Nonlinear Functional Analysis and Applications
Vol. 21, No. 4 (2016), pp. 561-566

http://nfaa.kyungnam.ac.kr/journal-nfaa

ON THE ZEROS OF DERIVATIVES OF A POLYNOMIAL

Nisar Ahmad Rather[1] and Faroz Ahmad Bhat[2]

[1]Department of Mathematics
University of Kashmir, Srinagar
e-mail: dr.narather@gmail.com

[2]Department of Mathematics
University of Kashmir, Srinagar
e-mail: farozmaths@rediffmail.com

Abstract. Let $\mathcal{P}_n$ denote the set of all polynomials of the form

$$P(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j)$$

of degree n with $|z_j| \geq 1$, $1 \leq\ j \leq n-1$ and $|a| \leq 1$. In this paper, we show that $P'(z) \neq 0$ in the region $|z-a| < \frac{1-|a|}{n}$ for all $P \in \mathcal{P}_n$. Some other results for critical points of a polynomial are also obtained.

1. Introduction and preliminaries

The Gauss-Lucass Theorem states that all the critical points of a polynomial lie in the convex hull containing all the zeros of that polynomial. This is best possible in the sense that, if $P(z)$ has all its zeros in the disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$, then no proper subset of D can be guaranteed to contain even one zero of $P'(z)$. Gauss Lucass theorem has been thoroughly investigated [4] and sharpened in several ways. However there is one related question that deserves attention, namely given one specific zero a of $P(z)$, how far from a lies a zero of $P'(z)$? In this connection the following conjecture was made by Bulgarrian Mathematician B L Sendov in 1962 but became later known as Illef's conjecture (see [3, Problem 4.5]).

[0]Received November 24, 2015. Revised February 24, 2016.
[0]2010 Mathematics Subject Classification: 49J40, 47H10, 47H17.
[0]Keywords: Zeros, derivatives, polynomials.

Conjecture. Let $P(z)$ be a polynomial of degree n having all its zeros in the unit disk $|z| \leq 1$. If a is any one of these zeros, then $P'(z)$ has atleast one zero in the disk $|z-a| \leq 1$. This conjecture has been fully verified for all polynomials of degree $n \leq 8$ (see [2]). Some special cases of this conjecture have also been proved (see [1, 3, 5]). Aziz and Zarger [1] have proved the following result.

Theorem 1.1. ([1]) *If* $P(z) = z\prod_{j=1}^{n-1}(z-z_j)$ *is a polynomial of degree* n *with* $|z_j| \geq 1$, $j = 1, 2, ..., n-1$, *then* $P'(z) \neq 0$ *in* $|z| < \frac{1}{n}$.

For the proofs of our theorems we need the following result known as coincidence theorem of Walsh (see [4]).

Lemma 1.2. *Let* $G(z_1, z_2, ..., z_n)$ *be a symmetric* n *linear form of total degree* n *in* $z_1, z_2, ..., z_n$ *and let* C *be a circular region containing the* n *points* $w_1, w_2, ..., w_n$. *Then there exists at least one point* α *belonging to* C *such that* $G(z_1, z_2, ..., z_n) = G(\alpha, \alpha, ..., \alpha)$.

2. Main results

In this paper we prove the following result of which Theorem 1.1 is a special case.

Theorem 2.1. *If* $P(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j)$ *is a polynomial of degree* n *with* $|z_j| \geq 1$ *and* $|a| \leq 1$, *then* $P'(z) \neq 0$ *in the region* $|z-a| < \frac{1-|a|}{n}$.

Proof. We have $P(z) = (z-a)Q(z)$, where $Q(z) = \prod_{j=1}^{n-1}(z-z_j)$ has all its zeros in $|z| \geq 1$. This gives

$$P'(z) = Q(z) + (z-a)Q'(z).$$

If w is any zero of $P'(z)$, then

$$0 = P'(w) = Q(w) + (w-a)Q'(w). \tag{2.1}$$

This is an equation which is linear and symmetric in the zeros $z_1, z_2, ..., z_{n-1}$ of $Q(z)$. Hence an application of the Lemma 1.2 with circular region $D = \{z \in \mathbb{C} : |z| \geq 1\}$ shows that w will also satisfy the equation obtained by substituting into the equation (2.1)

$$Q(z) = (z-\beta)^{n-1},$$

where β is suitably chosen point in the circular region $D = \{z \in \mathbb{C} : |z| \geq 1\}$. That is w satisfies the equation

$$(w-\beta)^{n-1} + (n-1)(w-a)(w-\beta)^{n-2} = 0$$

or

$$(w-\beta)^{n-2}[(w-\beta) + (n-1)(w-a)] = 0.$$

Which implies $w = \beta$ or $w = \frac{\beta + a(n-1)}{n}$. If $w = \beta$, then

$$|w-a| = |\beta - a| \geq 1 - |a| \geq \frac{1-|a|}{n}.$$

If $w = \frac{\beta + a(n-1)}{n}$, then

$$\begin{aligned} |w-a| &= \left| \frac{\beta + a(n-1)}{n} - a \right| \\ &= \left| \frac{\beta - a}{n} \right| \geq \frac{1-|a|}{n}. \end{aligned}$$

Thus $P'(z)$ has all its zeros in $|z-a| \geq \frac{1-|a|}{n}$. This completes the proof of Theorem 2.1. □

Remark 2.2. For $a = 0$, Theorem 2.1 reduces to Theorem 1.1.

Next we prove the following result.

Theorem 2.3. *If* $P(z) = z^k \prod_{j=1}^{n-k} (z - z_j)$ *is a polynomial of degree* n *where* $|z_j| \geq R$, $R > 0$, $1 \leq j \leq n-k$, $1 \leq k \leq n-1$, *then* $P^{(s)}(z)$ *has* $k-s$ *fold zeros at origin and the remaining* $n-k$ *zeros lie in*

$$|z| \geq \frac{k(k-1)...(k-s+1)}{n(n-1)...(n-s+1)} R,$$

where $1 \leq s \leq k$.

Proof. Let $T(z) = \prod_{j=1}^{n-k} (z - z_j)$ where $|z_j| \geq R$, $R > 0$, $1 \leq j \leq n-k$. Then by hypothesis we have $P(z) = z^k T(z)$ so that

$$\begin{aligned} P'(z) &= kz^{k-1}T(z) + z^k T'(z) \\ &= z^{k-1}Q(z), \end{aligned} \tag{2.2}$$

where $Q(z) = kT(z) + zT'(z)$. Let w be any zero of $Q(z)$, then

$$0 = Q(w) = kT(w) + wT'(w). \tag{2.3}$$

This is an equation which is linear and symmetric in the zeros $z_1, z_2, ..., z_{n-k}$ of $T(z)$. Hence an application of the Lemma 1.2 with circular region $C = \{z \in \mathbb{C} : |z| \geq R\}$ shows that w will also satisfy the equation obtained by substituting into the equation (2.3).

$$T(z) = (z - \beta)^{n-k},$$

where β is suitably chosen point in the circular region $C = \{z \in \mathbb{C} : |z| \geq R\}$. That is w satisfies the equation

$$k(w - \beta)^{n-k} + w(n - k)(w - \beta)^{n-k-1} = 0$$

or

$$(w - \beta)^{n-k-1}(nw - k\beta) = 0.$$

This implies $w = \beta$ or $w = \frac{k}{n}\beta$. If $w = \beta$, then

$$|w| = |\beta| \geq R \geq \frac{k}{n}R.$$

If $w = \frac{k}{n}\beta$, then

$$|w| = \frac{k}{n}|\beta| \geq \frac{k}{n}R.$$

Since w is arbitrary zero of $Q(z)$, it follows that all the zeros of $Q(z)$ lie in $|z| \geq \frac{k}{n}R$. Hence it follows that $P'(z)$ has $(k-1)$ fold zeros at origin and the remaining $(n-k)$ zeros lie in the region $|z| \geq \frac{k}{n}R$.

Again from (2.2), we have $P'(z) = z^{k-1}Q(z)$. This gives

$$P''(z) = (k - 1)z^{k-2}Q(z) + z^{k-1}Q'(z)$$

or equivalently,

$$P''(z) = z^{k-2}F(z),$$

where

$$F(z) = (k - 1)Q(z) + Q'(z). \tag{2.4}$$

If now w is any zero of $F(z)$, then

$$0 = F(w) = (k - 1)Q(w) + Q'(w).$$

This is an equation which is linear and symmetric in the zeros $z_1, z_2, ..., z_{n-k}$ of $Q(z)$. Hence by Lemma 1.2 with circular region $C = \{z \in \mathbb{C} : |z| \geq \frac{k}{n}R\}$, it follows that w will also satisfy the equation obtained by substituting into the equation (2.4),

$$Q(z) = (z - \beta_1)^{n-k}$$

where β_1 is suitably chosen point in the circular region $C = \{z \in \mathbb{C} : |z| \geq \frac{k}{n}R\}$. That is w satisfies the equation

$$(k - 1)(w - \beta_1)^{n-k} + w(n - k)(w - \beta_1)^{n-k-1} = 0$$

or

$$(w-\beta_1)^{n-k-1}[w(n-1)-\beta_1(k-1)]=0.$$

This implies either $w=\beta_1$ or $w=\frac{k-1}{n-1}\beta_1$. If $w=\beta_1$, then $|w|=|\beta_1|\geq\frac{k}{n}R\geq\frac{k(k-1)}{n(n-1)}R$. If $w=\frac{k-1}{n-1}\beta_1$, then

$$|w|=\frac{k-1}{n-1}|\beta_1|\geq\frac{k(k-1)}{n(n-1)}R.$$

This shows all the zeros of $F(z)$ lie in $|z|\geq\frac{k(k-1)}{n(n-1)}R$. Thus $P''(z)$ has $k-2$ fold zeros at origin and the remaining $n-k$ zeros lie in the region $|z|\geq\frac{k(k-1)}{n(n-1)}R$.

Continuing this way, it follows that $P^{(s)}(z)$ has $k-s$ fold zero at $z=0$ and the remaining $n-k$ zeros lie in

$$|z|\geq\frac{k(k-1)...(k-s+1)}{n(n-1)...(n-s+1)}R.$$

This completes the proof of Theorem 2.3. □

Remark 2.4. If we take $k=1$ and $s=1$ in Theorem 2.3, then we get Theorem 1.1.

We also present the following results which follow by similar arguments.

Theorem 2.5. *If* $P(z)=z^k\prod_{j=1}^{n-k}(z-z_j)$ *where* $Re(z_j)\geq R$, $R>0$, $1\leq j\leq n-k$ *and* $1\leq k\leq n-1$, *then* $P^{(s)}(z)$ *has* $k-s$ *fold zeros at origin and the remaining* $n-k$ *zeros lie in the region*

$$Re(z)\geq\frac{k(k-1)...(k-s+1)}{n(n-1)...(n-s+1)}R.$$

Theorem 2.6. *If* $P(z)=z\prod_{j=1}^{n-1}(z-z_j)$ *is a polynomial of degree* n *with* $Re(z_j)\geq R$, $R>0$, *then*

(i) $P'(z)\neq 0$ *in the region* $Re(z)<\frac{R}{n}$.
(ii) $P^{(s)}(z)\neq 0$ *in the region* $Re(z)<\frac{s}{n}R$.

References

[1] A. Aziz and B.A. Zargar, *On the critical points of a polynomial*, Austral. Math. Soc., **57** (1998), 173–174.
[2] J.E. Brown and Guangping Xiang, *Proof of the Sendov Conjecture for polynomials of degree at most eight*, J. Math. Anal. Appl., **232** (1999), 272–292.
[3] W.K. Hayman, Research Problem in Function Theory, Athlone Press London, 1967.
[4] M. Marden, *Geometry of polynomials*, 2nd edition Math. Surveys, 3, Amer. Math. Soc., Providence, Rhode Island, 1996.
[5] M. Marden, *Conjectures on the critical points of a polynomial*, Amer. Math. Monthly, **90** (1983), 267–276.
[6] G. Schmeisser, *On Ilieff's conjecture*, Math. Z., **156** (1977), 165–173.