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OPERATORS PRESERVING INEQUALITIES BETWEEN THE POLYNOMIALS

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Abstract. In this paper, by combining the operators B and $D\alpha$, we investigate the dependence of $B[D_\alpha(P(Rz) - \beta P(z))]$ on the maximum modulus of $P(z)$ on $|z| = 1$ for every real or complex numbers α and β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > 1$. Our results include not only some known polynomial inequalities as special case, but also the results recently proved by Bidkham and Mezerji as a particular case.

1. Introduction

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree at most n and $P'(z)$ is its derivatives, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{1.1}$$

and

$$\max_{|z|=R>1} |P'(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{1.2}$$

Inequality (1.1) is an immediate consequence of S. Bernstein's inequality on the derivative of a trigonometric polynomial (for reference see [6, 11]), where as inequality (1.2) is a simple deduction from the maximum modulus principle [12, p.346]. In both inequalities (1.1) and (1.2) equality holds only when $P(z)$ is a constant multiple of z^n.

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If, we restrict ourselves to a class of polynomials having no zero in $|z| < 1$, then the above inequality can be sharpened. In fact, Erdös conjectured and latter Lax [10] proved that if $P(z) \neq 0$ in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\max_{|z|=R>1} \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Turán [14] proved that, if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.5}$$

Concerning the minimum modulus of a polynomial $P(z)$ and its derivative $P'(z)$, Aziz and Dawood [2] proved that, if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\min_{|z|=1} |P'(z)| \geq n \min_{|z|=1} |P(z)|. \tag{1.6}$$

Let α be any complex number, the polynomial $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of the polynomial $P(z)$ of degree at most n with respect to α. The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Aziz [1] extended inequality (1.3) and (1.5) to the polar derivative of a polynomial and proved that if $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2}\{|\alpha z^{n-1}| + 1\} \max_{|z|=1} |P(z)| \text{ for } |z| \geq 1. \tag{1.7}$$

Rahman [11, p.538] introduced a class B_n of operators B that map $P \in P_n$ into itself. That is, the operator B carries $P \in P_n$ into

$$B[P(z)] = \lambda_o P(z) + \lambda_1 \left(\frac{nz}{2}\right) P'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!},$$

where $\lambda_o,\ \lambda_1,$ and λ_2 are real or complex numbers such that all the zeros of

$$u(z) := \lambda_o + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2,\ C(n,r) = \frac{n!}{r!(n-r)!}, \tag{1.8}$$

lie in the half plane

$$|z| \leq \left| z - \frac{n}{2} \right|.$$

Concerning this operator Shah and Liman [13] proved:

Theorem A. If $P(z) \in P_n$ and $P(z) \neq 0$ in $|z| > 1$, then for $|z| \geq 1$,

$$|B[P(z)]| \geq |B[z^n]| \min_{|z|=1} |P(z)|. \tag{1.9}$$

Theorem B. If $P(z) \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $|z| \geq 1$,

$$|B[P(z)]| \leq \frac{1}{2}\left[\{|B[z^n]|+|\lambda_o|\} \max_{|z|=1} |P(z)| - \{|B[z^n]|-|\lambda_o|\} \min_{|z|=1} |P(z)|\right]. \tag{1.10}$$

Concerning the dependence of $|P(Rz) - P(z)|$ on $|P(z)|$ Aziz and Rather [4] proved:

Theorem C. If $P(z)$ is a polynomial of degree n, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq 1$,

$$|P(Rz) - \beta P(z)| \leq |R^n - \beta||z|^n \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.11}$$

Theorem D. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq 1$,

$$|P(Rz) - \beta P(z)| \leq \left\{\frac{|R^n - \beta||z^n| + |1-\beta|}{2}\right\} \max_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.12}$$

Recently Bidkham and Mezerji [7] have generalised some of the above inequalities by combining B and D_α operators and proved the following results:

Theorem E. If $P(z)$ is a polynomial of degree at most n, having all its zeros in $|z| \leq 1$, then for every complex number α with $|\alpha| \geq 1$,

$$|B[D_\alpha P(z)]| \geq n|\alpha||B[z^{n-1}]| \min_{|z|=1} |P(z)| \quad \text{for } |z| \geq 1. \tag{1.13}$$

Theorem F. If $P(z)$ is a polynomial of degree at most n, having no zero in $|z| < 1$, then for every α with $|\alpha| \geq 1$,

$$\begin{aligned} |B[D_\alpha P(z)]| \leq \frac{n}{2}\Big\{ &\{|\alpha||B[z^{n-1}]| + |\lambda_o|\} \max_{|z|=1} |P(z)| \\ &- \{|\alpha||B[z^{n-1}]| - |\lambda_o|\} \min_{|z|=1} |P(z)|\Big\} \quad \text{for } |z| \geq 1. \end{aligned} \tag{1.14}$$

In this paper we combine the different ideas and techniques used above and consider the operator B and D_α such that the operator B carries $D_\alpha P(z)$ into

$$B[D_\alpha P(z)] = \lambda_o D_\alpha P(z) + \lambda_1 \left(\frac{mz}{2}\right) D_\alpha P'(z) + \lambda_2 \left(\frac{mz}{2}\right)^2 \frac{D_\alpha P''(z)}{2!},$$

where $0 \le m \le n-1$ and $\lambda_o,\ \lambda_1,$ and λ_2 are real or complex numbers such that all zeros of

$$u(z) := \lambda_o + C(m,1)\lambda_1 z + C(m,2)\lambda_2 z^2,\ C(m,r) = \frac{m!}{r!(m-r)!}, \tag{1.15}$$

lie in the half plane

$$|z| \le \left|z - \frac{m}{2}\right|$$

and obtain compact generalizations of some well-known polynomial inequalities. We first prove the following:

Theorem 1.1. *If $P(z)$ is a polynomial of degree n, then for every real or complex numbers $\alpha,\ \beta$ with $|\alpha| \ge 1,\ |\beta| \le 1$ and $R > 1$*

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \le |\alpha| n |R^n - \beta| |B[z^{n-1}]| \max_{|z|=1} |P(z)|, \tag{1.16}$$

for $|z| \ge 1$.

The result is sharp and equality holds in inequality (1.16) for $P(z) = az^n, a \ne 0$.

Substituting for $B[D_\alpha(P(Rz) - \beta P(z))]$, we have for $|z| \ge 1$,

$$\begin{aligned}
&\bigg|\lambda_o D_\alpha(P(Rz) - \beta P(z)) + \lambda_1\left(\frac{mz}{2}\right) D_\alpha(P(Rz) - \beta P(z))' \\
&\quad + \lambda_2\left(\frac{mz}{2}\right)^2 \frac{D_\alpha(P(Rz) - \beta P(z))''}{2!}\bigg| \\
&\le |\alpha| n |R^n - \beta| \bigg|\lambda_o z^{n-1} + \lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2} \\
&\quad + \lambda_2\left(\frac{(n-1)z}{2}\right)^2 \frac{(n-1)(n-2)z^{n-3}}{2!}\bigg| \max_{|z|=1} |P(z)|,
\end{aligned} \tag{1.17}$$

where $0 \le m \le n-1$ and $\lambda_o,\ \lambda_1$ and λ_2 are such that all the zeros of $u(z)$ defined by inequality (1.15) lie in the half plane $Re\ z \le \frac{m}{4}$.

If, we choose $\beta = 0$ and let $R \to 1$ in inequality (1.16) we get the following result:

Corollary 1.2. *If $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| \ge 1$,*

$$|B[D_\alpha P(z)]| \le |\alpha| n |B[z^{n-1}]| \max_{|z|=1} |P(z)| \ \textit{ for } |z| \ge 1.$$

The result is sharp and equality holds for the polynomial $P(z) = az^n,\ \ a \ne 0$.

Remark 1.3. If, we choose $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ inequality (1.17) will reduce to

$$|D_\alpha P(z)| \leq |\alpha| n |z^{n-1}| \max_{|z|=1} |P(z)| \text{ for } |z| \geq 1. \tag{1.18}$$

Dividing both side of inequality (1.18) by $|\alpha|$ and letting $|\alpha| \to \infty$, inequality (1.18) will reduce to inequality (1.1).

Choosing $\lambda_o = 0 = \lambda_2$ in inequality (1.17) will give the following result:

Corollary 1.4. *If $P(z)$ is a polynomial of degree n, then for every real or complex numbers α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > 1$,*

$$\left|\frac{m}{2} D_\alpha (P(Rz) - \beta P(z))'\right| \leq |\alpha| n |R^n - \beta| \left|\left(\frac{(n-1)^2}{2}\right) z^{n-2}\right| \max_{|z|=1} |P(z)|. \tag{1.19}$$

Dividing both side of inequality (1.19) by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n - 1$ and for $\beta = 0$ and $R \to 1$, inequality (1.19) will reduce to,

$$|P''(z)| \leq n(n-1) |z^{n-2}| \max_{|z|=1} |P(z)| \text{ for} |z| \geq 1. \tag{1.20}$$

The result is best possible and equality holds in inequality (1.20) for $P(z) = az^n$.

We now prove the theorem which gives the extension of [13, Lemma (2.3)] to the polar derivative.

Theorem 1.5. *If $P(z)$ is a polynomial of degree n, then for every real or complex numbers α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > 1$,*

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))]| \\ &\leq n(|\alpha||R^n - \beta||B[z^{n-1}]| + |1 - \beta||\lambda_o|) \max_{|z|=1} |P(z)|, \end{aligned} \tag{1.21}$$

for $|z| \geq 1$, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

The result is best possible and the equality holds in inequality (1.21) for $P(z) = z^n + 1$. Substituting for $B[D_\alpha(P(Rz) - \beta P(z))]$ in inequality (1.21), we have for $|z| \geq 1$,

$$\begin{aligned} &\Bigg|\lambda_o D_\alpha(P(Rz) - \beta P(z)) + \lambda_1 \left(\frac{mz}{2}\right) D_\alpha(P(Rz) - \beta P(z))' \\ &\quad + \lambda_2 \left(\frac{mz}{2}\right)^2 \frac{D_\alpha(P(Rz) - \beta P(z))''}{2!}\Bigg| + \Bigg|\lambda_o D_\alpha(Q(Rz) - \beta Q(z)) \\ &\quad + \lambda_1 \left(\frac{mz}{2}\right) D_\alpha(Q(Rz) - \beta Q(z))' + \lambda_2 \left(\frac{mz}{2}\right)^2 \frac{D_\alpha(Q(Rz) - \beta Q(z))''}{2!}\Bigg| \end{aligned}$$

$$\begin{aligned}&\leq n\bigg\{|\alpha||R^n-\beta|\bigg|\lambda_o z^{n-1}+\lambda_1\bigg(\frac{(n-1)z}{2}\bigg)(n-1)z^{n-2}\\&\quad+\lambda_2\bigg(\frac{(n-1)z}{2}\bigg)^2\frac{(n-1)(n-2)z^{n-3}}{2!}\bigg|+|1-\beta||\lambda_o|\bigg\}\max_{|z|=1}|P(z)|,\end{aligned} \tag{1.22}$$

where $0 \leq m \leq n-1$ and λ_o, λ_1 and λ_2 are such that all the zeros of $u(z)$ defined by inequality (1.15) lie in the half plane $Re\ z \leq \frac{m}{4}$.

If, we choose $\beta = 0$ and let $R \to 1$ in inequality (1.21), we get the following extension of [13, Lemma (2.3)] to polar derivatives.

Corollary 1.6. *If $P(z)$ is a polynomial of degree n, then for every real or complex numbers α with $|\alpha| \geq 1$ and for $|z| \geq 1$*

$$|B[D_\alpha P(z)]| + |B[D_\alpha Q(z)]| \leq n(|\alpha||B[z^{n-1}]| + |\lambda_o|)\max_{|z|=1}|P(z)|,$$

which implies

$$\begin{aligned}&|B[nP(z)+(\alpha-z)P'(z)]|+|B[nQ(z)+(\alpha-z)Q'(z)]|\\&\leq n(|B[\alpha z^{n-1}]|+|\lambda_o|)\max_{|z|=1}|P(z)|,\end{aligned}$$

taking $\alpha = z$ in the above inequality, we get [13, Lemma (2.3)] *that is*

$$|B[P(z)]| + |B[Q(z)]| \leq (|B[z^n]| + |\lambda_o|)\max_{|z|=1}|P(z)| \quad \textit{for } |z| \geq 1.$$

Taking $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality (1.22), we get the following result:

Corollary 1.7. *If $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| + |D_\alpha Q(z)| \leq n\{|\alpha||z^{n-1}| + 1\}\max_{|z|=1}|P(z)| \quad \textit{for } |z| \geq 1. \tag{1.23}$$

Dividing both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, inequality (1.23) will reduce to,

$$|P'(z)| + |Q'(z)| \leq n|z^{n-1}|\max_{|z|=1}|P(z)| \quad \text{for } |z| \geq 1. \tag{1.24}$$

The result is best possible and equality holds in inequality (1.24) for $P(z) = z^n + 1$. The above result is a special case of the result due to Govil and Rahman [8, Inequality (3.2)].

Taking $\lambda_o = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality (1.22), we get the following result:

Corollary 1.8. *If $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| \geq 1$,*

$$m\{|D_\alpha P'(z)|+|D_\alpha Q'(z)|\} \leq n|\alpha|(n-1)^2|z^{n-2}| \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1. \tag{1.25}$$

Dividing both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n-1$, inequality (1.25) will reduce to

$$|P''(z)| + |Q''(z)| \leq n(n-1)|z^{n-2}| \max_{|z|=1}|P(z)| \quad \text{for } |z| \geq 1. \tag{1.26}$$

The result is best possible and equality holds in inequality (1.26) for $P(z) = z^n + 1$.

Next, we prove a result for the class of polynomials not vanishing in a unit disc and obtain compact generalization of inequalities (1.7). Infact we prove:

Theorem 1.9. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every real or complex numbers α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > 1$,*

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| \\ &\leq \frac{n}{2}\{|\alpha||R^n - \beta||B[z^{n-1}]| + |1-\beta||\lambda_o|\} \max_{|z|=1}|P(z)|, \end{aligned} \tag{1.27}$$

for $|z| \geq 1$.

The result is best possible and equality in inequality (1.27) holds for $P(z) = z^n + 1$. Substituting for $B[D_\alpha(P(Rz) - \beta P(z))]$ in inequality (1.27), we have for $|z| \geq 1$,

$$\begin{aligned} &\left|\lambda_o D_\alpha(P(Rz) - \beta P(z)) + \lambda_1\left(\frac{mz}{2}\right)D_\alpha(P(Rz) - \beta P(z))'\right. \\ &\quad \left. + \lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha(P(Rz) - \beta P(z))''}{2!}\right| \\ &\leq \frac{n}{2}\left\{|\alpha||R^n - \beta|\left|\lambda_o z^{n-1} + \lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2}\right.\right. \\ &\quad \left.\left. + \lambda_2\left(\frac{(n-1)z}{2}\right)^2\frac{(n-1)(n-2)z^{n-3}}{2!}\right| + |1-\beta||\lambda_o|\right\} \max_{|z|=1}|P(z)|, \end{aligned} \tag{1.28}$$

where $0 \leq m \leq n-1$ and λ_o, λ_1 and λ_2 are such that all the zeros of $u(z)$ defined by inequality (1.15) lie in the half plane $Re\ z \leq \frac{m}{4}$.

Remark 1.10. If we take $\beta = 0$ and let $R \to 1$, inequality (1.27) will reduce to the following result due to Bidkham and Mezerji [7].

If $P(z)$ is a polynomial of degree at most n, having no zero in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$,

$$|B[D_\alpha P(z)]| \leq \frac{n}{2}\{|\alpha||B[z^{n-1}]| + |\lambda_o|\} \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1.$$

Remark 1.11. If we take $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$, inequality (1.28) reduces to inequality (1.7) that is

$$|D_\alpha P(z)| \leq \frac{n}{2}\{|\alpha z^{n-1}| + 1\} \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1.$$

On dividing both sides of above inequality by $|\alpha|$ and letting $|\alpha| \to \infty$, we get inequality (1.3).

Choosing $\lambda_o = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality (1.28), we get the following result:

Corollary 1.12. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \geq 1$*

$$|mD_\alpha P'(z)| \leq \frac{n(n-1)^2}{2}|\alpha||z^{n-2}| \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1. \tag{1.29}$$

Dividing both sides of inequality (1.29) by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n - 1$ and we have

$$|P''(z)| \leq \frac{n(n-1)}{2}|z^{n-2}| \max_{|z|=1}|P(z)| \text{ for } |z| \geq 1. \tag{1.30}$$

The result is best possible and equality in inequality (1.30) holds for $P(z) = z^n + 1$.

We now prove the following interesting result, which provides the compact generalisation of inequality (1.13).

Theorem 1.13. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for every real or complex numbers $\alpha,\ \beta$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > 1$*

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \geq |\alpha|n|R^n - \beta||B[z^{n-1}]| \min_{|z|=1}|P(z)|, \tag{1.31}$$

for $|z| \geq 1$.

The result is sharp and equality holds in inequality (1.31) for $P(z) = az^n$. Substituting for $B[D_\alpha(P(Rz) - \beta P(z))]$, we have for $|z| \geq 1$,

$$\Bigg|\lambda_o D_\alpha(P(Rz) - \beta P(z)) + \lambda_1\left(\frac{mz}{2}\right)D_\alpha(P(Rz) - \beta P(z))'$$

$$+\lambda_2\left(\frac{mz}{2}\right)^2\frac{D_\alpha(P(Rz)-\beta P(z))''}{2!}\Bigg|$$

$$\geq |\alpha|n|R^n-\beta|\Bigg|\lambda_o z^{n-1}+\lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2} \tag{1.32}$$

$$+\lambda_2\left(\frac{(n-1)z}{2}\right)^2\frac{(n-1)(n-2)z^{n-3}}{2!}\Bigg|\min_{|z|=1}|P(z)|,$$

where $0 \leq m \leq n-1$ and $\lambda_o,\ \lambda_1$ and λ_2 are such that all the zeros of $u(z)$ defined by (1.15) lie in the half plane $Re\ z \leq \frac{m}{4}$.

Remark 1.14. If we take $\beta = 0$ and let $R \to 1$, inequality (1.31) will reduce to inequality (1.13).

Taking $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality (1.32), we will get the following result from which result of Aziz and Dawood [2] follows as a special case.

Corollary 1.15. *If $P(z)$ is a polynomial of degree at most n having all its zeros in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| \geq n|\alpha||z^{n-1}|\min_{|z|=1}|P(z)| \quad \textit{for } |z| \geq 1. \tag{1.33}$$

The result is best possible and equality holds in inequality (1.33) for $P(z) = az^n$. Dividing the inequality (1.33) both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n-1$, we obtain the inequality (1.5) as a special case.

Choosing $\lambda_o = 0 = \lambda_2$ with $\beta = 0$ and letting $R \to 1$ in inequality (1.32), we get the following result:

Corollary 1.16. *If $P(z)$ is a polynomial of degree at most n, having all its zeros in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \geq 1$,*

$$|mD_\alpha P'(z)| \geq n(n-1)^2|\alpha||z^{n-2}|\min_{|z|=1}|P(z)|. \tag{1.34}$$

Dividing both sides of the inequality (1.34) by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n-1$ we obtain

$$|P''(z)| \geq n(n-1)|z^{n-2}|\min_{|z|=1}|P(z)|. \tag{1.35}$$

The result is best possible and the equality holds in inequality (1.35) for $P(z) = az^n$.

As an improvement of inequality (1.31) and generalisation of inequality (1.10), we prove the following result:

Theorem 1.17. *If $P(z)$ is a polynomial of degree at most n which does not vanish in $|z| < 1$, then for every real or complex numbers α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R > 1$*

$$
\begin{aligned}
&|B[D_\alpha(P(Rz) - \beta P(z))]| \\
&\leq \frac{n}{2}\bigg[\left\{|\alpha||R^n - \beta||B[z^{n-1}]| + |\lambda_o||1-\beta|\right\} \max_{|z|=1}|P(z)| \\
&\quad - \left\{|\alpha||R^n - \beta||B[z^{n-1}]| - |\lambda_o||1-\beta|\right\} \min_{|z|=1}|P(z)|\bigg] \quad \text{for } |z| \geq 1.
\end{aligned} \tag{1.36}
$$

The result is sharp and equality in inequality (1.36) holds for the polynomial having all the zeros on the unit disk. Substituting for $B[D_\alpha(P(Rz) - P(z))]$ in inequality (1.36), we have for $|z| \geq 1$,

$$
\begin{aligned}
&\bigg|\lambda_o D_\alpha(P(Rz) - \beta P(z)) + \lambda_1\left(\frac{mz}{2}\right) D_\alpha(P(Rz) - \beta P(z))' \\
&\quad + \lambda_2\left(\frac{mz}{2}\right)^2 \frac{D_\alpha(P(Rz) - \beta P(z))''}{2!}\bigg| \\
&\leq \frac{n}{2}\Bigg[\Bigg\{|\alpha||R^n - \beta|\bigg|\lambda_o z^{n-1} + \lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2} \\
&\quad + \lambda_2\left(\frac{(n-1)z}{2}\right)^2 \frac{(n-1)(n-2)z^{n-3}}{2!}\bigg| + |1-\beta||\lambda_o|\Bigg\} \max_{|z|=1}|P(z)| \\
&\quad - \Bigg\{|\alpha||R^n - \beta|\bigg|\lambda_o z^{n-1} + \lambda_1\left(\frac{(n-1)z}{2}\right)(n-1)z^{n-2} \\
&\quad + \lambda_2\left(\frac{(n-1)z}{2}\right)^2 \frac{(n-1)(n-2)z^{n-3}}{2!}\bigg| - |1-\beta||\lambda_o|\Bigg\} \min_{|z|=1}|P(z)|\Bigg],
\end{aligned} \tag{1.37}
$$

where $0 \leq m \leq n-1$ and λ_o, λ_1 and λ_2 are such that all the zeros of $u(z)$ defined by (1.15) lie in the half place $Re\ z \leq \frac{m}{4}$.

Remark 1.18. If we take $\beta = 0$ and letting $R \to 1$, inequality (1.36) will reduce to inequality (1.14).

Remark 1.19. Taking $\lambda_1 = 0 = \lambda_2$ with $\beta = 0$ and let $R \to 1$, inequality (1.37) will reduce to the following result due to Aziz and Shah [5].

If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$,

$$
\max_{|z|=1}|D_\alpha P(z)| \leq \frac{n}{2}\left\{(|\alpha|+1)\max_{|z|=1}|P(z)| - (|\alpha|-1)\min_{|z|=1}|P(z)|\right\}.
$$

Dividing both sides by $|\alpha|$ and letting $|\alpha| \to \infty$, in the above inequality, it follows that if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2}\left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}.$$

The above result is an interesting refinement of Erdös-Lax theorem (inequality (1.3)) and was proved by Aziz and Dawood [2].

If we take $\lambda_o = 0 = \lambda_2$ with $\beta = 0$ and let $R \to 1$ in (1.37), we get the following result:

Corollary 1.20. *If $P(z)$ is a polynomial of degree at most n, having no zero in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$ and $|z| \geq 1$,*

$$|mD_\alpha P(z)| \leq \frac{n(n-1)^2}{2} |\alpha||z^{n-2}| \left\{ \max_{|z|=1} p(z)| - \min_{|z|=1} |p(z)| \right\}. \tag{1.38}$$

The result is best possible and equality holds in inequality (1.38) for $P(z) = z^n + 1$. Dividing both sides of the inequality (1.38) by $|\alpha|$ and letting $|\alpha| \to \infty$, then $m = n-1$ and we get

$$|P''(z)| \leq \frac{n(n-1)}{2} |z^{n-2}| \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}. \tag{1.39}$$

2. Lemmas

For the proof of above theorems we need the following lemmas. The first lemma follows from [9].

Lemma 2.1. *If all the zeros of polynomial $P(z)$ of degree n lie in $|z| \leq k$, where $k \leq 1$, then for $|\alpha| \geq k$, the polar derivative $D_\alpha[P(z)]$ of $P(z)$ at the point α also has all its zeros in $|z| \leq k$.*

The following lemma which we need is in fact implicit in [11, Lemma 14.5.7, p.540].

Lemma 2.2. *If all the zeros of the polynomial $P(z)$ of degree n lie in a circle $|z| \leq 1$, then all the zeros of the polynomial $B[P(z)]$ also lie in $|z| \leq 1$.*

As an application of Lemmas 2.1 and 2.2 we have the following lemma.

Lemma 2.3. *If all the zeros of polynomial $P(z)$ of degree n lie in $|z| \leq 1$, then for $|\alpha| \geq 1$, all the zeros of the polynomial $B[D_\alpha P(z)]$ also lie in $|z| \leq 1$.*

Proof. From Lemma 2.1 for $k = 1$, all the zeros of the polynomial $D_\alpha P(z)$ lie in $|z| \leq 1$ and so from Lemma 2.2 the polynomial $B[D_\alpha P(z)]$ has all its zeros in $|z| \leq 1$. □

The next lemma is due to Aziz and Rather [3].

Lemma 2.4. *If $P(z)$ is a polynomial of degree at most n having all its zeros in $|z| < k$, where $k \leq 1$, then $|P(Rz)| > |P(z)|$, for $|z| \geq 1$ and $R > 1$.*

Lemma 2.5. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every real or complex numbers α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $R \geq 1$.*

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \leq |B[D_\alpha(Q(Rz) - \beta Q(z))]|, \tag{2.1}$$

for $|z| \geq 1$, where $Q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

Proof. For $R = 1$, the result reduces to Bidkham and Mezerji [7, Lemma 4, p.597]. Now we will prove the result for $R > 1$. Since all the zeros of $P(z)$ lie in $|z| \geq 1$ and for every real or complex number λ with $|\lambda| > 1$, the polynomial $G(z) = P(z) - \lambda Q(z)$, where $Q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$ has all its zeros in $|z| \leq 1$. Applying lemma 4 to the polynomial $G(z)$ with $k = 1$, we get

$$|G(z)| < |G(Rz)| \text{ for } |z| = 1 \text{ and } R > 1.$$

Since all the zeros $G(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, therefore for any real or complex number β with $|\beta| \leq 1$, the polynomial $H(z) = G(Rz) - \beta G(z)$, has all its zeros in $|z| < 1$, for every λ with $|\lambda| > 1$ and $R > 1$, by Lemma 2.3 all the zeros of $B[D_\alpha H(z)]$ lie in $|z| < 1$. This implies

$$\begin{aligned} &B[D_\alpha(G(Rz) - \beta G(z))] \\ &= B[D_\alpha(P(Rz) - \beta P(z))] - \lambda B[D\alpha(Q(Rz) - \beta Q(z))], \end{aligned} \tag{2.2}$$

for $|z| \geq 1$ and $R > 1$. Inequality (2.2) implies

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \leq |B[D\alpha(Q(Rz) - \beta Q(z))]|, \tag{2.3}$$

for $|z| \geq 1$ and $R > 1$. For if it is not true, then there is a point $z = z_o$ with $|z_o| \geq 1$, such that

$$|B[D_\alpha(P(Rz_o) - \beta P(z_0))]| \geq |B[D\alpha(Q(Rz_o) - \beta Q(z_o))]|, \tag{2.4}$$

for $|z| \geq 1$ and $R > 1$. Since all the zeros of $Q(z)$ lie in $|z| \leq 1$, therefore it follows that all the zeros of $Q(Rz) - \beta Q(z)$, lie in $|z| \leq 1$ for every β with $|\beta| \leq 1$. Hence $Q(Rz_o) - \beta Q(z_o) \neq 0$, for $|z_o| \geq 1$. Which implies

$$B[D\alpha(Q(Rz_o) - \beta Q(z_o))] \neq 0 \text{ for } |z| \geq 1 \text{ and } R > 1.$$

We take

$$\lambda = \frac{B[D_\alpha(P(Rz_o) - P(z_o))]}{B[D\alpha(Q(Rz_o) - Q(z_o))]},$$

so that $|\lambda| > 1$. Which shows that $B[D_\alpha H(z)]$ has a zero in $|z| \geq 1$. Which is contradiction to the fact that all the zeros of $B[D_\alpha H(z)]$ lie in $|z| < 1$. Thus

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \leq |B[D\alpha(Q(Rz) - \beta Q(z))]|,$$

for $|z| \geq 1$ and $R \geq 1$. □

3. Proof of theorems

Proof of Theorem 1.1. Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. Therefore, by Rouche's Theorem we have all the zeros of the polynomial $G(z) = P(z)+\lambda z^n M$, lie in $|z| < 1$ for every λ with $|\lambda| > 1$. Now from Lemma 2.4, we have

$$|G(z)| < |G(Rz)| \quad \text{for } |z| = 1 \quad \text{and} \quad R > 1.$$

Since all the zeros of $G(Rz)$ lie in $|z| < \frac{1}{R} < 1$, therefore if β is any real or complex number with $|\beta| \leq 1$, we have all the zeros of the polynomial

$$G(Rz) - \beta G(z) = (P(Rz) - \beta P(z)) + \lambda(R^n - \beta)z^n M,$$

also lie in $|z| < 1$ for every $R > 1$ and $|\lambda| > 1$. Therefore by Lemma 2.3, all the zeros of $B[D_\alpha(G(Rz) - \beta G(z))]$, lie in $|z| < 1$ for every $R > 1$ and $|\lambda| > 1$. Which implies

$$\begin{aligned} &B[D_\alpha(G(Rz) - \beta G(z))] \\ &= B[D_\alpha(P(Rz) - \beta P(z))] + \lambda\alpha n(R^n - \beta)MB[z^{n-1}], \end{aligned} \tag{3.1}$$

for $|z| < 1$ and $R > 1$. Inequality (3.1) implies

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| \\ &\leq |\alpha|n|R^n - \beta||B[z^{n-1}]|M \quad \text{for} |z| \geq 1 \quad \text{and} \quad R > 1, \end{aligned} \tag{3.2}$$

for if this is not true, then there is a point $z = z_o$ with $|z_o| \geq 1$ such that

$$|B[D_\alpha(P(Rz_o) - \beta P(z_o))]| > |\alpha|n|R^n - \beta||B[z_o^{n-1}|M.$$

We take

$$\lambda = -\frac{B[D_\alpha(P(Rz_o) - \beta P(z_o))]}{\alpha n(R^n - \beta)B[z^{n-1}]},$$

so that $|\lambda| > 1$, for this choice of $|\lambda|$, we have $B[D_\alpha(G(Rz_o) - \beta G(z_o))] = 0$ for $|z_o| \geq 1$. Which is a contradiction to the fact that all the zeros of $B[D_\alpha(G(Rz) - \beta G(z))]$ lie in $|z| < 1$. Thus

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \leq \alpha n|R^n - \beta||B[z^{n-1}| \max_{|z|=1} |P(z)|,$$

for $|z| \geq 1$ and $R > 1$. □

Proof of Theorem 1.5. Let $M = \max_{|z|=1} |P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. Now for every real or complex number γ with $|\gamma| > 1$, it follows from Rouche's Theorem, the polynomial $G(z) = P(z) + \gamma M$ does not vanish in $|z| < 1$. Now applying Lemma 2.4 and 2.5 to the polynomial $G(z)$, we have for every real or complex number β with $|\beta| \leq 1$,

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z) + \gamma(1-\beta)M)]| \\ &\leq |B[D_\alpha(Q(Rz) - \beta Q(z) + \overline{\gamma}(R^n - \beta)z^n M)]|, \end{aligned} \tag{3.3}$$

for $|z| \geq 1$ and $R > 1$, where $Q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$. Inequality (3.3) implies

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))] + n\gamma(1-\beta)M\lambda_o| \\ &\leq |B[D_\alpha(Q(Rz) - \beta Q(z))] + \alpha n\overline{\gamma}(R^n - \beta)B[z^{n-1}]M|, \end{aligned} \tag{3.4}$$

for $|z| \geq 1$ and $R > 1$.

Now choosing the argument of $\overline{\gamma}$ on the R.H.S of inequality (3.4), such that

$$\begin{aligned} &|B[D_\alpha(Q(Rz) - \beta Q(z))] + \alpha n\overline{\gamma}(R^n - \beta)B[z^{n-1}]M| \\ &= |\alpha|n|\gamma||R^n - \beta||B[z^{n-1}]|M - |B[D_\alpha(Q(Rz) - \beta Q(z))]|, \end{aligned} \tag{3.5}$$

for $|z| \geq 1$ and $R > 1$. Therefore we get from inequality (3.4),

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| - |n\gamma(1-\beta)\lambda_o M| \\ &\leq |\alpha|n|\gamma||R^n - \beta||B[z^{n-1}]|M - |B[D_\alpha(Q(Rz) - \beta Q(z))]|, \end{aligned} \tag{3.6}$$

for $|z| \geq 1$ and $R > 1$. Therefore, inequality (3.6) implies

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))]| \\ &\leq |\alpha|n|\gamma||R^n - \beta||B[z^{n-1}]|M + n|\gamma||1-\beta||\lambda_o|M, \end{aligned} \tag{3.7}$$

for $|z| \geq 1$ and $R > 1$. Letting $|\gamma| \to 1$, in inequality (3.7), we get

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))]| \\ &\leq n(|\alpha||R^n - \beta||B[z^{n-1}]| + |1-\beta||\lambda_o|) \max_{|z|=1} |P(z)|, \end{aligned}$$

for $|z| \geq 1$ and $R > 1$. Which proves the theorem. □

Proof of Theorem 1.9. We have from Lemma 2.5,

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \leq |B[D_\alpha(Q(Rz) - \beta Q(z))]|,$$

for $|z| \geq 1$ and $R \geq 1$, where $Q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$. Also from Theorem 1.5, we have

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))] \\ &\leq n(|\alpha||R^n - \beta||B[z^{n-1}]| + |1-\beta||\lambda_o|) \max_{|z|=1} |P(z)|, \end{aligned}$$

for $|z| \geq 1$ and $R > 1$, where $Q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$. Combining the above two inequalities, we get

$$\begin{aligned}&|B[D_\alpha(P(Rz) - \beta P(z))]|\\&\leq \frac{n}{2}\{|\alpha||R^n - \beta||B[z^{n-1}]| + |1-\beta||\lambda_o|\} \max_{|z|=1}|P(z)|,\end{aligned}$$

for $|z| \geq 1$ and $R > 1$. Which proves the theorem. □

Proof of Theorem 1.13. If $P(z)$ has a zero on $|z| = 1$, then the result is trivial. So we suppose that $P(z)$ has all its zeros in $|z| < 1$. If $m = \min_{|z|=1}|P(z)|$, then $m > 0$ and $m \leq |P(z)|$ for $|z| = 1$. Therefore, if γ is any complex number with $|\gamma| < 1$, we have the polynomial $G(z) = P(z) - \gamma m z^n$ of degree n has all its zeros in $|z| < 1$. Now from Lemma 2.4, we have

$$|G(z)| < |G(Rz)| \text{ for } |z| = 1 \text{ and } R > 1.$$

Since all the zeros of $G(Rz)$ lie in $|z| < \frac{1}{R} < 1$, therefore for any real or complex number β with $|\beta| \leq 1$ and $R > 1$, it follows from Rouche's Theorem, the polynomial $H(z) = G(Rz) - \beta G(z)$ has all its zeros in $|z| < 1$. Therefore from Lemma 2.3, all the zeros of $B[D_\alpha H(z)]$ lie in $|z| < 1$. This implies

$$\begin{aligned}&B[D_\alpha(G(Rz) - \beta G(z))]\\&= B[D_\alpha(P(Rz) - \beta P(z))] - \alpha n\gamma(R^n - \beta)B[z^{n-1}]m,\end{aligned} \tag{3.8}$$

for $|z| \geq 1$ and $R > 1$. Inequality (3.8) implies for $|z| \geq 1$ and $R > 1$,

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \geq |\alpha|n|R^n - \beta||B[z^{n-1}]|m. \tag{3.9}$$

If inequality (3.9) is not true, then there is a point $z = z_o$ with $|z_o| \geq 1$ such that

$$|B[D_\alpha(P(Rz_o) - \beta P(z_o))]| < |\alpha|n|R^n - \beta||B[z^{n-1}|m.$$

We take

$$\gamma = \frac{B[D_\alpha(P(Rz_o) - \beta P(z_o))]}{\alpha n(R^n - \beta)B[z_o^{n-1}]m},$$

so that $|\gamma| < 1$. For this choice of $|\gamma|$, we have $B[D_\alpha H(z)] = 0$, for $|z| \geq 1$. Which is a contradiction to the fact that all the zeros of $B[D_\alpha H(z)]$ lie in $|z| < 1$. Thus we have

$$|B[D_\alpha(P(Rz) - \beta P(z))]| \geq |\alpha|n|R^n - \beta||B[z^{n-1}]| \min_{|z|=1}|P(z)|.$$

Hence the theorem follows. □

Proof of Theorem 1.17. Since the polynomial $P(z)$ does not vanish in $|z| < 1$, therefore if $m = \min_{|z|=1}|P(z)|$, then $m \leq |P(z)|$ for $|z| \leq 1$. Now for any real or complex number λ with $|\lambda| \leq 1$, the polynomial $G(z) = P(z) + \lambda m z^n$ does not vanish in $|z| < 1$. For if this is not true, then there is a point $z = z_o$, with $|z_o| < 1$, such that $G(z_o) = P(z_o) + \lambda m z_o^n = 0$. Which implies $|P(z_o)| = |m\lambda z_o^n| \leq m|z_o|^n < m$, contradicting the fact that $m \leq |P(z)|$ for $|z| \leq 1$. Thus $G(z)$ has no zero in $|z| < 1$ for every λ with $|\lambda| \leq 1$. Applying Lemma 2.5 to the polynomial $G(z)$, we have for $|\beta| \leq 1$ and $R > 1$,

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z) + (R^n - \beta)\lambda m z^n)]| \\ &\leq |B[D_\alpha(Q(Rz) - \beta Q(z) + (1 - \beta)\overline{\lambda} m)]|, \end{aligned} \tag{3.10}$$

for $|z| \geq 1$ and $R > 1$, where $Q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$. Inequality (3.10) implies

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))] + \alpha(R^n - \beta)\lambda m n B[z^{n-1}]| \\ &\leq |B[D_\alpha(Q(Rz) - \beta Q(z))] + n\lambda_o(1 - \beta)\overline{\lambda} m|, \end{aligned} \tag{3.11}$$

for $|z| \geq 1$ and $R > 1$. Choosing λ in inequality (3.11) such that

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))] + \alpha n \lambda(R^n - \beta) B[z^{n-1}] m| \\ &= |B[D_\alpha(P(Rz) - \beta P(z))]| + |\alpha| n |\lambda| |R^n - \beta| |B[z^{n-1}]| m, \end{aligned} \tag{3.12}$$

for $|z| \geq 1$ and $R > 1$. Inequality (3.12) implies

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| + |\alpha| n |\lambda| |R^n - \beta| |B[z^{n-1}]| m \\ &\leq |B[D_\alpha(Q(Rz) - \beta Q(z))]| + n|\lambda_o| |1 - \beta| |\lambda| m, \end{aligned} \tag{3.13}$$

for $|z| \geq 1$ and $R > 1$. Inequality (3.13) implies

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| \\ &\leq |B[D_\alpha(Q(Rz) - \beta Q(z))]| \\ &\quad + n||\lambda_o|1 - \beta||\lambda| m - |\alpha| n |\lambda| |R^n - \beta| |B[z^{n-1}]| m, \end{aligned} \tag{3.14}$$

for $|z| \geq 1$ and $R > 1$. Letting $|\lambda| \to 1$, we have for $|z| \geq 1$ and $R > 1$

$$\begin{aligned} &2|B[D_\alpha(P(Rz) - \beta P(z))]| \\ &\leq |B[D_\alpha(P(Rz) - \beta P(z))]| + |B[D_\alpha(Q(Rz) - \beta Q(z))]| \\ &\quad + n|\lambda_o| |1 - \beta| m - |\alpha| n |R^n - \beta| |B[z^{n-1}]| m, \end{aligned} \tag{3.15}$$

for $|z| \geq 1$ and $R > 1$. Applying Theorem 1.5, we get from inequality (3.15)

$$\begin{aligned} &|B[D_\alpha(P(Rz) - \beta P(z))]| \\ &\leq \frac{n}{2}\Bigg[\left\{ |\alpha| |R^n - \beta| |B[z^{n-1}]| + |\lambda_o| |1 - \beta| \right\} \max_{|z|=1} |P(z)| \\ &\quad - \left\{ |\alpha| |R^n - \beta| |B[z^{n-1}]| - |\lambda_o| |1 - \beta| \right\} \min_{|z|=1} |P(z)| \Bigg] \quad \text{for } |z| \geq 1. \end{aligned} \tag{1.36}$$

Hence the Theorem follows. □

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