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ON EXISTENCE OF NONTRIVIAL SOLUTIONS OF NEUMANN BOUNDARY VALUE PROBLEMS FOR QUASI-LINEAR ELLIPTIC EQUATIONS

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Abstract. In the present paper, some new existence results of nontrivial solutions are obtained for the following Neumann boundary value problem involving the $p-$Laplacian

$$\begin{cases} \triangle_p(u) = f(x,u), & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega \end{cases}$$

and conditions in recent literature for guaranteeing the existence of solutions with saddle point character are improved.

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1. Introduction and Main Results

Consider the following Neumann boundary value problem involving the $p-$Laplacian

$$\begin{cases} \triangle_p(u) = f(x,u), & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a $C^{0,1}$ boundary domain, $N \geq 1, p > N, f : \Omega \times R \to R$ is a Carathéodory function, that is, $f(x,t)$ is measurable in x for every $t \in R$ and continuous in t for a. e. $x \in \Omega$. Moreover, for every $s > 0$ we assume $\sup_{|t|\leq s} |f(\cdot,t)| \in L^1(\Omega)$. Here $\triangle_p$ is the $p-$Laplacian, i. e. $\triangle_p(u) = \mathrm{div}(| \bigtriangledown u|^{p-2} \bigtriangledown u)$.

The energy functional associated to problem (1.1) given by

$$\varphi(u) = \frac{1}{p}\int_\Omega | \bigtriangledown u|^p dx - \int_\Omega F(x,u)dx$$

is continuously differentiable and weakly lower semi-continuous on $W^{1,p}(\Omega)$ (see [12] or Lemma 2.4 in section 2) and

$$\langle \varphi'(u), v\rangle = \int_\Omega | \bigtriangledown u|^{p-2} \bigtriangledown u \cdot \bigtriangledown v dx - \int_\Omega f(x,u)v dx$$

for all $u, v \in W^{1,p}(\Omega)$, where $F(x,t) = \int_0^t f(x,s)ds$. And the space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\| = [\int_\Omega | \bigtriangledown u|^p dx + \int_\Omega |u|^p dx]^{\frac{1}{p}}.$$

It is well known that the weak solutions of problem (1.1) correspond to the critical points of functional φ.

As $p = 2$ problem (1.1) were studied in [2-9], and some well known solvability conditions were given, such as sign condition (see [2] and its references), the monotonicity condition (see [4], [5] and their references) and sublinear condition (see [7-9]).

Recently, Wu and Tan [12] studied problem (1.1) under some another monotone type condition and some sub-order condition, they obtained the following theorem:

Theorem A: *If the following conditions hold:*
(i) there exist $g, h \in L^1(\Omega; R), M > 0$ *and* $\alpha \in [0, p-1)$ *such that*

$$\frac{f(x,t)}{t} \leq \frac{g(x)t^\alpha + h(x)}{t}$$

for all $|t| \geq M$ *and a. e.* $x \in \Omega$*;*
(ii) there exist constants $0 < \varepsilon < c_p$ *and some* $\eta \in R$ *such that*

$$\frac{f(x,s) - f(x,t)}{s-t} \leq \frac{(c_p - \varepsilon)\bar{\lambda}|s-t|^{p-2}(s-t) + \eta}{s-t}$$

for any $s \neq t \in R$ *and a. e.* $x \in \Omega$*, where* $\bar{\lambda}$ *and* c_p *are the constants appearing in the following Lemma 2.1 and Lemma 2.2;*
(iii)

$$\int_\Omega F(x,r)dx \to +\infty \quad as \quad |r| \to \infty;$$

then the problem (1.1) *has at least one solution in* $W^{1,p}(\Omega)$*.*
If adding the following condition
(iv) $f(x,0) = 0$ *for a. e.* $x \in \Omega$ *and there exists a* $\delta > 0$ *such that as* $0 < |t| < \delta$, $\frac{f(x,t)}{t} < 0$ *for a. e.* $x \in \Omega$*, then the problem* (1.1) *has at least one nontrivial solution in* $W^{1,p}(\Omega)$*.*

We point out that the condition (i) in Theorem A is unnecessary, and in this paper we will give a new approach to obtain the same result in Theorem A without condition (i), and will give some new conditions to guarantee the existence of the nontrivial solutions of problem (1.1) by using a new method motivated by [1] and [11]. To state our main results, we need the following conceptions:

If $W^{1,p}(\Omega) = X \bigoplus Y$, for each $x \in X$ and each $y \in Y$, let $\psi(x,y) = \varphi(x+y)$. The solution $u = x + y$ of problem (1.1) is said to be of correlated property if there exists a continuous function θ such that $y = \theta(x)$ and either $\psi(x, \theta(x)) = \min_{y \in Y} \psi(x,y)$ or $\psi(x,\theta(x)) = \sup_{y\in Y} \psi(x,y)$. The solution $u = x + y$ of problem (1.1) is said to possess saddle point character if it is correlated and is a saddle point of $\psi(x,y)$.

Briefly, we have the following main results:

Theorem 1.1. *If the following conditions hold:*
(i) there exist constants $0 < \varepsilon < c_p$ *and some* $\eta \in R$ *such that*

$$[f(x,s) - f(x,t)](s-t) \leq (c_p - \varepsilon)\bar{\lambda}|s-t|^p + \eta(s-t)$$

for any $s, t \in R$ *and a. e.* $x \in \Omega$*, where* c_p *and* $\bar{\lambda}$ *are the constants as in Theorem A;*
(ii)

$$\int_\Omega F(x,r)dx \to +\infty \quad as \quad |r| \to \infty;$$

then the problem (1.1) *has at least one solution with saddle point character in* $W^{1,p}(\Omega)$*.*

If adding the following condition

(iii) $f(x,0) = 0$ *for a. e.* $x \in \Omega$ *and there exists a* $\delta > 0$ *such that as* $0 < |t| < \delta, f(x,t)\ t < 0$ *for a. e.* $x \in \Omega$*, then the problem* (1.1) *has at least one nontrivial solution with saddle point character in* $W^{1,p}(\Omega)$*.*

Theorem 1.2. *Suppose that* f *satisfies the conditions (i), (ii) in Theorem 1.1 and the following condition hold:*
(iii) $f(x,0) = 0$ *for a. e.* $x \in \Omega$ *and there exists some* $t_0 \in R$ *such that* $f(x,t_0) = 0$ *for a. e.* $x \in \Omega$ *and* $\int_\Omega F(x,t_0)dx < 0$*.*

Then the problem (1.1) *has at least one nontrivial solution with saddle point character in* $W^{1,p}(\Omega)$*.*

Theorem 1.3. *Suppose that* f *satisfies the conditions (i), (ii) in Theorem 1.1 and the following condition hold:*
(iii) $f(x,0) = 0$ *for a. e.* $x \in \Omega$ *and there exists a* $\delta > 0$ *such that*

$$F(x,t) < 0$$

for all $t \in R$ *with* $0 < |t| \leq \delta$ *and for a. e.* $x \in \Omega$*.*

Then the problem (1.1) *has at least one nontrivial solution with saddle point character in* $W^{1,p}(\Omega)$*.*

Remark 1.1. In above statements we give the terminology of a solution with saddle point character, and it can be seen that the conditions for guaranteeing the existence of nontrivial solutions are different from that in other papers, hence our results are novel and are significant improvement, compare to the results in the list references.

Especially, for the solutions with correlated property, we have the following:

Theorem 1.4. *If the following conditions hold:*
(i) there exist constants $0 < \varepsilon \leq c_p$ *and some* $\eta \in R$ *such that*

$$[f(x,s) - f(x,t)](s-t) \leq (c_p - \varepsilon)\bar{\lambda}|s-t|^p + \eta(s-t)$$

for any $s, t \in R$ *and a. e.* $x \in \Omega$*, where* c_p *and* $\bar{\lambda}$ *are the constants as in Theorem A;*
(ii)there exist $g \in L^1(0,T;R^+)$ *and* $M > 1$ *such that*

$$F(x,t) \leq -g(x)|t|$$

for all $|t| \geq M$ *and a. e.* $x \in \Omega$*;*
then the problem (1.1) *has at least one solution with correlated property in* $W^{1,p}(\Omega)$ *which minimizing the energy functional* φ*.*

If adding the following condition
(iii) $f(x,0) = 0$ *for a. e.* $x \in \Omega$ *and there exists some* $t_0 \in R$ *such that* $f(x,t_0) = 0$ *for a. e.* $x \in \Omega$ *and* $\int_\Omega F(x,t_0)dx > 0$*.*

Then the problem (1.1) *has at least one nontrivial solution with correlated property in* $W^{1,p}(\Omega)$ *which minimizing the energy functional* φ*.*

Remark 1.2. Noticing that in condition (i) in Theorem 1.4, ε may equal c_p, this is the main distinction to the previous theorems, and if let $\varepsilon = c^p$ and $\eta = 0$, this condition implies that $-f(x,t)$ is monotone, which will contravene with condition (ii) in Theorem 1.1.

2. Lemmas for the proofs of theorems

To prove our main results, we need the following lemmas:

Lemma 2.1 ([12], Proposition 1)**.** *Let $E = \{u \in W^{1,p}(\Omega) : \int_\Omega u(x)dx = 0\}$ and $p > 1$. Then $W^{1,p}(\Omega) = R \bigoplus E$ and there exists a number $\bar{\lambda}$ such that*

$$\int_\Omega |\nabla u|^p dx \geq \bar{\lambda} \int_\Omega |u|^p dx$$

for all $0 \neq u \in E$. And the $\bar{\lambda}$ can be supposed to be the biggest constant satisfying the above inequality throughout this paper.

Lemma 2.2 ([12], Proposition 2, and see [7])**.** *There exists a positive constant c_p such that*

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq c_p|x - y|^p$$

for any $x, y \in R^N$.

Remark 2.1. Note that in Lemma 2.2, let $y = 0$ one has that $c_p \leq 1$.

Lemma 2.3 ([12], Proposition 3)**.** *If the sequence $\{u_n\}$ converges weakly to u in $W^{1,p}(\Omega)$ and $p > N$. Then $\{u_n\}$ converges to u uniformly in $\overline{\Omega}$.*

Lemma 2.4 ([12], in Theorem 1)**.** *The energy functional φ associated to problem* (1.1) *is weakly lower semi-continuous on $W^{1,p}(\Omega)$.*

For the sake of completeness and the convenience of quotation, here, we would like to introduce the proof of Lemma 2.4 in detail as in Theorem 1 in [12].

Proof of Lemma 2.4. It is enough to prove that the functional ω defined by

$$\omega(u) = \int_\Omega F(x, u(x))dx$$

is weakly upper semi-continuous. If $\{u_n\} \subset W^{1,p}(\Omega)$ is such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, then by Lemma 2.3 $\|u_n - u\|_\infty \to 0$ for $n \to \infty$. We claim that

$$\int_\Omega F(x, u(x))dx \geq \limsup_{n\to\infty} \int_\Omega F(x, u_n(x))dx.$$

Otherwise, there exists a positive number $\varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\int_\Omega F(x, u(x))dx + \varepsilon_0 < \int_\Omega F(x, u_{n_k}(x))dx,$$

that is,

$$\int_\Omega F(x, u_{n_k}(x))dx - \int_\Omega F(x, u(x))dx > \varepsilon_0$$

for all positive integer k. On the other hand, since $p > N$, by the embedding theorem there exists a constant c such that $\|v\|_\infty \le c\|v\|$ for all $v \in W^{1,p}(\Omega)$. Since again $\|u_n - u\|_\infty \to 0$ for $n \to \infty$, there is a constant $M_1 > 0$ such that

$$|u(x) + s[u_{n_k}(x) - u(x)]| \le M_1$$

for all $s \in (0,1), x \in \overline{\Omega}$ and for all n_k. Consequently, by the mean value theorem, one has

$$\begin{aligned}
&|\int_\Omega F(x, u_{n_k}(x))dx - \int_\Omega F(x, u(x))dx| \\
&= |\int_\Omega \int_0^1 f(x, u(x) + s[u_{n_k}(x) - u(x)])[u_{n_k}(x) - u(x)]dsdx| \\
&\le \int_\Omega \int_0^1 |f(x, u(x) + s[u_{n_k}(x) - u(x)])|\ |u_{n_k}(x) - u(x)|dsdx \\
&\le \|u_{n_k} - u\|_\infty \int_\Omega [\sup_{|t| \le M_1} |f(x,t)|]dx \\
&< \varepsilon_0
\end{aligned}$$

for large k, a contradiction proving the assertion. Therefore, φ is weakly lower semi-continuous on $W^{1,p}(\Omega)$.

3. Proofs Of Theorems

Now we can give the proofs of our main results.

Proof of Theorem 1.1. Let $W^{1,p}(\Omega) = R \bigoplus E$, for each $v \in R$, define the functional $J_v(w) : E \to R$ as follows:

$$J_v(w) = \varphi(v + w) = \frac{1}{p}\int_\Omega |\bigtriangledown w|^p dx - \int_\Omega F(x, v + w(x))dx. \tag{3.1}$$

For any $w_1, w_2 \in E$, by (i), Lemmas 2.1 and 2.2, there exists some constant number $C > 0$ such that

$$
\begin{aligned}
&(J_v'(w_1) - J_v'(w_2), w_1 - w_2)\\
&= (\varphi'(v + w_1) - \varphi'(v + w_2), w_1 - w_2)\\
&= c_p \int_\Omega | \bigtriangledown (w_1 - w_2)|^p dx - \int_\Omega [f(x, v + w_1) - f(x, v + w_2)]\ (w_1 - w_2)dx\\
&\geq c_p \int_\Omega | \bigtriangledown (w_1 - w_2)|^p dx - (c_p - \varepsilon)\overline{\lambda} \int_\Omega |w_1 - w_2|^p dt - \eta \int_\Omega (w_1 - w_2)dx\\
&\geq \varepsilon \int_\Omega | \bigtriangledown (w_1 - w_2)|^p dx \qquad (3.2)\\
&\quad - (c_p - \varepsilon)[\int_\Omega | \bigtriangledown (w_1 - w_2)|^p dx - \overline{\lambda} \int_\Omega |w_1 - w_2|^p] dx\\
&\geq \varepsilon \int_\Omega | \bigtriangledown (w_1 - w_2)|^p dx\\
&\geq C\|w_1 - w_2\|^p\\
&= \|w_1 - w_2\| h(\|w_1 - w_2\|),
\end{aligned}
$$

where $h(s) = Cs^{p-1}$ is a strictly increasing function from R^+ to R^+ such that $h(s) \to +\infty$ as $s \to +\infty$.

Since $\varphi \in C^1(W^{1,p}(\Omega), R)$, then $J_v(\cdot) \in C^1(E, R)$, by (3.2) J_v has at most one critical point. If $J_v'(0) = 0$, then J_v has the only critical point $w = 0$. If $J_v'(0) \neq 0$, we claim that $J_v(\cdot)$ is coercive on E. In fact, since

$$
\begin{aligned}
J_v(w) &= J_v(0) + \int_0^1 \langle J_v'(sw), w\rangle ds\\
&= J_v(0) + \int_0^1 \langle J_v'(0), w\rangle ds + \int_0^1 \langle J_v'(sw) - J_v'(0), w\rangle ds\\
&\geq J_v(0) - \|J_v'(0)\|\|w\| + \int_0^1 \|w\| h(\|sw\|) ds
\end{aligned}
$$

By the property of h, we may choose R large enough such that

$$h(\|sw\|) \geq 4\ \|J_v'(0)\| \ \text{ uniformly for } \|w\| \geq R, s \in [\frac{1}{2}, 1].$$

Therefore

$$J_v(w) \geq J_v(0) + \|J_v'(0)\|\ \|w\| \qquad (3.3)$$

which implies that $J_v(w) \to +\infty$ as $\|w\| \to \infty$. Hence J_v is coercive on E. Moreover, it follows from $\varphi(\cdot)$ is weakly lower semicontinuous that $J_v(\cdot)$ is weakly lower semicontinuous on E. By Theorem 1.1 in [6] we know that J_v has a unique critical point which minimize $J_v(\cdot)$. We denote the only critical point of $J_v(\cdot)$ by $\theta(v)$, then $\theta(\cdot)$ is a well defined mapping from R to E and such that $J_v(\theta(v)) = \min_{w\in E}\varphi(v+w)$, that is, for all $v \in R$, $\theta(v)$ is the unique critical point for J_v in E such that $J_v(\theta(v)) = \min_{w\in E}\varphi(v+w)$. Hence

$$\langle J_v'(\theta(v)), w\rangle = 0$$

for all $w \in E$.

For each $v \in R$, we set

$$\Psi(v) = \varphi(v + \theta(v)) = \min_{w\in E}\varphi(v + w).$$

We shall show that $\Psi(v)$ is continuously differentiable on R and $\Psi'(v) = \varphi'(v+\theta(v))|_R$ for every $v \in R$.

First we prove that $\theta(\cdot)$ is a bounded mapping. Otherwise, if there exists a bounded sequence $\{v_n\}$ in R such that $\{\theta(v_n)\}$ is unbounded. By the compactness of $\overline{\{v_n\}}$ and $\varphi \in C^1(W^{1,p}(\Omega), R)$, we know $\varphi(\overline{\{v_n\}})$ is compact, and hence $\varphi(\{v_n\})$ is bounded, i.e., there exists some $C_1 > 0$ such that

$$\varphi(v_n) \le C_1, \qquad n = 1, 2, \cdots . \tag{3.4}$$

On the other hand, (3.3) implies that

$$\varphi(v_n) \ge \varphi(v_n + \theta(v_n)) \ge J_{v_n}(0) + \|J_{v_n}'(0)\|\|\theta(v_n)\| \to +\infty$$

as $n \to \infty$; which is a contradiction with (3.4).

Next we prove that $\theta(\cdot)$ is continuous. Suppose that $\{v_n\}$ in R be such that $v_n \to v_0$. By $\varphi \in C^1(W^{1,p}(\Omega), R)$, and since

$$\begin{aligned} J_{v_n}'(\theta(v_0)) &= \varphi'(v_n + \theta(v_0))|_E \\ &\to \varphi'(v_0 + \theta(v_0))|_E = J_{v_0}'(\theta(v_0)) \end{aligned}$$

and

$$\begin{aligned} (J_{v_n}'(\theta(v_0)), \theta(v_0) - \theta(v_n)) =& (J_{v_n}'(\theta(v_0)) - J_{v_n}'(\theta(v_n)), \theta(v_0) - \theta(v_n)) \\ &\ge C\|\theta(v_0) - \theta(v_n)\|^p, \end{aligned}$$

one has $\|\theta(v_0) - \theta(v_n)\| \to 0$. Hence $\theta:\ R \to E$ is a continuous mapping, and it can be shown as in [11] that $\Psi(v) \in C^1(R, R)$ and

$$(\Psi'(v), z) = (\varphi'(v + \theta(v)), z), \qquad \forall v,\ z \in R.$$

For the sake of completeness we reproduce that rather short proof here.

Indeed, for $s > 0$,

$$\begin{aligned} \frac{\Psi(v + sz) - \Psi(v)}{s} &= \frac{\varphi(v + sz + \theta(v + sz)) - \varphi(v + \theta(v))}{s} \\ &\le \frac{\varphi(v + sz + \theta(v)) - \varphi(v + \theta(v))}{s} \\ &= \int_0^1 \langle \varphi'(v + \theta(v) + tsz), z\rangle dt \end{aligned}$$

In a similar way, we have

$$\frac{\Psi(v+sz)-\Psi(v)}{s} \geq \int_0^1 \langle \varphi'(v+\theta(v+sz)+tsz), z\rangle dt$$

Combining above two inequalities proving the assertions.

Hence it follows from above facts that $u \in W^{1,p}(\Omega)$ is a critical point of $\varphi(\cdot)$ if and only if $u = v + \theta(v)$ and v is a critical point of $\Psi(\cdot)$ on R.

Condition (ii) implies that $\varphi(v) \to -\infty$ (as $\|v\| \to \infty$) on R, then one has by $\Psi(v) = \varphi(v+\theta(v)) \leq \varphi(v)$ that

$$\Psi(v) \to -\infty \qquad \text{as} \quad \|v\| \to \infty.$$

Hence there exists a $v_0 \in R$ such that

$$\Psi(v_0) = \sup_{v \in R} \Psi(v),$$

so v_0 is a critical point of $\Psi(\cdot)$, and hence $u = v_0 + \theta(v_0)$ is a critical point of $\varphi(\cdot)$. Therefore problem (1.1) has at least a solution with saddle point character in $W^{1,p}(\Omega)$. Then the rest part is same as in Theorem 1 in [12], we omit it and complete the proof of Theorem 1.1.

Remark 3.1. From the above proof of Theorem 1.1 we see that no form need to impose upon the potential function in the process proving the coerciveness, the method is simple and elegant for proving the coerciveness and is different from the methods in the others, compare to the proofs in [6-8, 10].

Proof of Theorem 1.2. Under the present conditions, Theorem 1.1 has showed that there exists $v_0 \in R$ such that $u = v_0 + \theta(v_0)$ is a solution with saddle point character in $W^{1,p}(\Omega)$. To show that the solution u in Theorem 1.2 is nontrivial if adding condition (iii), we shall show that $\theta(t) = 0$ for all $t \in R$ such that $f(x,t) = 0$. For each $w \in E$ one has

$$\begin{aligned}\langle J_t'(\theta(t)), w\rangle &= \langle \varphi'(t+\theta(t)), w\rangle \\ &= \int_\Omega |\bigtriangledown(\theta(t))|^{p-2} \bigtriangledown(\theta(t)) \cdot \bigtriangledown w dx - \int_\Omega f(x, t+\theta(t)) w dx \\ &= 0.\end{aligned}$$

Set $w = \theta(t)$, hence one has

$$\int_\Omega |\bigtriangledown(\theta(t))|^p dx - \int_\Omega f(x, t+\theta(t))\ \theta(t) dx = 0.$$

That is

$$\begin{aligned}
0 &= \int_\Omega |\bigtriangledown (\theta(t))|^p dx - \int_\Omega f(x, t+\theta(t))\ \theta(t)dx \\
&= \int_\Omega |\bigtriangledown (\theta(t))|^p dx - \int_\Omega [f(x, t+\theta(t)) - f(x,t)]\ \theta(t)dx \\
&\geq c_p \int_\Omega |\bigtriangledown (\theta(t))|^p dx - (c_p - \varepsilon)\overline{\lambda} \int_\Omega |\theta(t)|^p dx - \eta \int_\Omega \theta(t) dx \\
&\geq C\|\theta(t)\|^p \geq 0
\end{aligned}$$

which implies that $\|\theta(t)\| = 0$, hence $\theta(t) = 0$. Hence by condition (iii) one has $\theta(t_0) = 0$ and $\theta(0) = 0$ and $\Psi(t_0) - \Psi(0) = \int_\Omega [F(x, 0) - F(x, t_0)]dx > 0$. This implies

$$\Psi(v_0) = \sup_{v\in R} \Psi(v) \geq \Psi(t_0) > \Psi(0).$$

Hence $v_0 \neq 0$ and $u = v_0 + \theta(v_0)$ is a nontrivial critical point of φ, and problem (1.1) has at least one nontrivial solution with saddle point character in $W^{1,p}(\Omega)$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Similarly, we only need to show that the solution u in Theorem 1.1 is nontrivial if adding condition (iii), we have showed that $\theta(0) = 0$ as $f(x, 0) = 0$, hence $\Psi(0) = 0$. Since $p > N$, by the embedding theorem there exists a constant c such that $\|u\|_\infty \leq c\|u\|$ for all $u \in W^{1,p}(\Omega)$, then by the continuity of $\theta(\cdot)$ there is $\delta > \rho > 0$ such that

$$0 < \|\rho + \theta(\rho)\|_\infty \leq \delta.$$

If $\theta(\rho) = 0$, then $\Psi(\rho) - \Psi(0) = \int_\Omega [F(x, 0) - F(x, \rho)]dx > 0$. If $\theta(\rho) \neq 0$, then we have

$$\begin{aligned}
\Psi(\rho) - \Psi(0) &= \varphi(\rho + \theta(\rho)) - \varphi(0) \\
&= \frac{1}{p}\int_\Omega |\bigtriangledown (\theta(\rho))|^p dx - \int_\Omega F(x, \rho + \theta(\rho))dx \\
&\geq \frac{1}{p}\int_\Omega |\bigtriangledown (\theta(\rho))|^p dx \\
&> 0.
\end{aligned}$$

Hence one has

$$\Psi(v_0) = \sup_{v\in R} \Psi(v) \geq \Psi(\rho) > \Psi(0).$$

Then $v_0 \neq 0$ and $u = v_0 + \theta(v_0)$ is a nontrivial critical point of φ, and problem (1.1) has at least one nontrivial solution with saddle point character in $W^{1,p}(\Omega)$. This completes the proof of Theorem 1.3.

Proof of Theorem 1.4. Let $W^{1,p}(\Omega) = R \bigoplus E$, for each $v \in R$, define the functional $J_v(w) : E \to R$ as follows:

$$J_v(w) = \varphi(v+w) = \frac{1}{p}\int_\Omega |\bigtriangledown w|^p dx - \int_\Omega F(x, v + w(x))dx.$$

As same as in Theorem 1.1, we know that for each $v \in R$, there exists a well defined continuous mapping $\theta(\cdot)$ from R to E such that

$$J_v(\theta(v)) = \min_{w \in E} \varphi(v+w),$$

and for each $v \in R$, we set

$$\Psi(v) = \varphi(v + \theta(v)) = \min_{w \in E} \varphi(v+w).$$

And hence we know that $u \in W^{1,p}(\Omega)$ is a critical point of $\varphi(\cdot)$ if and only if $u = v + \theta(v)$ and v is a critical point of $\Psi(\cdot)$ on R.

Moreover, for $u = v + w, v \in R, w \in E$, as $\|u\| \to \infty$ in $W^{1,p}(\Omega)$ if and only if $|v| + \int_\Omega |\bigtriangledown w|^p dx \to \infty$ by Lemma 2.1, and noticing that $|a+b|c \geq |a|c - |bc|$ for all real numbers a, b, c, hence for $|v|$ large, by condition (ii) we have

$$\begin{aligned}
\Psi(v) &= \varphi(v + \theta(v)) \\
&= \frac{1}{p}\int_\Omega |\bigtriangledown (\theta(v))|^p dx - \int_\Omega F(x, v + \theta(v))dx \\
&\geq \frac{1}{p}\int_\Omega |\bigtriangledown (\theta(v))|^p dx + \int_\Omega g(x)|v + \theta(v)|dx \\
&\geq \frac{1}{p}\int_\Omega |\bigtriangledown (\theta(v))|^p dx + \int_\Omega g(x)|v|dx - \int_\Omega |g(x)|\ |\theta(v)|dx \\
&\geq \frac{1}{p}\int_\Omega |\bigtriangledown (\theta(v))|^p dx + |v|\int_\Omega g(x)dx - [\int_\Omega |g(x)|^q]^{\frac{1}{q}} dx - [\int_\Omega |\theta(v)|^p]^{\frac{1}{p}} dx \\
&\geq \frac{1}{p}\int_\Omega |\bigtriangledown (\theta(v))|^p dx + c_1|v| - c_2 - c_3[\int_\Omega |\bigtriangledown (\theta(v))|^p dx]^{\frac{1}{p}}
\end{aligned}$$

which implies that $\Psi(v) \to +\infty$ as $|v| \to \infty$, where $c_1 = \int_\Omega g(x)dx > 0$. Hence there exists a $v_0 \in R$ such that

$$\Psi(v_0) = \inf_{v \in R} \Psi(v),$$

so v_0 is a critical point of $\Psi(\cdot)$, and hence $u = v_0 + \theta(v_0)$ is a critical point of $\varphi(\cdot)$. Moreover, for any $z = v + w, v \in R, w \in E$, one has

$$\varphi(z) = \varphi(v+w) \geq \inf_{w \in E} \varphi(v+w) = \Psi(v) \geq \Psi(v_0) = \varphi(u)$$

Therefore u minimize the energy functional φ.

It is easy to show that this solution $u = v_0 + \theta(v_0)$ is nontrivial if adding condition (iii). Indeed, we have $\theta(t_0) = 0$ and $\theta(0) = 0$ and

$$\Psi(t_0) - \Psi(0) = \int_\Omega [F(x,0) - F(x,t_0)]dx < 0.$$

This implies

$$\Psi(v_0) = \inf_{v\in R} \Psi(v) \le \Psi(t_0) < \Psi(0).$$

Hence $v_0 \neq 0$ and $u = v_0 + \theta(v_0)$ is a nontrivial critical point of φ, and problem (1.1) has at least one nontrivial solution with correlated property in $W^{1,p}(\Omega)$ which minimize the energy functional φ. This completes the proof of Theorem 1.4.

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