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EXPANDING THE APPLICABILITY OF THE SHADOWING LEMMA FOR OPERATORS WITH CHAOTIC BEHAVIOR USING RESTRICTED CONVERGENCE DOMAINS

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Abstract. We present a weaker version of the celebrated Newton–Kantorovich theorem based on our new restricted convergence domains to find solutions of discrete dynamical systems involving operators with chaotic behavior. Our results extend the application of the shadowing lemma and are given under the same computational cost as in earlier studies.

1. Introduction

It is very difficult to prove mathematically, in general, that a given system behaves chaotically [4]–[6]. However, complicated behavior of dynamical systems can easily be detected via numerical experiments [2], [4]–[6], and the references therein. The shadowing lemma [4, p.1684] proved via the Newton–Kantorovich theorem [3] was used in [4] to present a computer-assisted method that allows us to prove that a discrete dynamical system admits the shift operator as a subsystem. Motivated by this work and using a weaker version of the Newton–Kantorovich theorem [1, 2] (see Theorem 2.1 that follows) we show that it is possible to weaken the shadowing Lemma on which the work in [4]–[6] is based. Using restricted convergence domains (under the same

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computational cost) we obtained a larger upper bound on the crucial norm of operator M^{-1} (see (2.7)). Moreover, the information on location of the shadowing orbit is more precise. Other advantages have already been reported in [1, 2]. Clearly this approach widens the applicability of the shadowing lemma.

2. The shadowing lemma

Let $U(v,\rho)$, $\bar{U}(v,\rho)$ stand respectively for the open and closed balls in $\mathbf{R}^k$ with center $v \in \mathbf{R}^k$ and of radius $\rho > 0$. We need the definitions: Let $D \subseteq \mathbf{R}^k$ be an open subset of $\mathbf{R}^k$ (k a natural number), and let $f : D \to D$ be an injective operator. Then the pair (D, f) is a discrete dynamical system. Denote by $S = l^\infty\left(\mathbf{Z}, \mathbf{R}^k\right)$ the space of $\mathbf{R}^k$ valued bounded sequences $x = \{x_n\}$ with norm $\|x\| = \sup_{n\in\mathbf{Z}} |x_n|_2$. Here we use the Euclidean norm in $\mathbf{R}^k$ and denote it by $|\cdot|$, omitting the index 2. A δ_0–pseudo–orbit is a sequence $y = \{y_n\} \in D^{\mathbf{Z}}$ with $|y_{n+1} - f(y_n)| \leq \delta_0$ $(n \in \mathbf{Z})$. A r-shadowing orbit $x = \{x_n\}$ of a δ_0–pseudo–orbit y is an orbit of (D, f) with $|y_n - x_n| \leq 2$ $(n \in \mathbf{Z})$.

We need the following semilocal convergence theorem for Newton method [2].

Theorem 2.1. *Let X, Y be Banach spaces and D be an open convex subset of X. Let $F : D \subseteq X \to Y$ be a Fréchet differentiable operator. Suppose that there exist $x_0 \in D$, positive constant η, β, L_0 and L such that:*

$$F'(x_0)^{-1} \in L(Y, X), \quad \left\|F'(x_0)^{-1}\right\| \leq \beta, \tag{2.1}$$

$$\left\|F'(x_0)^{-1} F(x_0)\right\| \leq \eta, \tag{2.2}$$

$$\left\|F'(x) - F'(x_0)\right\| \leq L_0 \|x - x_0\| \quad \textit{for all } x \in D, \tag{2.3}$$

$$\left\|F'(x) - F'(y)\right\| \leq L \|x - y\| \quad \textit{for all } x, y \in D_0 = D \cap U\left(x_0, \frac{1}{L_0}\right), \tag{2.4}$$

$$h_A = \beta\ L_1\ \eta \leq 1, \tag{2.5}$$

and

$$\bar{U}(x_0, s^*) \subseteq D,$$

where,

$$s^* = \lim_{n\to\infty} s_n,$$

$$s_0 = 0, s_1 = \eta, s_2 = s_1 + \frac{L_0 (s_1 - s_0)^2}{2(1 - L_0 s_1)} s_{n+2} = s_{n+1} + \frac{L(s_{n+1} - s_n)}{2(1 - L_0 s_{n+1})} \quad (n \geq 1),$$

$$L_1 = \frac{1}{4}\ (4L_0 + \sqrt{L_0^2 + 8\ L_0\ L} + \sqrt{L_0\ L}).$$

Then, sequence $\{y_n\}$ $(n \geq 0)$ *generated by Newton's method*

$$y_{n+1} = y_n - F'(y_n)^{-1} F(y_n), \quad n \geq 0$$

is well defined, remains in $\bar{U}(x_0, s^*)$ *for all* $n \geq 0$ *and converges to a unique solution* $y^* \in \bar{U}(x_0, s^*)$, *so that estimates*

$$\|y_{n+1} - y_n\| \leq s_{n+1} - s_n$$

and

$$\|y_n - y^*\| \leq s^* - s_n \leq 2\eta - s_n$$

hold for all $n \geq 0$.

Moreover y^* *is the unique solution of equation* $F(y) = 0$ *in* $U(x_0, R)$ *provided that*

$$L_0 (s^* + R) \leq 2$$

and

$$U(x_0, R) \subseteq D.$$

The advantages of Theorem 2.1 over the Newton-Kantorovich theorem [3] have been explained in detail in [1] and [2].

From now on we set $X = Y = \mathbf{R}^k$.

Sufficient conditions for a δ_0-pseudo-orbit y to admit a unique r-shadowing orbit are given in the following main result.

Theorem 2.2. (Weak version of the shadowing lemma) *Let* $D \subseteq \mathbf{R}^k$ *be open,* $f \in C^{1,Lip}(D, D)$ *be injective,* $y = \{y_n\} \in D^{\mathbf{Z}}$ *be a given sequence,* $\{A_n\}$ *be a bounded sequence of* $k \times k$ *matrices and let* $\delta_0, \delta, \ell_0, \ell$ *be positive constants. Suppose that for the operator*

$$M : S \to S \quad \text{with} \quad \{M\ z\}_n = z_{n+1} - Az_n \tag{2.6}$$

is invertible and

$$\left\|M^{-1}\right\| \leq a = \frac{1}{\delta + \sqrt{\ell_1^*\ \delta_0}}, \tag{2.7}$$

where

$$\ell_1^* = \frac{1}{4}\ (4\ell_0 + \sqrt{\ell_0^2 + 8\ \ell_0\ \ell} + \sqrt{\ell_0\ \ell}).$$

Suppose that (2.5) *is satisfied for*

$$\ell_0 = L_0, \quad \beta = \left(\frac{1}{\|M^{-1}\|} - \delta\right)^{-1} \quad \text{and} \quad \|F'(0)^{-1}F(0)\| \leq \eta.$$

Then, the numbers t^*, R *given by*

$$t^* = \lim_{n \to \infty} t_n \tag{2.8}$$

and

$$R = \frac{2}{\ell_0} - t^* \tag{2.9}$$

satisfy $0 < t^* \leq R$, *where sequence* $\{t_n\}$ *is given by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_2 = t_1 + \frac{\ell \left(t_1 - t_0\right)^2}{2 \left(1 - \ell_0 t_1\right)},$$

$$t_{n+2} = t_{n+1} + \frac{\ell \left(t_{n+1} - t_n\right)^2}{2 \left(1 - \ell_0 t_{n+1}\right),} \quad n \geq 1 \tag{2.10}$$

and

$$\eta = \frac{\delta_0}{\frac{1}{\|M^{-1}\|} - \delta}. \tag{2.11}$$

Let $r \in [t^*, R]$. *Suppose that*

$$\overline{\bigcup_{n \in \mathbf{Z}} U \left(y_n, r\right)} \subseteq D \tag{2.12}$$

and for every $n \in \mathbf{Z}$

$$\left|y_{n+1} - f\left(y_n\right)\right| \leq \delta_0, \tag{2.13}$$

$$\left|A_n - Df\left(y_n\right)\right| \leq \delta, \tag{2.14}$$

$$\left|F'\left(u\right) - F'\left(0\right)\right| \leq \ell_0 \left|u\right| \tag{2.15}$$

and

$$\left|F'\left(u\right) - F'\left(v\right)\right| \leq \ell \left|u - v\right|, \tag{2.16}$$

for all $u, v \in U\left(y_n, r\right)$. *Then there is a unique* t^**-shadowing orbit* $x^* = \{x_n\}$ *of* y. *Moreover, there is no orbit* $\bar{x}$ *other than* x^* *such that*

$$\|\bar{x} - y\| \leq r. \tag{2.17}$$

Proof. We shall solve the difference equation

$$x_{n+1} = f\left(x_n\right), \quad n \geq 0 \tag{2.18}$$

provided that x_n is close to y_n. Setting

$$x_n = y_n + z_n \tag{2.19}$$

and

$$g_n\left(z_n\right) = f\left(z_n + y_n\right) - A_n z_n - y_{n+1} \tag{2.20}$$

we can have

$$z_{n+1} = A_n z_n + g_n\left(z_n\right). \tag{2.21}$$

Define $D_0 = \{z = \{z_n\} : \|z\| \leq 2\}$ and nonlinear operator $G : D_0 \to S$, by

$$\left(G\left(z\right)\right)_n = g_n\left(z_n\right). \tag{2.22}$$

Operator G can naturally be extended to a neighborhood of D_0. Equation (2.21) can be rewritten as

$$F\left(x\right) = M\ x - G\left(x\right) = 0, \tag{2.23}$$

where F is an operator from D_0 into S.

We will show the existence and uniqueness of a solution $x^* = \{x_n\}$ $(n \geq 0)$ of equation (2.23) with $\|x^*\| \leq r$ using Theorem 2.1. Clearly we need to express η, L_0, L and β in terms of $\left\|M^{-1}\right\|, \delta_0, \delta, \ell_0$ and ℓ.

(i) $\left\|F'(0)^{-1} F(0)\right\| \leq \eta$.

Using (2.13), (2.14) and (2.20) we get $\|F(0)\| \leq \delta_0$ and $\|G'(0)\| \leq \delta$, since $[G'(0)(w)]_n = (F'(y_n) - A_n) w_n$.

By (2.7) and the Banach lemma on invertible operators [3] we get $F'(0)^{-1}$ exists and

$$\left\|F'(0)^{-1}\right\| \leq \left(\frac{1}{\|M^{-1}\|} - \delta\right)^{-1}. \tag{2.24}$$

That is, η can be given by (2.11).

(ii) $\left\|F'(0)^{-1}\right\| \leq \beta$.

By (2.24) we can set

$$\beta = \left(\frac{1}{\|M^{-1}\|} - \delta\right)^{-1}. \tag{2.25}$$

(iii) $\|F'(u) - F'(v)\| \leq L\|u - v\|$.

We can have using (2.16)

$$\begin{aligned} \left|\left(F'(u) - F'(v)\right)(w)_n\right| &= \left|\left(F'(y_n + u_n) - F'(y_n + v_n)\right) w_n\right| \\ &\leq \ell\,|u_n - v_n|\,|w_n|. \end{aligned} \tag{2.26}$$

Hence we can set $L = \ell$.

(iv) $\|F'(u) - F'(0)\| \leq L_0\|u\|$.

By (2.17) we get

$$\begin{aligned} \left|\left(F'(u) - F'(0)\right)(w)_n\right| &= \left|\left(F'(y_n + u_n) - F'(y_n + 0)\right) w_n\right| \\ &\leq \ell_0\,|u_n|\,|w_n|. \end{aligned} \tag{2.27}$$

That is, we can take $L_0 = \ell_0$.

Crucial condition (2.5) is satisfied by (2.7) and with the above choices of η, β, L and L_0. Therefore the claims of Theorem 2.2 follow immediately from the conclusions of Theorem 2.1. That completes the proof of the theorem. □

Remark 2.3. Suppose that (2.4) hold on D (as in [3, 4, 5, 6]) and let L_1(i.e.. ℓ_1) be the corresponding constant. Then, we have that

$$L_0 \leq L_1 \text{ and } L \leq L_1. \tag{2.28}$$

The Kantorovich sufficient convergence condition corresponding to (2.5) and given in [4] is:

$$h \leq L_1\eta \leq 1. \tag{2.29}$$

Then, in view of (2.5), (2.28) and (2.29), we have that

$$h \leq 1 \Longrightarrow h_A \leq 1$$

but not necessarily vice versa unless, if $L_0 = L = L_1$. Otherwise our Theorem 2.2 improves Theorem 1 in [4]. Indeed, the upper bound in [4, p.1684] is given by

$$\left\| M^{-1} \right\| \leq b = \frac{1}{\delta + \sqrt{2\ell\delta_0}}. \tag{2.30}$$

By comparing (2.7) with (2.30), we deduce that

$$b < a.$$

Finally notice that the error bounds are tighter (use $\ell_0 = \ell$ in (2.10) to obtain the estimates in [4]) and the information on the location of the solution more precise than in [4], if $L_0 < L$. Examples where $L_0 < L$ can be found in [1, 2]. That is, we have justified the claims made in the introduction.

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