\&+\big\{(\lambda|a_k|-|a_{k+1}|)\cos\alpha+(\lambda|a_k|+|a_{k+1}|)\sin\alpha\big\}+|\lambda-1||a_k|\\&+\big\{(\lambda|a_k|-|a_{k-1}|)\cos\alpha+(\lambda|a_k|+|a_{k-1}|)\sin\alpha\big\}+\cdots\\&+\big\{(|a_2|-|a_1|)\cos\alpha+(|a_2|+|a_1|)\sin\alpha\big\}\\&+\big\{(|a_1|-\mu|a_0|)\cos\alpha+(|a_1|+\mu|a_0|)\sin\alpha\big\}+|\mu-1||a_0|\Bigg\}+|a_0|\\=|a_0|+R\Bigg\{&|a_n|(\sin\alpha-\cos\alpha+1)+2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha\\&+2\sin\alpha\sum_{j=1}^{n-1}|a_j|+|a_0|-\mu|a_0|(\cos\alpha-\sin\alpha+1)\Bigg\}.\end{aligned}$$

Since $F(z)$ is analytic for $|z|\leq R$, $F(0)=a_0\neq 0$, it follows by Lemma 3.2 that the number of zeros of $F(z)$ in $|z|\leq\frac{R}{c}$ $(c>0,R>0,c<R)$ is less than

or equal to

$$\frac{1}{\log c}\log\frac{1}{|a_0|}R^{n+1}\Big\{|a_n|(\sin\alpha-\cos\alpha+1)+2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha$$
$$+\alpha\sum_{j=0}^{n-1}|a_j|+2|a_0|-\mu|a_0|(\cos\alpha-\sin\alpha+1)\Big\},\quad\text{for}\quad R\geq 1$$

and

$$\frac{1}{\log c}\log\frac{1}{|a_0|}\Big\{|a_0|+R|a_n|(\sin\alpha-\cos\alpha+1)+2\lambda|a_k|\cos\alpha+2(\lambda-1)|a_k|\sin\alpha$$
$$+\alpha\sum_{j=0}^{n-1}|a_j|+|a_0|-\mu|a_0|(\cos\alpha-\sin\alpha+1)\Big\},\quad\text{for}\quad R\leq 1.$$

On the other hand, we have

$$F(z)=G(z)+a_0$$

where

$$\begin{aligned}G(z)=&-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots\\&+(a_{k+1}-a_k)z^{k+1}+\cdots+(a_2-a_1)z^2+(a_1-\mu a_0)z+(\mu a_0-a_0)z.\end{aligned}$$

For $|z|\leq R$, we have by using the hypothesis and Lemma 3.1,

$$\begin{aligned}|G(z)|\leq{}&|a_n|R^{n+1}+\big\{(|a_n|-|a_{n-1}|)\cos\alpha+(|a_n|+|a_{n-1}|)\sin\alpha\big\}R^n\\&+\big\{(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha\big\}R^{n-1}+...\\&+\big\{(|a_{k+1}|-\lambda|a_k|)\cos\alpha+(|a_{k+1}|+\lambda|a_k|)\sin\alpha\big\}R^{k+1}\\&+|\lambda-1||a_k|R^{k+1}+|\lambda-1||a_k|R^k\\&+\big\{(\lambda|a_k|-|a_{k-1}|)\cos\alpha+(\lambda|a_k|+|a_{k-1}|)\sin\alpha\big\}R^k+...\\&+\big\{(|a_2|-|a_1|)\cos\alpha+(|a_2|+|a_1|)\sin\alpha\big\}R^2\\&+\big\{(|a_1|-\mu|a_0|)\cos\alpha+(|a_1|+\mu|a_0|)\sin\alpha\big\}R^2+|1-\mu||a_0|R.\end{aligned}$$

This implies for $R\geq 1$ so that $R^j\leq R^{n+1}$, for all $j=0,1,2,...,n$,

$$
\begin{aligned}
|G(z)| \leq R^{n+1}\Big\{&|a_n| + \big\{(|a_{n-1}| - |a_n|)\cos\alpha + (|a_{n-1}| + |a_n|)\sin\alpha\big\} \\
&+ \big\{(|a_{n-2}| - |a_{n-1}|)\cos\alpha + (|a_{n-2}| + |a_{n-1}|)\sin\alpha\big\} + \cdots \\
&+ \big\{(\lambda|a_k| - |a_{k+1}|)\cos\alpha + (\lambda|a_k| + |a_{k+1}|)\sin\alpha\big\} + |\lambda - 1||a_k| \\
&+ \big\{(\lambda|a_k| - |a_{k-1}|)\cos\alpha + (\lambda|a_k| + |a_{k-1}|)\sin\alpha\big\} + \cdots \\
&+ \big\{(|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha\big\} \\
&+ \big\{(|a_1| - \mu|a_0|)\cos\alpha + (|a_1| + \mu|a_0|)\sin\alpha\big\} + |\mu - 1||a_0|\Big\} \\
= R^{n+1}\Big\{&|a_n|(\sin\alpha - \cos\alpha + 1) + 2\lambda|a_k|\cos\alpha + 2(\lambda - 1)|a_k|\sin\alpha \\
&+ 2\sin\alpha\sum_{j=1}^{n-1}|a_j| + |a_0| - \mu|a_0|(\cos\alpha - \sin\alpha + 1)\Big\} \\
= M_1&
\end{aligned}
$$

and for $R \leq 1$ so that $R^j \leq R$, for all $j = 1, 2, ..., n+1$,

$$
\begin{aligned}
|G(z)| \leq R\Big\{&|a_n| + \big\{(|a_{n-1}| - |a_n|)\cos\alpha + (|a_{n-1}| + |a_n|)\sin\alpha\big\} \\
&+ \big\{(|a_{n-2}| - |a_{n-1}|)\cos\alpha + (|a_{n-2}| + |a_{n-1}|)\sin\alpha\big\} + \cdots \\
&+ \big\{(\lambda|a_k| - |a_{k+1}|)\cos\alpha + (\lambda|a_k| + |a_{k+1}|)\sin\alpha\big\} + |\lambda - 1||a_k| \\
&+ \big\{(\lambda|a_k| - |a_{k-1}|)\cos\alpha + (\lambda|a_k| + |a_{k-1}|)\sin\alpha\big\} + \cdots \\
&+ \big\{(|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha\big\} \\
&+ \big\{(|a_1| - \mu|a_0|)\cos\alpha + (|a_1| + \mu|a_0|)\sin\alpha\big\} + |\mu - 1||a_0|\Big\} \\
= R\Big\{&|a_n|(\sin\alpha - \cos\alpha + 1) + 2\lambda|a_k|\cos\alpha + 2(\lambda - 1)|a_k|\sin\alpha \\
&+ 2\sin\alpha\sum_{j=1}^{n-1}|a_j| + |a_0| - \mu|a_0|(\cos\alpha - \sin\alpha + 1)\Big\} \\
= M_2&.
\end{aligned}
$$

Since $G(z)$ is analytic for $|z| \leq R$, $G(0) = 0$, it follows from Schwarz Lemma that for $|z| \leq R$,

$$|G(z)| \leq M_1|z|, \quad \text{for } R \geq 1$$

and

$$|G(z)| \leq M_2|z|, \quad \text{for } R \leq 1.$$

Hence, for $|z| \le R$, $R \ge 1$,

$$\begin{aligned}|F(z)| &= |G(z) + a_0| \\ &\ge |a_0| - |G(z)| \\ &\ge |a_0| - M_1|z| \\ &> 0, \quad \text{if } |z| < \frac{|a_0|}{M_1}\end{aligned}$$

and similarly for $|z| \le R$, $R \le 1$,

$$|F(z)| > 0, \quad \text{if } |z| < \frac{|a_0|}{M_2}.$$

This shows that $F(z)$ and hence $P(z)$ has all its zeros in $|z| \ge \frac{|a_0|}{M_1}$ for $R \ge 1$ and in $|z| \ge \frac{|a_0|}{M_2}$ for $R \le 1$. That completes the proof of Theorem 2.2. □

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