Nonlinear Functional Analysis and Applications
Vol. 21, No. 4 (2016), pp. 613-646

http://nfaa.kyungnam.ac.kr/journal-nfaa

BHASKAR-LAKSHMIKANTHAM TYPE COUPLED FIXED POINT RESULTS FOR RATIONAL CONTRACTIVE EXPRESSION USING g-MONOTONE MAPPING

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Abstract. In this paper, we prove some coupled coincidence point theorems for maps satisfying contractive conditions involving a rational expression in the setting of partially ordered metric spaces using monotone property instead of the often used mixed monotone property. We also give some sufficient conditions for the existence and uniqueness of coupled coincidence points. In particular, it is shown that the results existing in the literature are extended, generalized, unified and improved by using monotone property. Also, examples are given to support these improvements. As an application, we give a result of existence and uniqueness for the solutions of a class of nonlinear integral equations.

1. Introduction

Over the past ten decades, we are witness of flourishing of the field of nonlinear functional analysis and it particular, fixed point theory in variety of

[0]Received February 10, 2016. Revised June 22, 2016.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Partially ordered metric space, coupled coincidence point, uniqueness of coupled coincidence point, rational contractive expression, g-monotone mappings.

directions either in the form of deriving new results or applications. In the journey of fixed point theory, in year 1912, Polish Mathematician, Stephen Banach [1] introduced the notion of Banach Contraction Principle and achieved fixed point theorems. The Banach Contraction Principle is one of the cornerstones in the development of Nonlinear Analysis, in general, and metric fixed point theory, in particular. This principle was extended and improved in many directions and various fixed point theorems were established. Two usual ways for extending and improving the Banach Contraction Principle are obtained by:

(1) replacing the underlying metric space by certain generalized metric space;

(2) changing the contraction condition to more general ones.

The Banach contraction principle is a power tool for solving many problems in applied mathematics and sciences, it has been improved and extended in many ways. It is well known that the metric fixed point theory is still very actual, important and useful in all area of Mathematics. It can be applied, for instance in variational inequalities, optimization, dynamic programming, approximation theory, etc.

On the other hand, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [22] who presented its applications to matrix equations. Subsequently, Nieto and Rodrguez-Lpez [19] extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed point theorems in ordered metric spaces. For more details see [8, 9, 14, 18, 20, 25] and the references cited therein.

The study of mixed monotone mapping is an active area of research due to its wide scope of application. The theory of mixed monotone mapping in ordered Banach space was extensively investigated in [28]. Guo and Lakshmikantham [4] introduced the notion of a coupled fixed point for two mappings. Bhaskar and Lakshmikantham [3] proved some interesting coupled fixed point theorems for mappings satisfying a mixed monotone property and coupled coincidence point in partial ordered metric spaces. Coupled common fixed point and coincidence point problems were first addressed by Lakshmikantham and Ciric [11] in which the authors extended the work of Bhaskar and Lakshmikantham [3] by defining the mixed g-monotone property and proved the existence and uniqueness of a coupled coincidence point for such property in partially ordered metric spaces. Some of the coupled fixed point results on mixed monotone property can be seen in the papers [2, 5, 6, 7], [12]-[18], [23]-[27].

Recently, Radenović [21] introduced a notion of monotone mappings and derived coupled fixed results without use of mixed monotone property. After that Kadelburg and Kumam [10] proved common coupled fixed point theorems for Geraghty-type contraction mappings using this property.

Motivated with the notion of monotone mappings, we establish two coupled coincidence point results for mappings $F : X \times X \to X$ and $g : X \to X$ for two different form of contraction conditions involving rational terms in the frame of partially ordered metric space using monotone property. Examples are given in support of results and it is showed that it is distinguish from the results having mixed monotone property. Basically in first result compatibility of F and g is taken on the condition of continuity and closeness of g, while in second result w^*-compatible is assumed with completeness of g in X. Finally we apply the obtained result to investigate the existence of unique solutions to a class of nonlinear integral equations.

2. Preliminaries

We recall the following definitions used throughout the paper.

Definition 2.1. Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space iff

(i) (X, d) is a metric space,
(ii) $(X, \preceq)$ is a partial order.

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set. Let $F : X \times X \to X$ and $g : X \to X$. Then for all $x, y \in X$

(1) X is comparable if $x \preceq y$ or $y \preceq x$ holds.
(2) F is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y; that is
$$x_1, x_2 \in X, (x_1 \preceq x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y)$$
and
$$y_1, y_2 \in X, (y_1 \preceq y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2).$$
(3) F is said to have the mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in x in its first argument and is monotone g-nonincreasing in y in its second argument; that is; for any $x, y \in X$,
$$g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y) \text{ for all } x_1, x_2 \in X$$
and
$$g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2) \text{ for all } y_1, y_2 \in X.$$
Note that if g is the identity mapping, then F is said to have the mixed monotone property.

(4) X is regular, if
   (a) a non-decreasing sequence $\{x_n\}$ holds $d(x_n, x)$ imply 0, then $x_n \preceq x$ for all n, and
   (b) a non-decreasing sequence $\{y_n\}$ holds $d(y_n, y)$ imply 0, then $y_n \succeq y$ for all n.

(5) $F : X \times X \to X$ is said to have *monotone property* ([21]) if the following two conditions are satisfied:

$$(\forall x_1, x_2, y \in X)\ x_1 \preceq x_2 \ \Rightarrow F(x_1, y) \preceq F(x_2, y),$$
$$(\forall x, y_1, y_2 \in X)\ y_1 \preceq y_2 \ \Rightarrow F(x, y_1) \preceq F(x, y_2).$$

(6) F is said to have the *g-monotone property* ([10]) if F is monotone g-nondecreasing in both of its arguments, that is, $x, y \in X$,

$$g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y) \text{ for all } x_1, x_2 \in X$$

and

$$g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \preceq F(x, y_2) \text{ for all } y_1, y_2 \in X.$$

Note that if g is the identity mapping, then F is said to have the monotone property.

Definition 2.3. ([4, 24]) Let X be a nonempty set and $F : X \times X \to X$, $g : X \to X$ be two mappings. A pair $(x, y) \in X \times X$ is called:

(a) a coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$;

(b) a coupled coincidence point of mappings g and F if

$$gx = F(x, y) \text{ and } gy = F(y, x).$$

and in this case (gx, gy) is called a coupled point of coincidence;

(c) a common coupled fixed point of mappings g and F if

$$x = gx = F(x, y) \text{ and } y = gy = F(y, x).$$

Definition 2.4. ([17]) Let (X, d) be a metric space, $F : X \times X \to X$ and $g : X \to X$. Then, we say that F and g are compatible if

$$d(gF(x_n, y_n), F(gx_n, gy_n)) \to 0 \text{ and}$$
$$d(gF(y_n, x_n)), F(gy_n, gx_n)) \to 0, \text{ as } n \to \infty$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$$

for all $x, y \in X$.

Definition 2.5. Let X be a nonempty set. Mappings $F : X \times X \to X$ and $g : X \to X$ are said to be

(W_1) w-compatible if $gF(x,y) = F(gx, gy)$ whenever $gx = F(x,y)$ and $gy = F(y,x)$;
(W_2) w^*-compatible if $gF(x,x) = F(gx, gx)$ whenever $gx = F(x,x)$.

It is note that F and g may be w^*-compatible but not w-compatible.

3. Result-I

Now we are standing in a position to prove our first main result.

Theorem 3.1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the g-monotone property. Suppose that the following hold:*

(i) *g is continuous and $g(X)$ is closed;*
(ii) *$F(X \times X) \subseteq g(X)$ and g and F are compatible;*
(iii) *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \preceq F(y_0, x_0)$;*
(iv) *there exists $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + 2\gamma + 2\delta < 1$ for all $(x,y), (u,v) \in X \times X$ satisfying $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$ (or $g(x) \succeq g(u)$ and $g(y) \succeq g(v)$),*

$$\begin{aligned} &d(F(x,y), F(u,v)) \\ &\leq \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] + \beta M((x,y),(u,v)) \\ &\quad + \frac{\gamma}{2}[d(gx, F(x,y)) + d(gu, F(u,v)) + d(gy, F(y,x)) + d(gv, F(v,u))] \\ &\quad + \frac{\delta}{2}[d(gx, F(u,v)) + d(gy, F(v,u)) + d(gu, F(x,y)) + d(gv, F(y,x))], \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &M((x,y),(u,v)) \\ &= \min\left\{ d(gx, F(x,y)) \frac{2 + d(gu, F(u,v)) + d(gv, F(v,u))}{2 + d(gx, gu) + d(gy, gv)}, \right. \\ &\qquad \left. d(gu, F(u,v)) \frac{2 + d(gx, F(x,y)) + d(gy, F(y,x))}{2 + d(gx, gu) + d(gy, gv)} \right\}; \end{aligned} \tag{3.2}$$

(v) *F is continuous.*

Then there exists $\bar{x}, \bar{y} \in X$, $g\bar{x} = F(\bar{x}, \bar{y})$ and $g\bar{y} = F(\bar{y}, \bar{x})$, that is, F and g have a coupled coincidence point $(\bar{x}, \bar{y}) \in X \times X$.

Proof. Starting from x_0, y_0 by condition(iii) and using that $F(X \times X) \subseteq g(X)$ (condition (ii)), we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Then $g(x_0) \preceq g(x_1)$ and $g(y_0) \preceq g(y_1)$. Analogously, there exist $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$.

Continuing the above procedure, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$F(x_n, y_n) = g(x_{n+1}) \text{ and } F(y_n, x_n) = g(y_{n+1})$$

for all $n \in \{0, 1, 2, \ldots\}$. Now by induction, we prove that

$$g(x_n) \preceq g(x_{n+1}) \text{ and } g(y_n) \preceq g(y_{n+1}). \tag{3.3}$$

By using g-monotone property of F and g, we have

$$g(x_2) = F(x_1, y_1) \preceq F(x_0, y_1) \preceq F(x_0, y_0) \preceq g(x_1)$$

and

$$g(y_2) = F(y_1, x_1) \preceq F(y_0, x_1) \preceq F(y_0, x_0) \preceq g(y_1)$$

for all $n \geq 0$. Since $g(x_2) \preceq g(x_1)$ and $g(y_2) \preceq g(y_1)$, so the initial step of the induction is true. Suppose that (3.3) holds. Then using the g-monotone property of F and (3.3), we obtain, for $n = 1, 2, \ldots,$

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \preceq F(x_{n+1}, y_{n+1}) = g(x_{n+2}),$$

and consequently $g(x_{n+1}) \preceq g(x_{n+2})$. Similarly, we can show that $g(y_{n+1}) \preceq g(y_{n+2})$. In general, we conclude that

$$g(x_n) \preceq g(x_{n+1}) \text{ and } g(y_n) \preceq g(y_{n+1}) \text{ for all } n \geq 0.$$

Thus by the mathematical induction, we conclude that (3.3) holds for all $n \geq 0$. We easily check

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \cdots \preceq g(x_{n+1}) \preceq \cdots \tag{3.4}$$

and

$$g(y_0) \preceq g(y_1) \preceq g(y_2) \preceq \cdots \preceq g(y_{n+1}) \preceq \cdots . \tag{3.5}$$

If $g(x_{n+1}) = g(x_n)$ and $g(y_{n+1}) = g(y_n)$ for some n, then $F(x_n, y_n) = g(x_n)$ and $F(y_n, x_n) = g(y_n)$, hence (gx_n, gy_n) is a coupled coincidence point of F and g. Suppose, further, that

$$g(x_n) \neq g(x_{n+1}) \text{ or } g(y_n) \neq g(y_{n+1}) \text{ for each } n \in \mathbb{N}_0.$$

Now, we claim that, for $n \in \mathbb{N}_0$,

$$\begin{aligned} &d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n)) \\ &\leq \left(\frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - \delta}\right)^n [d(g(x_1), g(x_0)) + d(g(y_1), g(y_0))]. \end{aligned} \tag{3.6}$$

Indeed, for $n=1$, using $g(x_1) \preceq g(x_0)$, $g(y_1) \preceq g(y_0)$ and (3.1), we get:

$$\begin{aligned}
&d(g(x_2), g(x_1)) \qquad (3.7)\\
&= d(F(x_1, y_1), F(x_0, y_0))\\
&\le \frac{\alpha}{2}[d(g(x_1), g(x_0)) + d(g(y_1), g(y_0))] + \beta M((x_1, y_1), (x_0, y_0))\\
&\quad + \frac{\gamma}{2}[d(g(x_1), F(x_1, y_1)) + d(g(x_0), F(x_0, y_0)) + d(g(y_1), F(y_1, x_1))\\
&\quad + d(g(y_0), F(y_0, x_0))]\\
&\quad + \frac{\delta}{2}[d(g(x_1), F(x_0, y_0)) + d(g(y_1), F(y_0, x_0)) + d(g(x_0), F(x_1, y_1))\\
&\quad + d(g(y_0), F(y_1, x_1))]\\
&\le \frac{\alpha}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))]\\
&\quad + \beta d(g(x_1), F(x_1, y_1))\, \frac{2 + d(g(x_0), F(x_0, y_0)) + d(g(y_0), F(y_0, x_0))}{2 + d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))}\\
&\quad + \frac{\gamma}{2}[d(g(x_1), g(x_2)) + d(g(x_0), g(x_1)) + d(g(y_1), g(y_2)) + d(g(y_0), g(y_1))]\\
&\quad + \frac{\delta}{2}[d(g(x_1), g(x_1)) + d(g(y_1), g(y_1)) + d(g(x_0), g(x_2)) + d(g(y_0), g(y_2))]\\
&\le \frac{\alpha}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))] + \beta d(g(x_1), g(x_2))\\
&\quad + \frac{\gamma+\delta}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1)) + d(g(x_1), g(x_2)) + d(g(y_1), g(y_2))].
\end{aligned}$$

Similarly, using that

$$d(g(y_2), g(y_1)) = d(F(y_1, x_1), \quad F(y_0, x_0)) = d(F(y_0, x_0), F(y_1, x_1))$$

and

$$\begin{aligned}
&M((x_1, y_1), (x_0, y_0))\\
&\le d(g(y_1), F(y_1, x_1))\, \frac{2 + d(g(y_0), F(y_0, x_0)) + d(g(x_0), F(x_0, y_0))}{2 + d(g(y_0), g(y_1)) + d(g(x_0), g(x_1))}\\
&= d(g(y_1), g(y_2)),
\end{aligned}$$

we get

$$\begin{aligned}
d(g(y_2), g(y_1)) &\le \frac{\alpha}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))] + \beta d(g(y_1), g(y_2)) \qquad (3.8)\\
&\quad + \frac{\gamma+\delta}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))\\
&\quad + d(g(x_1), g(x_2)) + d(g(y_1), g(y_2))].
\end{aligned}$$

Adding (3.7) and (3.8), we have

$$\begin{aligned}&d(g(x_2),g(x_1))+d(g(y_2),g(y_1))\\&\le\left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)[d(g(x_0),g(x_1))+d(g(y_0),g(y_1))].\end{aligned}$$

In a similar way, proceeding by induction, if we assume that (3.6) holds, we get that

$$\begin{aligned}&d(g(x_{n+2}),g(x_{n+1}))+d(g(y_{n+2}),g(y_{n+1}))\\&\le\left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)[d(g(x_{n+1}),g(x_n))+d(g(y_{n+1}),g(y_n))]\\&\le\left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)^{n+1}[d(g(x_0),g(x_1))+d(g(y_0),g(y_1))].\end{aligned}$$

Hence, by induction, (3.6) is proved.

Set

$$h_n:=d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1})),\quad n\in\mathbb{N}$$

and $\Delta:=\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}<1$. Then, the sequence $\{h_n\}$ is decreasing and

$$h_n\le\Delta^n h_0$$

which implies that

$$\lim_{n\to\infty}h_n=\lim_{n\to\infty}[d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))]=0.$$

Thus,

$$\lim_{n\to\infty}d(g(x_n),g(x_{n+1}))=0\text{ and }\lim_{n\to\infty}d(g(y_n),g(y_{n+1}))=0.$$

We shall prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. By assumption (3.4), $h_n>0$ for $n\in\mathbb{N}_0$. Then, for each $n\ge m$ we have

$$\begin{aligned}&d(g(x_n),g(x_m))\\&\le d(g(x_n),g(x_{n-1}))+d(g(x_{n-1}),g(x_{n-2}))+\cdots+d(g(x_{m+1}),g(x_m))\end{aligned}$$

and

$$\begin{aligned}&d(g(y_n),g(y_m))\\&\le d(g(y_n),g(y_{n-1}))+d(g(y_{n-1}),g(y_{n-2}))+\cdots+d(g(y_{m+1}),g(y_m)).\end{aligned}$$

Therefore,

$$\begin{aligned}d(g(x_n),g(x_m))+d(g(y_n),g(y_m))&\le h_{n-1}+h_{n-2}+\cdots+h_m\\&\le(\Delta^{n-1}+\Delta^{n-2}+\cdots+\Delta^m)h_0\\&\le\frac{\Delta^m}{1-\Delta}h_0\end{aligned}$$

which implies that

$$\lim_{m,n\to\infty}[d(g(x_n), g(x_m)) + d(g(y_n), g(y_m))] = 0.$$

This imply that $\{g(x_n)\}$ and $\{g(y_n)\}$ are cauchy sequences in X since $0 \leq \Delta < 1$. Since $g(X)$ is a closed subset of a complete metric space, there exists $\bar{x}, \bar{y} \in g(X)$ such that

$$\lim_{n\to\infty} F(\bar{x}_n, \bar{y}_n) = \lim_{n\to\infty} g(\bar{x}_n) = \bar{x} \quad \text{and} \quad \lim_{n\to\infty} F(\bar{y}_n, \bar{x}_n) = \lim_{n\to\infty} g(\bar{y}_n) = \bar{y}. \tag{3.9}$$

Using (3.9) and the continuity of g, we get

$$\lim_{n\to\infty} g(g(\bar{x}_n)) = g\bar{x} \quad \text{and} \quad \lim_{n\to\infty} g(g(\bar{y}_n)) = g\bar{y}. \tag{3.10}$$

From $g(\bar{x}_{n+1}) = F(\bar{x}_n, \bar{y}_n)$ and $g(\bar{y}_{n+1}) = F(\bar{y}_n, \bar{x}_n)$, and by condition (ii), the compatibility of F and g, we have

$$\begin{aligned} &\lim_{n\to\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0, \\ &\lim_{n\to\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0. \end{aligned} \tag{3.11}$$

Now, we claim that $(\bar{x}, \bar{y})$ is a coupled coincidence point of F and g.

Passing to the limit as $n \to \infty$ in (3.11), by using (3.9), (3.10) and the continuity of F we get

$$\begin{aligned} g(\bar{x}) &= \lim_{n\to\infty} g(g(\bar{x}_{n+1})) = \lim_{n\to\infty} F(g(\bar{x}_n), g(\bar{y}_n)) \\ &= F(\lim_{n\to\infty}(g\bar{x}_n), \lim_{n\to\infty}(g\bar{y}_n)) = F(\bar{x}, \bar{y}), \end{aligned}$$

$$\begin{aligned} g(\bar{y}) &= \lim_{n\to\infty} g(g(\bar{y}_{n+1})) = \lim_{n\to\infty} F(g(\bar{y}_n), g(\bar{x}_n)) \\ &= F(\lim_{n\to\infty}(g\bar{y}_n), \lim_{n\to\infty}(g\bar{x}_n)) = F(\bar{y}, \bar{x}). \end{aligned}$$

Thus, we proved that $g\bar{x} = F(\bar{x}, \bar{y})$ and $g\bar{y} = F(\bar{y}, \bar{x})$. This completes the proof of the theorem. □

Remark 3.2. In Theorem 3.1, the condition that F has the g-monotone property is a substitution for the g-mixed monotone property that was used in most of the coupled fixed point results so far. Note that this condition may be more natural than the mixed g-monotone property and can be used in various examples.

Remark 3.3. Comparing the conditions in Theorem 3.1 and Theorem 2.3 of Nashine and Zoran [18], we see that our result is a generalization of Theorem 2.3 in [18].

If $g = I_X$, the identity mapping in Theorem 3.1, then we deduce the following result of coupled fixed point.

Corollary 3.4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ has the monotone property. Suppose that the following hold:*

(i) *F is continuous;*
(ii) *there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \preceq F(y_0, x_0)$;*
(iii) *there exists $\alpha, \beta, \gamma, \delta \geq 0$ for all $(x, y), (u, v) \in X$ satisfying $x \preceq u$ and $y \preceq v$ (or $x \succeq u$ and $y \succeq v$),*

$$\begin{aligned} & d(F(x,y), F(u,v)) \\ & \leq \frac{\alpha}{2}[d(x,u) + d(y,v)] + \beta M((x,y),(u,v)) \\ & \quad + \frac{\gamma}{2}[d(x, F(x,y)) + d(u, F(u,v)) + d(y, F(y,x)) + d(v, F(v,u))] \\ & \quad + \frac{\delta}{2}[d(x, F(u,v)) + d(y, F(v,u)) + d(u, F(x,y)) + d(v, F(y,x))], \end{aligned}$$

where

$$\begin{aligned} & M((x,y),(u,v)) \\ & = \min\left\{ d(x, F(x,y))\, \frac{2 + d(u, F(u,v)) + d(v, F(v,u))}{2 + d(x,u) + d(y,v)}, \right. \\ & \qquad \left. d(u, F(u,v))\, \frac{2 + d(x, F(x,y)) + d(y, F(y,x))}{2 + d(x,u) + d(y,v)} \right\} \end{aligned}$$

with $\alpha + \beta + 2\gamma + 2\delta < 1$.

Then there exists $x_0, y_0 \in X$, $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

By choosing α, β, γ and δ suitably, one can deduce some corollaries from Theorem 3.1. For example, if we take $\beta = \delta = 0$ in Theorem 3.1, then we obtain the following corollary.

Corollary 3.5. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mapping having the g-monotone property. Suppose that the following hold:*

(i) *g is continuous and $g(X)$ is closed;*
(ii) *$F(X \times X) \subseteq g(X)$ and g and F are compatible;*
(iii) *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \preceq F(y_0, x_0)$;*
(iv) *there exists $\alpha, \gamma, \geq 0$ with $\alpha + 2\gamma < 1$, for all $(x, y), (u, v) \in X \times X$ satisfying $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$,*

$$
\begin{aligned}
&d(F(x,y),F(u,v))\\
&\leq \frac{\alpha}{2}[d(gx,gu)+d(gy,gv)]\\
&\quad+\frac{\gamma}{2}[d(gx,F(x,y))+d(gu,F(u,v))+d(gy,F(y,x))+d(gv,F(v,u))]
\end{aligned}
$$

holds true. Then there exists $x_0, y_0 \in X$, $gx = F(x,y)$ and $gy = F(y,x)$, that is, F and g have a coupled coincidence point $(\bar{x},\bar{y}) \in X \times X$.

Theorem 3.6. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the g-monotone property on X. Suppose that the following hold:*

(i) *g is continuous and $g(X)$ is closed;*
(ii) *$F(X \times X) \subseteq g(X)$ and g and F are compatible;*
(iii) *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \preceq F(y_0, x_0)$;*
(iv) *there exists $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha+\beta+2\gamma+2\delta < 1$, for all $(x,y),(u,v) \in X \times X$ satisfying $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$,*

$$
\begin{aligned}
&d(F(x,y),F(u,v)) \qquad (3.12)\\
&\leq \frac{\alpha}{2}[d(gx,gu)+d(gy,gv)]+\beta M((x,y),(u,v))\\
&\quad+\frac{\gamma}{2}[d(gx,F(x,y))+d(gu,F(u,v))+d(gy,F(y,x))+d(gv,F(v,u))]\\
&\quad+\frac{\delta}{2}[d(gx,F(u,v))+d(gy,F(v,u))+d(gu,F(x,y))+d(gv,F(y,x))]
\end{aligned}
$$

where

$$
\begin{aligned}
&M((x,y),(u,v))\\
&=\min\Big\{d(gx,F(x,y))\,\frac{2+d(gu,F(u,v))+d(gv,F(v,u))}{2+d(gx,gu)+d(gy,gv)},\\
&\qquad d(gu,F(u,v))\,\frac{2+d(gx,F(x,y))+d(gy,F(y,x))}{2+d(gx,gu)+d(gy,gv)}\Big\}
\end{aligned}
$$

holds true. Finally, we assume that X has the following properties:

(a) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $gx_n \preceq gx$ for all n,*
(b) *if a non nondecreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $gy_n \preceq gy$ for all n.*

Then, F and g have coupled coincidence point $(\bar{x},\bar{y}) \in X \times X$.

Proof. Following the proof of Theorem 3.1, we only have to show that $(g\bar{x}, g\bar{y})$ is a coupled coincidence point of F and g. We have

$$\begin{aligned} d(F(\bar{x},\bar{y}),g\bar{x}) &\le d(F(\bar{x},\bar{y}),gx_{n+1}) + d(gx_{n+1},g\bar{x}) \\ &= d(F(\bar{x},\bar{y}),F(x_n,y_n)) + d(gx_{n+1},g\bar{x}). \end{aligned} \tag{3.13}$$

Since the nondecreasing sequence $\{x_n\}$ converges to $\bar{x}$ and the nondecreasing sequence $\{y_n\}$ converges to $\bar{y}$, by (a)–(b), we have:

$$g\bar{x} \preceq gx_n \quad \text{and} \quad g\bar{y} \preceq gy_n, \quad \forall\, n.$$

Now, from the contractive condition (3.1), we have:

$$\begin{aligned} &d(F(\bar{x},\bar{y}),F(x_n,y_n)) \\ &\le \frac{\alpha}{2}[d(g\bar{x},gx_n) + d(g\bar{y},gy_n)] + \beta M((\bar{x},\bar{y}),(x_n,y_n)) \\ &\quad + \frac{\gamma}{2}[d(g\bar{x},F(\bar{x},\bar{y})) + d(gx_n,F(x_n,y_n)) + d(g\bar{y},F(\bar{y},\bar{x})) + d(gy_n,F(y_n,x_n))] \\ &\quad + \frac{\delta}{2}[d(g\bar{x},F(x_n,y_n)) + d(g\bar{y},F(y_n,x_n)) + d(gx_n,F(\bar{x},\bar{y})) + d(gy_n,F(\bar{y},\bar{x}))] \\ &\le \frac{\alpha}{2}[d(g\bar{x},gx_n) + d(g\bar{y},gy_n)] \\ &\quad + \beta d(g\bar{x},F(\bar{x},\bar{y}))\,\frac{2 + d(gx_n,gx_{n+1}) + d(gy_n,gy_{n+1})}{2 + d(g\bar{x},gx_n) + d(g\bar{y},gy_n)} \\ &\quad + \frac{\gamma}{2}[d(g\bar{x},F(\bar{x},\bar{y})) + d(gx_n,gx_{n+1}) + d(g\bar{y},F(\bar{y},\bar{x})) + d(gy_n,gy_{n+1})] \\ &\quad + \frac{\delta}{2}[d(g\bar{x},gx_{n+1}) + d(g\bar{y},gy_{n+1}) + d(gx_n,F(\bar{x},\bar{y})) + d(gy_n,F(\bar{y},\bar{x}))]. \end{aligned}$$

Then, from (3.13), we get:

$$\begin{aligned} &d(F(\bar{x},\bar{y}),g\bar{x})) \\ &\le d(gx_{n+1},g\bar{x}) + \frac{\alpha}{2}[d(g\bar{x},gx_n) + d(g\bar{y},gy_n)] \\ &\quad + \beta d(g\bar{x},F(\bar{x},\bar{y}))\,\frac{2 + d(gx_n,gx_{n+1}) + d(gy_n,gy_{n+1})}{2 + d(g\bar{x},gx_n) + d(g\bar{y},gy_n)} \\ &\quad + \frac{\gamma}{2}[d(g\bar{x},F(\bar{x},\bar{y})) + d(gx_n,gx_{n+1}) + d(g\bar{y},F(\bar{y},\bar{x})) + d(gy_n,gy_{n+1})] \\ &\quad + \frac{\delta}{2}[d(g\bar{x},gx_{n+1}) + d(g\bar{y},gy_{n+1}) + d(gx_n,F(\bar{x},\bar{y})) + d(gy_n,F(\bar{y},\bar{x}))]. \end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$\begin{aligned} d((F(\bar{x},\bar{y}),g\bar{x})) &\le \beta d(g\bar{x},F(\bar{x},\bar{y})) \\ &\quad + \frac{\gamma+\delta}{2}[d(g\bar{x},F(\bar{x},\bar{y})) + d(g\bar{y},F(\bar{y},\bar{x}))]. \end{aligned} \tag{3.14}$$

Similarly,

$$d(g\bar{y}, F(\bar{y}, \bar{x})) \leq \beta d(g\bar{y}, F(\bar{y}, \bar{x})) + \frac{\gamma+\delta}{2}[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x}))]. \tag{3.15}$$

Adding (3.14) and (3.15), we have

$$\begin{aligned} &d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x})) \\ &\leq (\beta+\gamma+\delta)[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x}))] \\ &\leq (\alpha+\beta+2\gamma+2\delta)[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x}))]. \end{aligned}$$

Since $0 \leq \alpha+\beta+2\gamma+2\delta < 1$, we obtain $d(F(\bar{x}, \bar{y}), g\bar{x}) = 0$ and $d(g\bar{y}, F(\bar{y}, \bar{x}))$, i.e., $F(\bar{x}, \bar{y}) = g\bar{x}$ and $F(\bar{y}, \bar{x}) = g\bar{y}$. This completes the proof of the theorem. □

Now we shall prove a uniqueness theorem for the coupled coincidence point. Note that, if $(X, \preceq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, \quad (u, v) \preceq (x, y) \Leftrightarrow gx \preceq gu,\ gy \preceq gv.$$

Theorem 3.7. *Assume that*

$$\forall (x, y), (x^*, y^*) \in X \times X,\ \exists\, (u, v) \in X \times X \textit{ such that} \tag{3.16}$$

$(F(u, v), F(v, u))$ is comparable to $(F(\bar{x}, \bar{y}), F(\bar{y}, \bar{x}))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then F and g have unique coupled coincidence point that is there exists a unique $(x, y) \in X \times X$ such that $g\bar{x} = F(\bar{x}, \bar{y})$ and $g\bar{y} = F(\bar{y}, \bar{x})$, $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$.

Adding (3.16) *to the hypotheses of Theorem* 3.1*, we obtain the uniqueness of the coupled coincidence point of F and g.*

Proof. From Theorem 3.1 we know that there exists the set of coupled coincidence point of F and g is non empty. Suppose that $(g\bar{x}, g\bar{y})$ and (gx^*, gy^*) are coupled coincidence point of F and g, that is $g\bar{x} = F(\bar{x}, \bar{y})$ and $g\bar{y} = F(\bar{y}, \bar{x})$, $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$ which is obtained as $g\bar{x} = \lim_{n\to\infty} F^n(x_0, y_0)$ and $g\bar{y} = \lim_{n\to\infty} F^n(y_0, x_0)$. Then we have to show that

$$d(g\bar{x}, gx^*) + d(g\bar{y}, gy^*) = 0. \tag{3.17}$$

implies that $g\bar{x} = gx^*$ and $g\bar{y} = gy^*$. We distinguish two cases.

Case I: $(F(\bar{x}, \bar{y}), F(\bar{y}, \bar{x}))$ is comparable with $(F(x^*, y^*), F(y^*, x^*))$ with respect to the ordering in $X \times X$. Let, e.g., $g\bar{x} \preceq gx^*$ and $g\bar{y} \preceq gy^*$. Then, we can apply the contractive condition (3.1) to obtain

$$\begin{aligned} d(g\bar{x}, gx^*) &= d(F(\bar{x},\bar{y}), F(x^*,y^*)) \\ &\leq \frac{\alpha}{2}[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)] + \delta[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)], \end{aligned}$$

and

$$\begin{aligned} d(g\bar{y}, gy^*) &= d(F(\bar{y},\bar{x}), F(y^*,x^*)) = d(F(y^*,x^*), F(\bar{y},\bar{x})) \\ &\leq \frac{\alpha}{2}[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)] + \delta[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)]. \end{aligned}$$

Adding up, we get that

$$d(g\bar{x},gx^*) + d(g\bar{y},gy^*) \leq (\alpha + 2\delta)[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)].$$

Since $0 \leq \alpha + 2\delta < 1$, (3.17) holds.

Case II: $(F(\bar{x},\bar{y}), F(\bar{y},\bar{x}))$ is not comparable with $(F(x^*,y^*), F(y^*,x^*))$. In this case, by assumption there exists $(u,v) \in X \times X$ that is comparable both to $(F(\bar{x},\bar{y}), F(\bar{y},\bar{x}))$ and $(F(x^*,y^*), F(y^*,x^*))$. Then, for all $n \in \mathbb{N}$, $(F^n(u,v), F^n(v,u))$ is comparable both to $(F^n(\bar{x},\bar{y}), F^n(\bar{y},\bar{x})) = (g\bar{x}, g\bar{y})$ and $(F^n(x^*,y^*), F^n(y^*,x^*)) = (gx^*, gy^*)$. We have

$$\begin{aligned} &d(g\bar{x},gx^*) + d(g\bar{y},gy^*) \\ &= d(F^n(\bar{x},\bar{y}), F^n(x^*,y^*)) + d(F^n(\bar{y},\bar{x}), F^n(y^*,x^*)) \\ &\leq d(F^n(\bar{x},\bar{y}), F^n(u,v)) + d(F^n(u,v), F^n(x^*,y^*)) \\ &\quad + d(F^n(\bar{y},\bar{x}), F^n(v,u)) + d(F^n(v,u), F^n(y^*,x^*)) \\ &\leq (\alpha^n + 2\delta^n)[d(g\bar{x},u) + d(g\bar{y},v) + d(gx^*,u) + d(gy^*,v)]. \end{aligned}$$

Since $0 < \alpha, \delta < 1$, (3.17) holds. We deduce that in all cases (3.17) holds. Thus we obtained $(g\bar{x}, g\bar{y}) = (gx^*, gy^*)$. This implies that $g\bar{x} = gx^*$ and $g\bar{y} = gy^*$ and the uniqueness of the coupled coincidence point of F and g is proved. □

If $g = I$, the identity mapping in Theorem 3.7, then we deduce the following corollary.

Corollary 3.8. *In addition to the hypotheses of Corollary* 3.4, $\forall(x,y),(x^*,y^*) \in X \times X$, $\exists\,(u,v) \in X \times X$ *such that* $(F(u,v), F(v,u))$ *is comparable to* $(F(\bar{x},\bar{y}), F(\bar{y},\bar{x}))$ *and* $(F(x^*,y^*), F(y^*,x^*))$*. Then F has a unique coupled fixed point that is there exists a unique* $(x,y) \in X \times X$ *such that* $\bar{x} = F(\bar{x},\bar{y})$ *and* $\bar{y} = F(\bar{y},\bar{x})$, $x^* = F(x^*,y^*)$ *and* $y^* = F(y^*,x^*)$.

Theorem 3.9. *In addition to the hypotheses of Theorem* 3.1 *(resp. Theorem* 3.6*), suppose that x_0, y_0 in X are comparable. Then* $g\bar{x} = g\bar{y}$.

Proof. Suppose that $gx_0 \preceq gy_0$. We claim that

$$gx_n \preceq gy_n, \ \forall\, n \in \mathbb{N}. \tag{3.18}$$

From the monotone property of F and g, we have

$$gx_1 = F(x_0, y_0) \preceq F(y_0, y_0) \preceq F(y_0, x_0) = gy_1.$$

Assume that $gx_n \preceq gy_n$ for some n. Now,

$$\begin{aligned} gx_{n+1} &= F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)) \\ &= F(x_n, y_n) \preceq F(y_n, y_n) \preceq F(y_n, x_n) = gy_{n+1}. \end{aligned}$$

Hence, (3.18) holds.

Now, using (3.18) and the contractive condition, we get

$$\begin{aligned} &d(g\bar{x}, g\bar{y}) \\ &\leq d(g\bar{x}, gx_{n+1}) + d(gx_{n+1}, gy_{n+1}) + d(gy_{n+1}, g\bar{y}) \\ &= d(g\bar{x}, gx_{n+1}) + d(F(y_n, x_n), F(x_n, y_n)) + d(gy_{n+1}, g\bar{y}) \\ &\leq d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) + \beta M((y_n, x_n), (x_n, y_n)) \\ &\quad + \frac{\gamma}{2}[d(gx_n, F(x_n, y_n)) + d(gy_n, F(y_n, x_n)) + d(gy_n, F(y_n, x_n)) \\ &\quad + d(gx_n, F(x_n, y_n)))] \\ &\quad + \frac{\delta}{2}[d(gx_n, F(y_n, x_n)) + d(gy_n, F(x_n, y_n)) + d(gy_n, F(x_n, y_n)) \\ &\quad + d(gx_n, F(y_n, x_n))] \\ &\leq d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\ &\quad + \beta d(gy_n, gy_{n+1}) \frac{2 + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2 + 2d(gy_n, gx_n)} \\ &\quad + \gamma[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] + \delta[d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1})] \\ &\leq d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\ &\quad + \beta d(gy_n, gy_{n+1})[2 + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\ &\quad + \gamma[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] + \delta[d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1})]. \end{aligned}$$

Passing to the limit as $n \to \infty$, we get that

$$d(g\bar{x}, g\bar{y}) \leq (\alpha + 2\delta) d(g\bar{x}, g\bar{y}).$$

Since $0 \leq \alpha + 2\delta < 1$, this implies that $d(g\bar{x}, g\bar{y}) = 0$, i.e., $g\bar{x} = g\bar{y}$. This completes the proof of the theorem. □

We illustrate our results by the following example which also distinguishes these result from the known ones.

Example 3.10. Let $X = [0, +\infty)$. Then $(X, \preceq)$ is a partially ordered set with the standard metric of real numbers. Let $d(x, y) = |x - y|$ for $x, y \in X$. Define $g : X \to X$ by $g(x) = \frac{x^2}{8\min\{\alpha,\beta,\gamma,\delta\}}$ with $0 < \alpha + \beta + 2\gamma + 2\delta < 1$, and the continuous mapping $F : X \times X \to X$ given by

$$F(x, y) = \begin{cases} \frac{x^2-y^2}{8}, & if \quad x \geq y \\ 0, & if \quad x < y. \end{cases}$$

Denote $\min\{\alpha, \beta, \gamma, \delta\} = \lambda$. By routine calculations, the reader can easily verify that the following assumptions hold:

(i) F and g is continuous and $g(X)$ is closed;
(ii) $F(X \times X) \subseteq g(X)$;

Here, we show only that F has the g-monotone property and F and g are compatible and condition (3.1) in Theorem 3.1 is satisfied for all real numbers.

- Condition (3.1) holds, for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$ or $g(x) \succeq g(u)$ and $g(y) \succeq g(v)$.

Let $\alpha, \beta, \gamma, \delta > 0$ be nonnegative numbers satisfying $\frac{1}{2} \leq \alpha < 1$ with $\alpha+\beta+2\gamma+2\delta < 1$, and denote by L and R, respectively, the left-hand and right-hand side of contraction condition (3.1).

Let $x, u, y, v \in [0, 1]$ and hence $gx \preceq gu$, $gy \preceq gv$ and using that $x + y \leq 1$ and $u + v \leq 1$. We get that in each case

$$L \leq \frac{x^2 - y^2}{8} \leq \left|\frac{x - y}{8}\right| \leq \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] \leq R.$$

Consider the following four cases.

Case I: If $x \leq y$ and $u \leq v$, and we have For example, if $0 \preceq u \preceq x \preceq y \preceq v \preceq 1$ then

$$L = d(F(x, y), F(u, v)) = d\left(\frac{x^2 - y^2}{8}, \frac{u^2 - v^2}{8}\right) = d(0, 0) = 0 \leq R$$

we get $L = 0$ and the contractive condition is trivially satisfied.

Case II: If $x > y$ and $u \leq v$, (and hence $y \leq u \leq v \leq x$), then we have

$$L = d(F(x, y), F(u, v)) = d\left(\frac{x^2 - y^2}{8}, 0\right) = \frac{x^2 - y^2}{8} \leq \left|\frac{x - y}{8}\right| \leq R$$

Case III: If $x > y$ and $u > v$, without loss of generality we suppose $x = u > 1$, and then we have

$$\begin{aligned} L &= d\left(\frac{x^2-y^2}{8}, \frac{u^2-v^2}{8}\right) = \frac{x^2-y^2}{8} - \frac{x^2-v^2}{8} \\ &= \left|\frac{x^2-y^2-x^2+v^2}{8}\right| = \left|\frac{v^2-y^2}{8}\right| \le \left|\frac{v-y}{8}\right| \\ &\le \frac{\alpha}{2}[d(gx,gu)+d(gy,gv)] \le R, \end{aligned}$$

since $\frac{1}{2} \le \alpha < 1$.

Case IV: If $x \le y$ and $u > v$, (and hence $x \le v < u \le y$).

Then also we get $L \le R$ and obviously condition is satisfied.

$$\begin{aligned} L = d(F(x,y),F(u,v)) = d\left(0, \frac{u^2-v^2}{8},\right) &= \frac{u^2-v^2}{8} \le \left|\frac{u-v}{8}\right| \\ &\le \frac{\alpha}{2}[d(gx,gu)+d(gy,gv)] \le R \end{aligned}$$

Thus condition (3.1) holds in all the cases. Hence by Theorem 3.1, F and g have a coupled coincidence point $(0,0) \in X \times X$. (Moreover, $(0,0)$ is a coupled common fixed point of F and g).

- Now we prove that F and g are compatible.

$$\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} g(x_n) = x \in X. \tag{3.19}$$

and

$$\lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} g(y_n) = y \in X. \tag{3.20}$$

We have to prove that

$$\begin{cases} \lim_{n\to\infty} d(gF(x_n,y_n), F(gx_n,gy_n)) = 0, \\ \lim_{n\to\infty} d(gF(y_n,x_n), F(gy_n,gx_n)) = 0. \end{cases} \tag{3.21}$$

We claim that $(x,y) = (0,0)$. From (3.19) and (3.20) and the definition of F, we get

$$\lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} g(y_n) = y.$$

From the definition of g, this implies that

$$\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} \frac{y_n^2}{8} = \frac{y}{8}.$$

that is,

$$\lim_{n\to\infty} y_n = y \tag{3.22}$$

Now, using (3.19) and the definition of F, we obtain

$$\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} \frac{(x_n^2-y_n^2)}{8} = \lim_{n\to\infty} \frac{x_n^2}{8} = x.$$

From (3.19) and the definition of g, we have

$$\lim_{n\to\infty} \frac{x_n^2}{8\lambda} = x.$$

Uniqueness of the limit, implies that

$$8x = 8\lambda x,$$

that is,

$$(1-\lambda)x = 0.$$

Since $0 < \lambda < 1$, Then, $x = 0$. Similarly, one can also show that $y = 0$. Then, we have

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = 0. \qquad (3.23)$$

Now, (3.21) follows immediately from (3.23), the continuity of F, the continuity of g and the continuity of d. Thus we proved that F and g are compatible.

Also, for all $x, y \in X$ F and g do not commute. Hence, Theorem 2.1 of Lakshmikantham and Cirić in [11] cannot be applied to this example.

Example 3.11. Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ and order. we define a partial order " $\preceq$ " on X as $x \preceq y$ if and only if $x \leq y$ for all $x, y \in X$. Consider the mapping $F(x, y) = \frac{x^2-2y^2}{12}$ and $g(x) = x^2$. All the condition of Theorems 3.1 are satisfied and now we will prove that F and g are compatible.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = p \text{ and } \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = q.$$

Then $\frac{p-2q}{4} = p$ and $\frac{q-2p}{4} = q$, where from it follows that $p = q = 0$. Then

$$d(gF(x_n, y_n), F(gx_n, gy_n)) = \left| \left(\frac{x_n^2 - 2y_n^2}{12}\right)^2 - \frac{x_n^4 - 2y_n^4}{12} \right| \to 0, \text{ as } n \to \infty$$

and similarly

$$d(gF(y_n, x_n), F(gy_n, gx_n)) = \left| \left(\frac{y_n^2 - 2x_n^2}{12}\right)^2 - \frac{y_n^4 - 2x_n^4}{12} \right| \to 0, \text{ as } n \to \infty.$$

Then it is clear that F and g are compatible. Since

$$g(F(x, y)) = g(\frac{x^2 - 2y^2}{12}) = \left(\frac{x^2 - 2y^2}{12}\right)^2 \neq \frac{x^4 - 2y^4}{12} = F(g(x), g(y))$$

for all $x, y \in X$, we have F and g are not commute.

Denote L and R respectively left and right hand side of contractive condition

Contractive condition (3.1) is satisfied with $\alpha = \frac{1}{3}, \beta = 0, \gamma = \frac{1}{6}, \delta = \frac{1}{12}$ and $\alpha + \beta + 2\gamma + 2\delta < 1$ which follows from

$$\begin{aligned}
L = d(F(x,y), F(u,v)) &= \left|\frac{x^2 - 2y^2}{12} - \frac{u^2 - 2v^2}{12}\right| \\
&= \frac{1}{12}\left|(x^2 - u^2) - 2(y^2 - v^2)\right| \\
&\leq \frac{1}{12}(d(gx, gu) + 2d(gy, gv)) \leq \frac{4}{12}\frac{(d(gx, gu) + d(gy, gv))}{2} \\
&\leq \frac{1}{3}\frac{(d(gx, gu) + d(gy, gv))}{2} \leq \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] \\
&\leq \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] + \beta M((gx, gy), (gu, gv)) \\
&\quad + \frac{\gamma}{2}[d(gx, F(x,y)) + d(gu, F(u,v)) + d(gy, F(y,x)) + d(gv, F(v,u))] \\
&\quad + \frac{\delta}{2}[d(gx, F(u,v)) + d(gy, F(v,u)) + d(gu, F(x,y)) + d(gv, F(y,x))] \\
&\leq R.
\end{aligned}$$

for all $x, y, u, v \in X$ for which is $gx \preceq gu$ and $gy \preceq gv$.

This shows that all the hypothesis of Theorem 3.1 and 3.6 are satisfied. Therefore, we conclude that F and g have a coupled coincidence point in X. This coupled fixed point is $(x, y) = (0, 0)$, that is, $g(0) = F(0, 0) = 0$.

4. Result-II

In this section, we are gearing up new results for w^*-compatible mapping with completeness of g in X in the underlying space.

Theorem 4.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the g-monotone property on X and satisfying*

$$\begin{aligned}
&d(F(x,y), F(u,v)) \qquad (4.1)\\
&\leq \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] + \beta N((x,y), (u,v)) \\
&\quad + \frac{\gamma}{2}[d(gx, F(x,y)) + d(gu, F(u,v)) + d(gy, F(y,x)) + d(gv, F(v,u))],
\end{aligned}$$

for all $(x, y), (u, v) \in X \times X$ with $gx \preceq gu$ and $gy \preceq gv$, when $D_1 = d(gx, F(u,v)) + d(gu, F(x,y)) \neq 0$ and $D_2 = d(gy, F(v,u)) + d(gv, F(y,x)) \neq$

0, *where*

$$N((x,y),(u,v)) \tag{4.2}$$
$$= \min\left\{\frac{d^2(gx,F(u,v))+d^2(gu,F(x,y))}{d(gx,F(u,v))+d(gu,F(x,y))}, \frac{d^2(gy,F(v,u))+d^2(gv,F(y,x))}{d(gy,F(v,u))+d(gv,F(y,x))}\right\}$$

and $\alpha,\beta,\gamma \geq 0$ *with* $\alpha+2\beta+2\gamma<1$. *Further,*

$$d(F(x,y),F(u,v)) = 0 \quad \text{if} \quad D_1 = 0 \text{ or } D_2 = 0. \tag{4.3}$$

Further, suppose

(1) $F(X\times X)\subseteq g(X)$;
(2) $g(X)$ *is a complete subspace of* X.

Also, suppose that X *has the following properties:*

(i) *if a non-decreasing sequence* $\{x_n\}$ *in* X *is such that* $x_n \to x$, *then* $x_n \preceq x$ *for all* $n\in\mathbb{N}$,
(ii) *if a non-decreasing sequence* $\{y_n\}$ *in* X *is such that* $y_n \to y$, *then* $y_n \preceq y$ *for all* $n\in\mathbb{N}$.

Then there exist $x_0,y_0\in X$ *such that*

$$gx_0 \preceq F(x_0,y_0) \quad \text{and} \quad gy_0 \preceq F(y_0,x_0). \tag{4.4}$$

Then, F *and* g *have a coupled coincidence point* $(\bar{x},\bar{y})\in X\times X$.

Proof. Following the proof in the line of Theorem 3.1, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$F(x_n,y_n) = g(x_{n+1}) \text{ and } F(y_n,x_n) = g(y_{n+1})$$

for all $n\in\{0,1,2,\dots\}$ and

$$g(x_n)\preceq g(x_{n+1}) \text{ and } g(y_n)\preceq g(y_{n+1}). \tag{4.5}$$

By using g-monotone property of F and g, we have $g(x_{n+1})\preceq g(x_{n+2})$ and $g(y_{n+1})\preceq g(y_{n+2})$. In general, we conclude that

$$g(x_n)\preceq g(x_{n+1}) \text{ and } g(y_n)\preceq g(y_{n+1}) \text{ for all } n\geq 0.$$

Thus by the mathematical induction, we conclude that (3.3) holds for all $n\geq 0$. We check easily that

$$g(x_0)\preceq g(x_1)\preceq g(x_2)\preceq\cdots\leq g(x_n)\preceq\cdots \tag{4.6}$$

and

$$g(y_0)\preceq g(y_1)\preceq g(y_2)\preceq\cdots\preceq g(y_n)\preceq\cdots. \tag{4.7}$$

If $g(x_{n+1})=g(x_n)$ and $g(y_{n+1})=g(y_n)$ for some n, then $F(x_n,y_n)=g(x_n)$ and $F(y_n,x_n)=g(y_n)$, hence (gx_n,gy_n) is a coupled coincidence point of F and g. Suppose, further, that $g(x_{n+1})\neq g(x_n)$ and $g(y_{n+1})\neq g(y_n)$ for some $n\in\mathbb{N}_0$.

Now, we claim that, for $n \in \mathbb{N}$ for contractive condition,

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^n [d(gx_1, gx_0) + d(gy_1, gy_0)]. \tag{4.8}$$

Indeed, for $n = 1$, consider the following possibilities.

Case I: $gx_0 \neq gx_2$ and $gy_0 \neq gy_2$. Then $d(gx_1, F(x_0, y_0)) + d(gx_0, F(x_1, y_1)) \neq 0$ and $d(gy_1, F(y_0, x_0)) + d(gy_0, F(y_1, x_1)) \neq 0$. Hence, using $gx_1 \preceq gx_0$, $gy_1 \preceq gy_0$ and (4.1), we get:

$$\begin{aligned}
d(gx_2, gx_1) &= d(F(x_1, y_1), F(x_0, y_0)) \\
&\leq \frac{\alpha}{2}[d(gx_1, gx_0) + d(gy_1, gy_0)] + \beta N(x_1, y_1), (x_0, y_0)) \\
&\quad + \frac{\gamma}{2}[d(gx_1, F(x_1, y_1)) + d(gx_0, F(x_0, y_0)) + d(gy_1, F(y_1, x_1)) \\
&\quad + d(gy_0, F(y_0, x_0))] \\
&\leq \frac{\alpha}{2}[d(gx_0, gx_1) + d(gy_0, gy_1)] + \beta \frac{d^2(gx_1, F(x_0, y_0)) + d^2(gx_0, F(x_1, y_1))}{d(gx_1, F(x_0, y_0)) + d(gx_0, F(x_1, y_1))} \\
&\quad + \frac{\gamma}{2}[d(gx_1, gx_2) + d(gx_0, gx_1) + d(gy_1, gy_2) + d(gy_0, gy_1)] \\
&\leq \frac{\alpha}{2}[d(gx_0, gx_1) + d(gy_0, gy_1)] + \beta[d(gx_0, gx_1) + d(gx_1, gx_2)] \\
&\quad + \frac{\gamma}{2}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gx_1, gx_2) + d(gy_1, gy_2)].
\end{aligned} \tag{4.9}$$

Similarly, using that

$$d(gy_2, gy_1) = d(F(y_1, x_1), \quad F(y_0, x_0)) = d(F(y_0, x_0), F(y_1, x_1))$$

and

$$\begin{aligned}
N((y_1, x_1), (y_0, x_0)) &\leq \frac{d^2(gy_1, F(y_0, x_0) + d^2(gy_0, F(y_1, x_1))}{d(gy_1, F(y_0, x_0)) + d(gy_0, F(y_1, x_1))} \\
&= d(gy_0, gy_2) \leq d(gy_0, gy_1) + d(gy_1, gy_2),
\end{aligned}$$

we get

$$\begin{aligned}
&d(gy_2, gy_1) \\
&\leq \frac{\alpha}{2}[d(gx_0, gx_1) + d(gy_0, gy_1)] + \beta[d(gy_0, gy_1) + d(gy_1, gy_2)] \\
&\quad + \frac{\gamma}{2}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gx_1, gx_2) + d(gy_1, gy_2)].
\end{aligned} \tag{4.10}$$

Adding (4.9) and (4.10), we have

$$d(gx_2, gx_1) + d(gy_2, gy_1) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right) [d(gx_0, gx_1) + d(gy_0, gy_1)]. \tag{4.11}$$

Case II: $gx_0 = gx_2$ and $gy_0 \neq gy_2$. The first equality implies that $d(gx_1, F(x_0, y_0)) + d(gx_0, F(x_1, y_1)) = 0$, and hence $d(gx_1, gx_2) = d(F(x_0, y_0), F(x_1, y_1)) = 0$, by (4.3). This means that $gx_0 = gx_1 = gx_2$. From $gy_0 \neq gy_2$, as in the first case, we get that (4.10) holds true. As a consequence

$$d(gy_1, gy_2) \leq \frac{\frac{\alpha}{2} + \beta + \frac{\gamma}{2}}{1 - \beta - \frac{\gamma}{2}} d(gy_0, gy_1) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(gy_0, gy_1),$$

since $\frac{\frac{\alpha}{2}+\beta+\frac{\gamma}{2}}{1-\beta-\frac{\gamma}{2}} \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$. But then $d(gx_0, gx_1) = d(gx_1, gx_2) = 0$ implies that (4.11) holds.

The case $gx_0 \neq gx_2$ and $gy_0 = gy_2$ is treated analogously.

Case III: If $gx_0 = gx_2$ and $gy_0 = gy_2$, then $d(gx_1, F(x_0, y_0)) + d(gx_0, F(x_1, y_1)) = 0$ and $d(gy_1, F(y_0, x_0)) + d(gy_0, F(y_1, x_1)) = 0$. Hence, (4.3) implies that $gx_1 = gx_2 = gx_3$ and $gy_1 = gy_2 = gy_3$, and so (4.11) holds trivially. Thus, (4.8) holds for $n = 1$. In a similar way, proceeding by induction, if we assume that (4.8) holds, we get that

$$\begin{aligned}
&d(gx_{n+2}, gx_{n+1}) + d(gy_{n+2}, gy_{n+1}) \\
&\leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right) [d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] \\
&\leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)^{n+1} [d(gx_0, gx_1) + d(gy_0, gy_1)].
\end{aligned}$$

Hence, by induction, (4.8) is proved.

Using similar arguments as in the proof of Theorem 3.1, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. By completeness of $g(X)$, there exists $gx, gy \in g(X)$ such that $gx_n \to gx$ and $gy_n \to gy$ as $n \to \infty$.

Since $\{gx_n\}$ and $\{gy_n\}$ are nondecreasing, using the conditions (i) and (ii), we have $gx_n \preceq gx$ and $gy_n \preceq gy$ for all $n \geq 0$. If $gx_n = gx$ and $gy_n = gy$ for some $n \geq 0$, then $gx = gx_n \preceq gx_{n+1} \preceq gx = gx_n$ and $gy \preceq gy_n \preceq gy_{n+1} = gy$, which implies that

$$gx_n = gx_{n+1} = F(x_n, y_n) \text{ and } gy_n = gy_{n+1} = F(y_n, x_n),$$

that is, (x_n, y_n) is a coupled coincidence point of F and g. Then, we suppose that $(gx_n, gy_n) \neq (gx, gy)$ for all $n \geq 0$.

Now we prove that $F(x, y) = gx$ and $F(y, x) = gy$. For this, Now, suppose that (i-ii) holds. Since $\{gx_n\}$ and $\{gy_n\}$ are non-decreasing and $g(x_n) \to x$, and $g(y_n) \to y$, by assumption (i-ii), we have $g(x_n) \preceq g(x)$ and $g(y_n) \preceq g(y)$

for all n. Then, we get

$$\begin{aligned} d(gx, F(x,y)) &\le d(gx, gx_{n+1}) + d(gx_{n+1}, F(x,y)) \\ &\le d(gx, gx_{n+1}) + d(gx_{n+1}, F(x_n, y_n)) + d(F(x_n, y_n), F(x,y)) \\ &\to d(gx, gx) + d(gx_{n+1}, gx_{n+1}) + d(F(x,y), F(x,y)) \\ &= 0 \text{ as } n \to \infty. \end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality, we get $d(g(x), F(x,y)) = 0$. Hence $g(x) = F(x,y)$. Similarly, one can show that $g(y) = F(y,x)$. Thus F and g have a coupled coincidence point. This completes the proof of the theorem. □

Example 4.2. Let $X = \mathbb{R}$ and defined partial ordered relation by $x \preceq y$. Define a mapping $d : X \times X \to [0,\infty)$ by $d(x,y) = |x-y|$ for all $x, y \in X$. Let mapping $F : X \times X \to X$ and $g : X \to X$ be defined by

$$F(x,y) = 0.5$$

and

$$g(x) = x - 0.5$$

for all $x \in X$. Then, $g(X) = [-0.5, \infty)$ is a complete subspace of X.

By simple checking, we see that F and g satisfy the contractive condition (4.1) for given value

Denote L and R respectively left and right hand side of contractive condition

$$\begin{aligned} L &= d(F(x,y), F(u,v)) \\ &\le d(0.5, 0.5) = 0 \\ &\le \frac{\alpha}{2}[d(x-0.5, y-0.5) + d(u-0.5, v-0.5)] \\ &\le \frac{\alpha}{2}[|x-u| + |y-v|] \le \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] \le R \end{aligned}$$

where $0 < \alpha < 1$ and also F satisfy the g-monotone property. Moreover, g and F are continuous For example there exists two points $0, 0.5 \in X$ such that

$$g(0) = -0.5 \le 0.5 = F(0, 0.5)$$

and

$$g(0.5) = 0 \le 0.5 = F(0.5, 0).$$

This shows that F has the g-monotone property. Hence this example does not satisfy the Theorem 2.3 of Sintunavarat and Kumam [27] and there is no function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ and $\lim_{r \to t+} \varphi < t$ for each $t > 0$. Since

$$g(F(x,y)) = g(0.5) = 0 \ne 0.5 = F(g(x), g(y))$$

for all $x, y \in X$, we have F and g are not commutative. Therefore, Theorem 2.1 of Lakshmikantham and Ćirić [11] not satisfied for this example.

This, shows that all the hypothesis of Theorem 4.1 is satisfies. Therefore, we conclude that F and g have a coupled coincidence point in X. This coupled fixed point is $(x, y) = (1, 1)$, that is, $g(1) = F(1, 1) = 0.5$.

In the next theorem, we will substitute the continuity hypothesis on F and g by an additional property satisfied by the space $(X, d, \preceq)$.

Theorem 4.3. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings having the g-monotone property. Assume that there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$\begin{aligned} & d(F(x,y), F(u,v)) \\ & \leq \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] + \beta N((x,y),(u,v)) \\ & \quad + \frac{\gamma}{2}[d(gx, F(x,y)) + d(gu, F(u,v)) + d(gy, F(y,x)) + d(gv, F(v,u))], \end{aligned}$$

for all $(x, y), (u, v) \in X \times X$ with $gx \preceq gu$ and $gy \preceq gv$, when $D_1 = d(gx, F(u, v)) + d(gu, F(x, y)) \neq 0$ and $D_2 = d(gy, F(v, u)) + d(gv, F(y, x)) \neq 0$, where

$$\begin{aligned} & N((x,y),(u,v)) \\ & = \min\left\{ \frac{d^2(gx, F(u,v)) + d^2(gu, F(x,y))}{d(gx, F(u,v)) + d(gu, F(x,y))}, \frac{d^2(gy, F(v,u)) + d^2(gv, F(y,x))}{d(gy, F(v,u)) + d(gv, F(y,x))} \right\}. \end{aligned}$$

Further, $d(F(x, y), F(u, v)) = 0$ if $D_1 = 0$ or $D_2 = 0$.

Suppose that there exist $x_0, y_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0) \quad \text{and} \quad gy_0 \preceq F(y_0, x_0).$$

Finally, assume that X has the following properties:

- (i) *if a nondecreasing sequence $\{gx_n\}$ in X converges to $x \in X$, then $gx_n \preceq gx$ for all n,*
- (ii) *if a nondecreasing sequence $\{gy_n\}$ in X converges to $y \in X$, then $gy_n \preceq gy$ for all n.*

Then, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Following the proof of Theorem 4.1, we only have to show that $(g\bar{x}, g\bar{y})$ is a coupled coincidence point of F and g. Suppose this is not the case, i.e., $F(\bar{x}, \bar{y}) \neq g\bar{x}$ or $F(\bar{y}, \bar{x}) \neq g\bar{y}$ (e.g., let the first one of these holds). We have

$$\begin{aligned} d(F(\bar{x}, \bar{y}), g\bar{x}) & \leq d(F(\bar{x}, \bar{y}), gx_{n+1}) + d(gx_{n+1}, g\bar{x}) \\ & = d(F(\bar{x}, \bar{y}), F(x_n, y_n)) + d(gx_{n+1}, g\bar{x}). \end{aligned} \tag{4.12}$$

Since the nondecreasing sequence $\{gx_n\}$ converges and $g(x_n) \to \bar{x}$ and the nonincreasing sequence $\{gy_n\}$ converges and $g(y_n) \to \bar{y}$, by (i)–(ii), we have:

$$g\bar{x} \preceq gx_n \quad \text{and} \quad g\bar{y} \preceq gy_n, \quad \forall\, n.$$

Now, from the contractive condition, we have:

$$\begin{aligned}
&d(F(\bar{x},\bar{y}), F(x_n,y_n))\\
&\le \frac{\alpha}{2}[d(g\bar{x}, gx_n) + d(g\bar{y}, gy_n)] + \beta N((\bar{x},\bar{y}),(x_n,y_n))\\
&\quad + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x},\bar{y})) + d(gx_n, F(x_n,y_n)) + d(g\bar{y}, F(\bar{y},\bar{x})) + d(gy_n, F(y_n,x_n))]\\
&\le \frac{\alpha}{2}[d(g\bar{x}, gx_n) + d(g\bar{y}, gy_n)] + \beta\frac{d^2(g\bar{x}, gx_{n+1}) + d^2(gx_n, F(\bar{x},\bar{y}))}{d(g\bar{x}, gx_{n+1}) + d(gx_n, F(\bar{x},\bar{y}))}\\
&\quad + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x},\bar{y})) + d(gx_n, gx_{n+1}) + d(g\bar{y}, F(\bar{y},\bar{x})) + d(gy_n, gy_{n+1})].
\end{aligned}$$

We note that the case $d(g\bar{x}, gx_{n+1}) + d(gx_n, F(\bar{x},\bar{y})) = 0$ is impossible, since otherwise the condition (4.3) would imply $g\bar{x} = F(\bar{x},\bar{y})$, which is excluded. Then, from (4.12), we get:

$$\begin{aligned}
&d(F(\bar{x},\bar{y}), g\bar{x})\\
&\le d(gx_{n+1}, g\bar{x}) + \frac{\alpha}{2}[d(g\bar{x}, gx_n) + d(g\bar{y}, gy_n)]\\
&\quad + \beta\frac{d^2(g\bar{x}, gx_{n+1}) + d^2(gx_n, F(\bar{x},\bar{y}))}{d(g\bar{x}, gx_{n+1}) + d(gx_n, F(\bar{x},\bar{y}))}\\
&\quad + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x},\bar{y})) + d(gx_n, gx_{n+1}) + d(g\bar{y}, F(\bar{y},\bar{x})) + d(gy_n, gy_{n+1})].
\end{aligned}$$

Passing to the limit as $n \to \infty$ (and again using that $F(\bar{x},\bar{y}) \ne g\bar{x}$), we have

$$d(F(\bar{x},\bar{y}), g\bar{x}) \le \beta d(g\bar{x}, F(\bar{x},\bar{y})) + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x},\bar{y})) + d(g\bar{y}, F(\bar{y},\bar{x}))]. \quad (4.13)$$

Now, if $g\bar{y} = F(\bar{y},\bar{x})$, using that $\beta + \frac{\gamma}{2} < 1$, it follows immediately that $g\bar{x} = F(\bar{x},\bar{y})$, a contradiction. If this is not the case, we similarly get

$$d(g\bar{y}, F(\bar{y},\bar{x})) \le \beta d(g\bar{y}, F(\bar{y},\bar{x})) + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x},\bar{y})) + d(g\bar{y}, F(\bar{y},\bar{x}))]. \quad (4.14)$$

Adding (4.13) and (4.14), we have

$$\begin{aligned}
&d(g\bar{x}, F(\bar{x},\bar{y})) + d(g\bar{y}, F(\bar{y},\bar{x}))\\
&\le (\beta+\gamma)[d(g\bar{x}, F(\bar{x},\bar{y})) + d(g\bar{y}, F(\bar{y},\bar{x}))]\\
&\le (\alpha+2\beta+2\gamma)[d(g\bar{x}, F(\bar{x},\bar{y})) + d(g\bar{y}, F(\bar{y},\bar{x}))].
\end{aligned}$$

Since $0 \le \alpha+2\beta+2\gamma < 1$, we obtain $d(F(\bar{x},\bar{y}), g\bar{x}) = 0$ and $d(g\bar{y}, F(\bar{y},\bar{x})) = 0$, i.e., $F(\bar{x},\bar{y}) = g\bar{x}$ and $F(\bar{y},\bar{x}) = g\bar{y}$, again a contradiction. This completes the proof of the theorem. □

Theorem 4.4. *Assume that*

$$\forall (x,y),(x^*,y^*) \in X \times X,\ \exists\,(u,v) \in X \times X \text{ such that} \tag{4.15}$$

$(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. If F and g have unique coupled coincidence point, that is there exists a unique $(x,y) \in X \times X$ such that $gx = F(x,y)$ and $gy = F(y,x)$, $gx^* = F(x^*,y^*)$ and $gy^* = F(y^*,x^*)$.

Moreover, if F and g are w^*-compatible and unique coupled common fixed point in $X \times X$ then $x = gx = F(x,y)$ and $y = gy = F(y,x)$.

Adding (4.15) *to the hypotheses of Theorem* 4.1*, we obtain the uniqueness of the coupled coincidence and coupled common point of F and g.*

Proof. From Theorem 4.1 we know that there exists the set of coupled coincidence point of F and g is non empty. Suppose that $(g\bar{x},g\bar{y})$ and (gx^*,gy^*) are coupled coincidence point of F and g, that is $g\bar{x} = F(\bar{x},\bar{y})$ and $g\bar{y} = F(\bar{y},\bar{x})$, $gx^* = F(x^*,y^*)$ and $gy^* = F(y^*,x^*)$, which is obtained as $g\bar{x} = \lim_{n\to\infty} F^n(x_0,y_0)$ and $g\bar{y} = \lim_{n\to\infty} F^n(y_0,x_0)$. Let us prove that

$$d(g\bar{x},gx^*) + d(g\bar{y},gy^*) = 0. \tag{4.16}$$

We distinguish two cases.

Case I: $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ is comparable with $(F(x^*,y^*),F(y^*,x^*))$ with respect to the ordering in $X \times X$. Let, e.g., $g\bar{x} \preceq gx^*$ and $g\bar{y} \preceq gy^*$. Then, we can apply the contractive condition (4.1) to obtain

$$\begin{aligned} d(g\bar{x},gx^*) &= d(F(\bar{x},\bar{y}),F(x^*,y^*)) \\ &\le \frac{\alpha}{2}[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)] + \beta d(g\bar{x},gx^*), \end{aligned}$$

and

$$\begin{aligned} d(g\bar{y},gy^*) &= d(F(\bar{y},\bar{x}),F(y^*,x^*)) = d(F(y^*,x^*),F(\bar{y},\bar{x})) \\ &\le \frac{\alpha}{2}[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)] + \beta d(g\bar{y},gy^*). \end{aligned}$$

Adding up, we get that

$$d(g\bar{x},gx^*) + d(g\bar{y},gy^*) \le (\alpha+\beta)[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)].$$

Since $0 \le \alpha + \beta < 1$, (4.16) holds.

Case II: $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ is not comparable with $(F(x^*,y^*),F(y^*,x^*))$. In this case, By assumption there exists $(u,v) \in X \times X$ that is comparable both to $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then, for all $n \in \mathbb{N}$, $(F^n(u,v),F^n(v,u))$ is comparable both to $(F^n(\bar{x},\bar{y}),F^n(\bar{y},\bar{x})) = (g\bar{x},g\bar{y})$ and

$(F^n(x^*, y^*), F^n(y^*, x^*)) = (gx^*, gy^*)$. We have

$$\begin{aligned} &d(g\bar{x}, gx^*) + d(g\bar{y}, gy^*) \\ &= d(F^n(\bar{x}, \bar{y}), F^n(x^*, y^*)) + d(F^n(\bar{y}, \bar{x}), F^n(y^*, x^*)) \\ &\le d(F^n(\bar{x}, \bar{y}), F^n(u, v)) + d(F^n(u, v), F^n(x^*, y^*)) \\ &\quad + d(F^n(\bar{y}, \bar{x}), F^n(v, u)) + d(F^n(v, u), F^n(y^*, x^*)) \\ &\le (\alpha^n + \beta^n)[d(g\bar{x}, u) + d(g\bar{y}, v) + d(gx^*, u) + d(gy^*, v)]. \end{aligned}$$

Since $0 < \alpha, \beta < 1$, (4.16) holds.

We deduce that in all cases (4.16) holds. This implies that $(g\bar{x}, g\bar{y}) = (gx^*, gy^*)$ and the coupled coincidence point of F and g is proved. Note that if $(g\bar{x}, g\bar{y})$ is a coupled point of coincidence of F and g, then $(g\bar{y}, g\bar{x})$ is also a coupled points of coincidence of F and g. Then $(g\bar{x} = g\bar{y})$ and therefore $(g\bar{x}, g\bar{x})$ is the unique coupled point of coincidence of F and g.

Now we show that F and g have a unique common coupled fixed point. For this, let $g\bar{x} = x$. Then we have $x = g\bar{x} = F(\bar{x}, \bar{x})$.

By w^*-compatibility of F and g, we have

$$gx = gg\bar{x} = gF(\bar{x}, \bar{x}) = F(g\bar{x}, g\bar{x}) = F(x, x).$$

Thus (gx, gx) is a coupled point of coincidence of F and g. Consequently, $gx = g\bar{x}$. Therefore $x = gx = F(x, x)$. Hence (x, x) is a common coupled fixed point of F and g.

To prove the uniqueness of common coupled fixed point, let $x^* \in X$ with $x^* \neq x$ such that

$$x^* = gx^* = F(x^*, x^*).$$

Then (gx, gx) and (gx^*, gx^*) are two common coupled points of coincidence of F and g and, as was previously proved, it must be $gx = gx^*$, and so $x = gx = gx^* = x^*$. □

Our next result is as follows:

Theorem 4.5. *In addition to the hypotheses of Theorem* 4.1 *(resp. Theorem* 4.3*), suppose that* $g(x_0), g(y_0)$ *in* X *are comparable. Then* $g\bar{x} = g\bar{y}$*.*

Proof. Suppose that $x_0 \preceq y_0$. We claim that

$$gx_n \preceq gy_n, \ \forall\, n \in \mathbb{N}. \tag{4.17}$$

By the monotone property of F, we have

$$gx_1 = F(x_0, y_0) \preceq F(y_0, y_0) \preceq F(y_0, x_0) = gy_1.$$

Assume that $x_n \preceq y_n$ for some n. Now,

$$\begin{aligned} gx_{n+1} &= F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)) \\ &= F(x_n, y_n) \preceq F(y_n, y_n) \preceq F(y_n, x_n) = gy_{n+1}. \end{aligned}$$

Hence, (4.17) holds.

Now, using (4.17) and the contractive condition, we get

$$\begin{aligned}
d(g\bar{x}, g\bar{y}) &\leq d(g\bar{x}, gx_{n+1}) + d(gx_{n+1}, gy_{n+1}) + d(gy_{n+1}, g\bar{y}) \\
&= d(g\bar{x}, gx_{n+1}) + d(F(y_n, x_n), F(x_n, y_n)) + d(gy_{n+1}, g\bar{y}) \\
&\leq d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\
&\quad + \beta N((y_n, x_n), (x_n, y_n)) \\
&\quad + \frac{\gamma}{2}[d(gx_n, F(x_n, y_n)) + d(gy_n, F(y_n, x_n)) \\
&\qquad + d(gy_n, F(y_n, x_n)) + d(gx_n, F(x_n, y_n)))] \\
&\leq d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\
&\quad + \beta \frac{d^2(gx_n, F(y_n, x_n)) + d^2(gy_n, F(x_n, y_n))}{d(gx_n, F(y_n, x_n)) + d(gy_n, F(x_n, y_n))} \\
&\quad + \gamma[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\
&\leq d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\
&\quad + \beta \frac{d^2(gx_n, gy_{n+1}) + d^2(gy_n, gx_{n+1})}{d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1})} \\
&\quad + \gamma[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\
&(\text{ provided } d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1}) \neq 0).
\end{aligned}$$

Passing to the limit as $n \to \infty$, we get that

$$d(g\bar{x}, g\bar{y}) \leq (\alpha + \beta) d(g\bar{x}, g\bar{y}).$$

Since $0 \leq \alpha + \beta < 1$, this implies that $d(g\bar{x}, g\bar{y}) = 0$, i.e., $g\bar{x} = g\bar{y}$.

In the case when $d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1}) = 0$, the conditions of the theorem readily imply that $d(g\bar{x}, g\bar{y}) = 0$. This completes the proof of the theorem. □

Remark 4.6. The results of this paper can be easily modified in a way to obtain the existence of a coupled coincidence point of the mapping $F : X \times X \to X$ and an additional mapping $G : X \times X \to X$, in the case when F and G have the g-monotone property.

Remark 4.7. Similar corollaries can be derived as Result-I.

5. An application to nonlinear integral equations

In this section, Theorem 3.1 is used to guarantee the existence theorem for solution of the following nonlinear integral equations:

$$\begin{cases} x(t) = \int_0^T f(t, x(s), y(s))ds, & t \in [0, T]; \\ y(t) = \int_0^T f(t, y(s), x(s))ds, & t \in [0, T]. \end{cases} \tag{5.1}$$

where T is a real number such that $T > 0$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

This section is inspired by Sintunavarat and Kumam [27].

Definition 5.1. Let $C([0, T], \mathbb{R})$, denote the class of $\mathbb{R}$-valued continuous functions on the interval $[0, T]$, where T is a real number such that $T > 0$. An element $\alpha, \beta \in C([0, T], \mathbb{R})$ is called a coupled upper solution of the integral equation (5.1) if $\alpha(t) \leq \beta(t)$ and

$$\alpha(t) \leq \int_0^T f(t, \alpha(s), \beta(s))ds$$

and

$$\beta(t) \leq \int_0^T f(t, \beta(s), \alpha(s))ds$$

for all $t \in [0, T]$.

Now, we consider the following assumptions:

(I) $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous;

(II) for all $t \in [0, T]$ and for all $x, y, u, v \in \mathbb{R}$ for which $x \leq u$ and $y \leq v$, we have

$$\begin{aligned} 0 &\leq f(t, u, v) - f(t, x, y) \\ &\leq \frac{\alpha}{2}[|g(x(t)) - g(u(t))| + |g(y(t)) - g(v(t))|] \\ &\quad + \frac{\gamma}{2}[|g(x(t)) - F(x, y)(t)| + |g(u(t)) - F(u, v)(t)| \\ &\quad + |g(y(t)) - F(y, x)(t)| + |g(v(t)) - F(v, u)(t)|] \\ &\quad + \frac{\delta}{2}[|g(x(t)) - F(u, v)(t)| + |g(y(t)) - F(v, u)(t)| \\ &\quad + |g(u(t)) - F(x, y)(t)| + |g(v(t)) - F(y, x)(t)|], \end{aligned}$$

for $\alpha, \gamma, \delta \geq 0$ with $\alpha + 2\gamma + 2\delta < 1$.

Now, we are in position to furnish the existence theorem for solution of the integral equation (5.1).

Theorem 5.2. *Suppose that* (I) *and* (II) *hold. Then, the existence of a coupled upper solution for* (5.1) *provides the existence of solution* $(\bar{x}, \bar{y}) \in C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R})$ *for the integral equation* (5.1)*.*

Proof. Let $X := C([0,T], \mathbb{R})$ denote the class of $\mathbb{R}$-valued continuous functions on the interval $[0,T]$. We endowed X with the metric $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \sup_{t \in [0,T]} |x(t) - y(t)|, \quad \forall\, x, y \in X.$$

It is clear that (X, d) is a complete metric space. Moreover, we also have X is a partially ordered set if we define the following order relation in X:

$$x, y \in X, x \leq y \Leftrightarrow x(t) \leq y(t), \quad \forall\, t \in [0,T].$$

Suppose u_n is a monotone non-decreasing in X that converges to $u \in X$. Then, for every $t \in [0,T]$, the sequence of real numbers

$$u_1(t) \leq u_2(t) \leq u_3(t) \leq \cdots \leq u_n(t) \leq \cdots$$

converges to $u(t)$. Therefore, for all $t \in [0,T]$, $n \in N$, $u_n(-t) \leq u(t)$. Thus, $u_n \leq u$, for all $n \in N$. Similarly, we can verify that limit $v(t)$ of a monotone non-decreasing sequence $v_n(t)$ in X is a upper bound for all the elements in the sequence. That is, $v_n \geq u$, for all $n \in \mathbb{N}$. Therefore, condition (b) given in Theorem 3.6 is satisfied.

Define the mapping $F : C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$ and $g : C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$ by

$$F(x,y)(t) = \int_0^T f(t, x(s), y(s))ds, \quad x, y \in C([0,T], \mathbb{R}), \quad t \in [0,T].$$

and $g(x)(t) = x(t)$.

We first prove that F has the g-monotone property. By (I), for any $x, y \in X$ and for all $t \in [0,T]$, we have

$$\begin{aligned} x_1, x_2 \in X, g(x_1)(t) \leq g(x_2)(t) &\Rightarrow 0 \leq f(t, x_2(t), y(t)) - f(t, x_1(t), y(t)) \\ &\Rightarrow f(t, x_1(t), y(t)) \leq f(t, x_2(t), y(t)) \end{aligned}$$

$$\int_0^T f(t, x_1(s), y(s))ds \leq \int_0^T f(t, x_2(s), y(s))ds$$

$$F(x_1, y)(t) \leq F(x_2, y)(t). \tag{5.2}$$

Similarly, we can prove that for any $x, y \in X$ and for all $t \in [0,T]$, we have

$$y_1, y_2 \in X, g(y_1)(t) \leq g(y_2)(t) \Rightarrow F(x, y_1)(t) \leq F(x, y_2)(t). \tag{5.3}$$

From (5.2) and (5.3), we get F has the g-monotone property.

Now, let $x, y, u, v \in X$ for which $x \leq u$ and $y \leq v$. Using (II), for all $t \in [0, T]$, we have

$$\begin{aligned}
&|(F(x,y)(t) - F(u,v)(t)| \\
&= \int_0^T [f(t, u(s), v(s)) - f(t, x(s), y(s))]ds \\
&\leq \frac{\alpha}{2}[|g(x(s)) - g(u(s)|) + |g(y(s)) - g(v(s))|] \\
&\quad + \frac{\gamma}{2}[|g(x(s)) - F(x,y)(s)| + |g(u(s)) - F(u,v)(s)| \\
&\quad + |g(y(s)) - F(y,x)(s)| + |g(v(s)) - F(v,u)(s)|] \\
&\quad + \frac{\delta}{2}[|g(x(s)) - F(u,v)(s)) + |g(y(s)) - F(v,u)(s)| \\
&\quad + |g(u(s)) - F(x,y)(s)| + |g(v(s)) - F(y,x)(s)|], \\
&\leq \frac{\alpha}{2}[\sup_{z\in[0,T]} |g(x(z)) - g(u(z))| + \sup_{z\in[0,T]} |g(y(z)) - g(v(z))|] \\
&\quad + \frac{\gamma}{2}[\sup_{z\in[0,T]} |g(x(z)) - F(x,y)(z)| + \sup_{z\in[0,T]} |g(u(z)) - F(u,v)(z)| \\
&\quad + \sup_{z\in[0,T]} |g(y(z)) - F(y,x)(z)| + \sup_{z\in[0,T]} |g(v(z)) - F(v,u)(z)|] \\
&\quad + \frac{\delta}{2}[\sup_{z\in[0,T]} |g(x(z)) - F(u,v)(z)| + \sup_{z\in[0,T]} |g(y(z)) - F(v,u)(z)| \\
&\quad + \sup_{z\in[0,T]} |g(u(z)) - F(x,y)(z)| + \sup_{z\in[0,T]} |g(v(z)) - F(y,x)(z)|],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sup_{t\in[0,T]} |(F(x,y)(t) - F(u,v)(t)| \\
&\leq \frac{\alpha}{2}[\sup_{z\in[0,T]} |g(x(z)) - g(u(z))| + \sup_{z\in[0,T]} |g(y(z)) - g(v(z))|] \\
&\quad + \frac{\gamma}{2}[\sup_{z\in[0,T]} |g(x(z)) - F(x,y)(z)| + \sup_{z\in[0,T]} |g(u(z)) - F(u,v)(z)| \\
&\quad + \sup_{z\in[0,T]} |g(y(z)) - F(y,x)(z)| + \sup_{z\in[0,T]} |g(v(z)) - F(v,u)(z)|] \\
&\quad + \frac{\delta}{2}[|g(x(z)) - F(u,v)(z)| + \sup_{z\in[0,T]} |g(y(z)) - F(v,u)(z)| \\
&\quad + \sup_{z\in[0,T]} |g(u(z)) - F(x,y)(z)| + \sup_{z\in[0,T]} |g(v(z)) - F(y,x)(z)|].
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
& d(F(x,y),F(u,v)) \\
& \leq \frac{\alpha}{2}[d(gx,gu)+d(gy,gv)] \\
& \quad + \frac{\gamma}{2}[d(gx,F(x,y))+d(gu,F(u,v))+d(gy,F(y,x))+d(gv,F(v,u))] \\
& \quad + \frac{\delta}{2}[d(gx,F(u,v))+d(gy,F(v,u))+d(gu,F(x,y))+d(gv,F(y,x))],
\end{aligned}$$

$x, y, u, v \in X$ for which $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$. This implies that the condition (3.12) (for $\beta = 0$) of Theorem 3.6 is satisfied.

Now, let (α, β) be a coupled upper solution of the integral equation (5.1) then we have $\alpha(t) \leq F(\alpha,\beta)(t)$ and $\beta(t) \leq F(\beta,\alpha)(t)$ for all $t \in [0,T]$, that is, $\alpha \leq F(\alpha,\beta)$ and $\beta \leq F(\beta,\alpha)$.

Thus all clauses in Theorem 3.6 are satisfied. Therefore, we can apply Theorem 3.6 and thus there exists a point $(\widetilde{x},\widetilde{y}) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}$ such that

$$g(\widetilde{x}) = F(\widetilde{x},\widetilde{y}) \text{ and } g(\widetilde{y}) = F(\widetilde{y},\widetilde{x}).$$

Since $g(x) = x$ for all $x \in X$, we get $(\widetilde{x},\widetilde{y}) \in C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$ is the solution for the integral equation (5.1). □

Acknowledgments: The authors express their gratitude to the Professor J. K. Kim for the support and comments on the manuscript. The first author is thankful to the United State-India Education Foundation, New Delhi, India and IIE/CIES, Washington, DC, USA for Fulbright-Nehru PDF Award (No. 2052/FNPDR/2015).

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