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A CONSTRUCTIVE ALGORITHM FOR A COUPLED FIXED POINT PROBLEM

Binayak S. Choudhury[1] and Subhajit Kundu[2]

[1]Department of Mathematics
Indian Institute of Engineering Science and Technology, Shibpur
P.O.-Botanic Garden, Howrah-711103, West Bengal, India
e-mail: binayak12@yahoo.co.in

[2]Department of Mathematics
Government College of Engineering and Leather Technology
Salt Lake City, LB Block, Sector-III, Kolkata-700106, West Bengal, India
e-mail: subhajit.math@gmail.com

Abstract. Coupled fixed points have come to the focus of interest in recent times especially for their potential applications. Very recently the idea of coupled fixed point iterations has been introduced for approximating coupled fixed points in linear spaces. Here a coupled Mann type iteration is defined and is applied to the problem of finding coupled fixed points of certain mappings. The discussion of the paper is in the context of Hilbert spaces. An illustration of the main result is also given.

1. Introduction

The idea of coupled fixed point was given by Guo and Lakshmikantham [15] in 1987. For any nonempty set X and a mapping $F : X \times X \to X$, $(x, y) \in X \times X$ is said to be a coupled fixed point of F, if

$$F(x, y) = x \text{ and } F(y, x) = y. \tag{1.1}$$

In more recent times the topic of coupled fixed points and related topics have come to the focus of interest, especially after Bhaskar *et. al.* [14], had established a coupled contraction mapping principle in partially ordered metric spaces in 2006. Some instances of the works following the work of Bhaskar *et. al.* are noted in [5, 6, 7, 8, 10, 12, 13, 17, 18]. One of the reasons

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for such extensive interest in this category of problems is their potentiality of applications. An application to boundary value problems appeared in the work of Bhaskar *et. al.* [14]. Several other applications were done in subsequent works like [2, 3, 16].

Our purpose in this paper is to introduce an algorithm for arriving at a coupled fixed point for certain functions. There is a vast literature in fixed point theory on fixed point iterations. These iterations are discussed often, but not always, in Banach and Hilbert spaces and use the geometric concepts associated with these spaces very extensively. A good survey of these works is given in the book of Berinde [1]. Here we introduce a coupled Mann iteration scheme. Mann iteration [19] is the earliest known iteration discussed in linear spaces except the most widely used Picard iteration. Some most recent references on Mann iteration process are [11, 20, 21]. The Mann iteration is as follows:

Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a selfmap. Then for $x_0 \in C$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0.$$

where $\{\alpha_n\} \subset (0, 1)$ satisfy certain suitable control conditions.

In a recent paper [9], the present authors have initiated the study of coupled fixed point iterations by introducing a two step coupled iteration and applied the same to approximate coupled fixed points of certain mappings in the context of Hilbert spaces.

2. Preliminaries

The following are some technical descriptions which we require in this paper. We use the following contractive inequality condition on F which we write as Condition A.

Condition A. Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $F : C \times C \to C$ be any mapping. F is said to satisfy Condition A if for all $x, y, p, q \in C$ such that

$$\begin{aligned}
&\|F(x,y) - F(p,q)\|^2 + \|F(y,x) - F(q,p)\|^2 \\
&\leq a[\, \|x-p\|^2 + \|y-q\|^2 \,] \\
&+ b[\, (\|p - F(p,q)\|^2 + \|q - F(q,p)\|^2)(1 + \|x - F(x,y)\|^2 + \|y - F(y,x)\|^2) \\
&+ (\|x - F(x,y)\|^2 + \|y - F(y,x)\|^2)(1 + \|p - F(p,q)\|^2 + \|q - F(q,p)\|^2)\,],
\end{aligned}$$

where $a, b > 0$ and $b < \frac{1}{4}$.

A similar condition in the corresponding uncoupled case has been considered in [4].

Coupled Mann iteration scheme: Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $F : C \times C \to C$ be any mapping. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C iteratively defined as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n F(x_n, y_n), \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n F(y_n, x_n), \quad n \geq 0, \end{cases} \tag{2.1}$$

$$0 < \alpha_n < 1, \quad n = 0, 1, 2, \cdots, \tag{2.2}$$

$$0 < \lim_{n\to\infty} \alpha_n = d < 1. \tag{2.3}$$

It is known that coupled fixed point problems can be viewed as problems in product spaces. In this paper we do not adopt this view point. The reason is that the way of presentation in this paper is more suitable for coupled iteration we consider here.

3. Main results

Theorem 3.1. *Let $F : C \times C \to C$ be any function defined on a closed nonempty convex subset C of a Hilbert space H such that F satisfies Condition A. Then the Coupled Mann iteration scheme constructed in* (2.1)-(2.3) *with d in* (2.3) *satisfying $\frac{7}{4(2-b)} < d < 1$, if convergent, converges to a coupled fixed point of F.*

Proof. Let $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Then using parallelogram law,

$$\begin{aligned} &\|F(x,y) - x\|^2 + \|F(y,x) - y\|^2 \\ &= \|F(x,y) - x_{n+1} + x_{n+1} - x\|^2 + \|F(y,x) - y_{n+1} + y_{n+1} - y\|^2 \\ &\leq 2\|F(x,y) - x_{n+1}\|^2 + 2\|x_{n+1} - x\|^2 + 2\|F(y,x) - y_{n+1}\|^2 \\ &\quad + 2\|y_{n+1} - y\|^2. \end{aligned} \tag{3.1}$$

Now,

$$\begin{aligned} &\|F(x,y) - x_{n+1}\|^2 \\ &= \|(1-\alpha_n)(F(x,y) - x_n) + \alpha_n(F(x,y) - F(x_n,y_n))\|^2 \\ &\leq 2(1-\alpha_n)^2\|F(x,y) - x_n\|^2 + 2\alpha_n^2\|F(x,y) - F(x_n,y_n)\|^2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\|F(y,x)-y_{n+1}\|^2\\
&=\|(1-\alpha_n)(F(y,x)-y_n)+\alpha_n(F(y,x)-F(y_n,x_n))\|^2\\
&\leq 2(1-\alpha_n)^2\|F(y,x)-y_n\|^2+2\alpha_n^2\|F(y,x)-F(y_n,x_n)\|^2. \qquad (3.3)
\end{aligned}$$

Using (3.2) and (3.3) in (3.1) we have,

$$\begin{aligned}
&\|F(x,y)-x\|^2+\|F(y,x)-y\|^2\\
&\leq 2\|x_{n+1}-x\|^2+2\|y_{n+1}-y\|^2+4(1-\alpha_n)^2\|F(x,y)-x_n\|^2\\
&\quad+4\alpha_n^2\|F(x,y)-F(x_n,y_n)\|^2+4(1-\alpha_n)^2\|F(y,x)-y_n\|^2\\
&\quad+4\alpha_n^2\|F(y,x)-F(y_n,x_n)\|^2\\
&=2\|x_{n+1}-x\|^2+2\|y_{n+1}-y\|^2\\
&\quad+4(1-\alpha_n)^2\|F(x,y)-x+x-x_n\|^2+4\alpha_n^2\|F(x,y)-F(x_n,y_n)\|^2\\
&\quad+4(1-\alpha_n)^2\|F(y,x)-y+y-y_n\|^2\\
&\quad+4\alpha_n^2\|F(y,x)-F(y_n,x_n)\|^2\\
&\leq 2\|x_{n+1}-x\|^2+2\|y_{n+1}-y\|^2\\
&\quad+4\alpha_n^2[\,\|F(x,y)-F(x_n,y_n)\|^2+\|F(y,x)-F(y_n,x_n)\|^2\,]\\
&\quad+4(1-\alpha_n)^2[\,2\|F(x,y)-x\|^2+2\|x-x_n\|^2\,]\\
&\quad+4(1-\alpha_n)^2[\,2\|F(y,x)-y\|^2+2\|y-y_n\|^2\,]\\
&\leq 2\|x_{n+1}-x\|^2+2\|y_{n+1}-y\|^2\\
&\quad+4\alpha_n^2[\,a(\|x-x_n\|^2+\|y-y_n\|^2)+b\{(\|x_n-F(x_n,y_n)\|^2\\
&\quad+\|y_n-F(y_n,x_n)\|^2)(1+\|x-F(x,y)\|^2+\|y-F(y,x)\|^2)\\
&\quad+(\|x-F(x,y)\|^2+\|y-F(y,x)\|^2)(1+\|x_n-F(x_n,y_n)\|^2\\
&\quad+\|y_n-F(y_n,x_n)\|^2)\}]+8(1-\alpha_n)^2[\,\|F(x,y)-x\|^2\\
&\quad+\|F(y,x)-y\|^2+\|x-x_n\|^2+\|y-y_n\|^2\,]\ \text{(by Condition A)}. \qquad (3.4)
\end{aligned}$$

Now, $\|x_n-F(x_n,y_n)\|^2=\frac{\|x_n-x_{n+1}\|^2}{\alpha_n^2}$ and $\|y_n-F(y_n,x_n)\|^2=\frac{\|y_n-y_{n+1}\|^2}{\alpha_n^2}$. Using the above facts in (3.4) we have,

$$\begin{aligned}
&\|F(x,y)-x\|^2+\|F(y,x)-y\|^2\\
&\leq 2\|x_{n+1}-x\|^2+2\|y_{n+1}-y\|^2+8(1-\alpha_n)^2[\,\|F(x,y)-x\|^2\\
&\quad+\|F(y,x)-y\|^2+\|x-x_n\|^2+\|y-y_n\|^2\,]\\
&\quad+4\alpha_n^2\Big[\,a(\|x-x_n\|^2+\|y-y_n\|^2)
\end{aligned}$$

$$
\begin{aligned}
&+ b\Big\{ \Big(\frac{\|x_n - x_{n+1}\|^2}{\alpha_n^2} + \frac{\|y_n - y_{n+1}\|^2}{\alpha_n^2} \Big) \\
&\qquad \times (1 + \|x - F(x,y)\|^2 + \|y - F(y,x)\|^2) \\
&+ (\|x - F(x,y)\|^2 + \|y - F(y,x)\|^2) \\
&\qquad \times \Big(1 + \frac{\|x_n - x_{n+1}\|^2}{\alpha_n^2} + \frac{\|y_n - y_{n+1}\|^2}{\alpha_n^2} \Big) \Big\} \Big]. \qquad (3.5)
\end{aligned}
$$

Making $n \to \infty$ in (3.5) we have,

$$
\begin{aligned}
&\|F(x,y) - x\|^2 + \|F(y,x) - y\|^2 \\
&\le 8(1-d)^2 [\, \|F(x,y) - x\|^2 + \|F(y,x) - y\|^2 \,] \\
&\quad + 4d^2 b [\, \|F(x,y) - x\|^2 + \|F(y,x) - y\|^2 \,] \\
&= \{8(1-d)^2 + 4d^2 b\} [\, \|F(x,y) - x\|^2 + \|F(y,x) - y\|^2 \,]. \qquad (3.6)
\end{aligned}
$$

Now, $b < \frac{1}{4}$. We observe that for $0 < d < 1$,

$$
\begin{aligned}
2(1-d)^2 + bd^2 &= 2(1 - 2d + d^2) + bd^2 \\
&= 2 - 4d + 2d^2 + bd^2 \\
&\le 2 - 4d + 2d + bd \quad (\text{since } 0 < d < 1) \\
&= 2 - 2d + bd \\
&= 2 - (2-b)d.
\end{aligned}
$$

Since $d > \frac{7}{4(2-b)}$, then from the above considerations,

$$
2(1-d)^2 + bd^2 < \frac{1}{4}. \qquad (3.7)
$$

From (3.6) and (3.7), we obtain, $\|F(x,y) - x\|^2 + \|F(y,x) - y\|^2 = 0$. Hence $\|F(x,y) - x\|^2 = 0$ and $\|F(y,x) - y\|^2 = 0$. Therefore, $F(x,y) = x$ and $F(y,x) = y$. Hence (x,y) is a coupled fixed point of F. This completes the proof. □

Illustration: Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as

$$
F(x,y) = \begin{cases} x - y, & x \ge y, \\ 0, & x < y, \end{cases}
$$

where $\mathbb{R}$ is the set of real numbers. Clearly $(0,0)$ is the only coupled fixed point of the function $F(x,y)$. F satisfies Condition A for $a > 1$ and $b > 0$. Then Theorem 3.1 is applicable to the Coupled Mann iteration with $\alpha_n = \frac{1}{\sqrt{n+2}}$, for all $n \ge 0$.

Note: The function F need not to be continuous.

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