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CALCULUS OF VARIATIONS WITH COMBINATION OF CLASSICAL AND FRACTIONAL OPERATORS

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Abstract. In this paper, we present the problem of generalized fractional calculus of variations. Proposed generalization differs in terms of describing the objective function, which involves a combination of classical and fractional (differential and integral) operators. We obtain the necessary conditions in order to find an extremizer of the problem. Provided examples illustrate fractional Euler-Lagrange equations with noticeable consequences. Additionally, generalized fractional isoperimetric problem is discussed. This paper conjointly presents a formulation of the solution scheme for fractional calculus of variations. Construction of this scheme is in terms of approximating the composition of fractional derivatives. This method shows that the solution of Euler-Lagrange equations can also be obtained by the approximation of the composition of left and right fractional derivatives occurring in fractional Euler-Lagrange equations. Moreover, examples demonstrating the formulation are given with sufficient numerical information.

1. Introduction

The perception of fractional calculus is to explore and examine the applications of integrals and derivatives of a non-integer order. It permits the order of a derivative (or integral) to be any real or complex number. This topic

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was first introduced by Leibniz in 1695. After that, several eminent mathematicians like Euler, Lagrange, Laplace, Fourier, Liouville, Riemann further analyzed and contributed in the development of this area. The first exceptional application of fractional calculus was given by Abel in 1823 for solving the tautochrone problem. This subject encompasses a large history of over 300 years. The historical development is given in [21] and generally it can be found in any textbook on fractional calculus. Fractional derivative is a non-local property, thus proved very useful in numerous topics like viscoelasticity, control theory, signal and image processing etc.

In recent years considerable work has been dedicated to the problem of calculus of variations involving fractional derivatives and fractional integrals. Problem of fractional calculus of variations (FCOV) was born in 1996-1997 with the work of Riewe [20]. Riewe obtained a version of Euler-Lagrange equations for problems of the Calculus of Variations with fractional derivatives. Afterwards, Agrawal [1] proved a formulation for a variational problem with right and left fractional derivatives in Riemann-Liouville sense.

It was the first simplest (FCOV) problem defined to find the extremizer $y(t),\ t \in [a,b]$ for the functional

$$J(y) = \int_a^b F(t, y, {}_aD_x^{\alpha}, {}_xD_b^{\beta})\, dt,$$

subject to the boundary conditions

$$y(a) = y_a \quad \text{and} \quad y(b) = y_b,$$

where $0 < \alpha, \beta \leq 1$. It is to be noted that above problem reduces to the classical variational problem when $\alpha = \beta = 1$. This problem is further modified by employing Riesz and Caputo fractional operators in place of Riemann-liouville fractional derivatives.

In year 2004, Euler-Lagrange equations of Agrawal [1] were employed by Baleanu and Avkar [10] to investigate problems with Lagrangians which are linear on the velocities. During the period (2006-2011), researchers like Agrawal, Baleanu, Almeida and Torres have studied (FCOV) problems (see [2],[5],[6], [7],[8] and references therein). These problems include (FCOV) with various fractional operators, formulation of Noether's theorem for (FCOV), generalized natural boundary conditions for fractional variational problems with Caputo derivatives, solution schemes for Euler-Lagrange equations.

The paper is divided into two different parts. The first part is intended to explain the problem of generalized fractional calculus of variations (GFCOV) followed by formulating Euler-Lagrange equations. Present investigation is an extension of the work done by Odzijewicz and Torres [17]. Odzijewicz and Torres studied the problem of fractional calculus of variations involving a combination of classical and fractional derivative. In the present study, we formulate the problem not only with classical and fractional derivatives but also with fractional integrals. While the second part deals with constructing a solution scheme for fractional Euler-Lagrange equations. The construction of solution scheme is due to the appearance of composition of left and right fractional derivatives in necessary extremality conditions. The second part of the paper is motivated by the work done by Blaszczyk and Ciesielski [11].

Organization of the paper: Section 2 presents some preliminaries that will be required throughout the paper. We consider our main results in Section 3. Implementation of solution scheme is presented in Section 4, followed by Section 5 giving the conclusions and scope of future work.

2. Preliminaries

We give the definitions of fractional derivatives/integrals and their main properties. A variety of definitions for fractional derivative are present in literature which includes Riemann-Liouville, Grunwald-Letnikov, Weyl, Caputo, and Riesz fractional derivatives for example (see, [12],[16],[18],[19]). We have considered Riemann-Liouville fractional derivatives/integrals and Caputo fractional order derivatives throughout the paper.

Let $f \in C[a,b]$, where $C[a,b]$ is the space of all continuous functions defined over the closed interval $[a,b]$.

Definition 2.1. ([21]) For all $t \in [a,b]$ and $\alpha > 0$, Left Riemann-Liouville Fractional Integral (LRLFI) of order α is defined as

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > a. $$

Definition 2.2. ([21]) For all $t \in [a,b]$ and $\alpha > 0$, Right Riemann-Liouville Fractional Integral (RRLFI) of order α is defined as

$$ {}_tI_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau, \quad t < b. $$

Let us consider $f \in C^n[a,b]$, where $C^n[a,b]$ is the space of n times continuously differentiable functions defined over $[a,b]$.

Definition 2.3. ([21]) For all $t \in [a,b]$ and $n-1 \le \alpha < n$, Left Riemann-Liouville Fractional Derivative (LRLFD) of order α is defined as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau. $$

Definition 2.4. ([21]) For all $t \in [a,b]$ and $n-1 \leq \alpha < n$, Right Riemann-Liouville Fractional Derivative (RRLFD) of order α is defined as

$$ {}_tD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau)\, d\tau. $$

Definition 2.5. ([21]) For all $t \in [a,b]$ and $n-1 \leq \alpha < n$, Left Caputo Fractional Derivative (LCFD) of order α is defined as

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^n(\tau)\, d\tau. $$

Definition 2.6. ([21]) For all $t \in [a,b]$ and $n-1 \leq \alpha < n$, Right Caputo Fractional Derivative (RCFD) of order α is defined as

$$ {}_t^cD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} f^n(\tau)\, d\tau. $$

Definition 2.7. ([21], Relationship between Riemann-Liouville and Caputo derivative) For $n-1 \leq \alpha < n$,

$$ {}_aD_t^\alpha f(t) = {}_a^cD_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{f^k(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} $$

Thus,

$$ {}_aD_t^\alpha f = {}_a^cD_t^\alpha f \quad \text{iff} \quad f^k(a) = 0, \quad 0 \leq k \leq n-1. $$

Definition 2.8. ([21], Integration by parts) If f, g and the fractional derivatives ${}_aD_t^\alpha g$ and ${}_tD_b^\alpha f$ are continuous at every point $t \in [a,b]$, then

$$ \int_a^b f(t)\, {}_aD_t^\alpha g\, dt = \int_a^b g(t)\, {}_tD_b^\alpha f\, dt, $$

for any $0 < \alpha < 1$.

Definition 2.9. ([15]) Let $\alpha, \beta > 0$. The Mittag-Leffler function is defined by the following series expansion

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}. $$

Definition 2.10. ([14], Fundamental lemma of calculus of variations) Let $f \in C^k[a,b]$. Further

$$ \int_a^b f(x)\, h(x)\, dx = 0, $$

for every function $h \in C^k[a,b]$ with $h(a) = 0 = h(b)$. Then the fundamental lemma of the calculus of variations states that $f(x)$ is identically zero on $[a,b]$.

3. Main results

Calculus of variations with a variety of fractional operators has been studied by many researchers, for example see ([1],[2],[5],[9]). We propose a generalization for the problem of (FCOV) by allowing the objective function consisting of a combination of classical and fractional derivatives or integrals. Suggested generalization not only comprises of a combination of classical and fractional derivative but also deals with fractional integrals at the same time. We prove the necessary extremality conditions followed by some examples to discuss Euler-Lagrange equations. Furthermore, we construct a numerical scheme to approximate composition of fractional derivatives. By this approximation we formulate the solution scheme for fractional Euler-Lagrange equations.

3.1. Generalized Fractional Calculus of Variations (GFCOV).

Problem Statement: Let $0 < \alpha, \beta < 1$. Consider the problem to find a function $y \in C^1[a,b]$ which extremize the functional

$$J(y) = \int_a^b F(t, y(t) + k\, {}_aI_t^{1-\alpha} y(t), y'(t) + l\, {}_aD_t^{\beta} y(t))\, dt, \tag{3.1}$$

subject to the boundary conditions

$$y(a) = y_a\,, \quad y(b) = y_b\,. \tag{3.2}$$

Here k, l are fixed real numbers, $F \in C^2([a,b] \times \mathbb{R}^2; \mathbb{R})$. $\partial_2 F$, $\partial_3 F$ (the partial derivative of $F(\cdot\,,\,\cdot\,,\,\cdot)$ with respect to its second and third argument respectively) have continuous (RRLFI) of order $(1-\alpha)$ and continuous (RRLFD) of order β.

Note: This is to be noted that the integrand f contains a combination of classical and fractional derivative of y in its third argument. f also contains a combination of y and its fractional integral in the second argument.

Definition 3.1. A function $y \in C^1[a,b]$ that satisfies the given boundary conditions (3.2) is said to be admissible for problem (3.1)-(3.2).

For simplicity of notation, we introduce the operator $[\cdot]_{k,l}^{\alpha,\beta}$ defined by

$$[y]_{k,l}^{\alpha,\beta}(t) = (t, y(t) + k\, {}_aI_t^{1-\alpha} y(t), y'(t) + l\, {}_aD_t^{\beta} y(t))$$

with this notion, (3.1) can be simply written as

$$J(y) = \int_a^b F[y]_{k,l}^{\alpha,\beta}(t)\, dt.$$

Suppose that y is a solution of (3.1)-(3.2). We know that admissible functions $\tilde{y}$ can be written in the form $\tilde{y}(t) = y(t) + \epsilon\, \eta(t)$, where $\eta \in C^1[a, b]$, $\eta(a) = 0 = \eta(b)$ and $\epsilon \in \mathbb{R}$. Thus J can be considered as a function of ϵ

$$\begin{aligned} J(\epsilon) \;\; = \;\; & \int_a^b F[t,\ y(t) + \epsilon\, \eta(t) + k\, {}_aI_t^{1-\alpha}(y(t) + \epsilon\, \eta(t)), \\ & \quad \frac{d}{dt}(y(t) + \epsilon\, \eta(t)) + l\, {}_aD_t^{\beta}(y(t) + \epsilon\, \eta(t))]\, dt. \end{aligned}$$

Since fractional derivative and integral operators are linear

$$\begin{aligned} {}_aI_t^{1-\alpha}(y(t) + \epsilon\, \eta(t)) &= {}_aI_t^{1-\alpha}y(t) + \epsilon\, {}_aI_t^{1-\alpha}\eta(t), \\ {}_aD_t^{\beta}(y(t) + \epsilon\, \eta(t)) &= {}_aD_t^{\beta}y(t) + \epsilon\, {}_aD_t^{\beta}\eta(t). \end{aligned}$$

On the other hand,

$$\begin{aligned} \left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} &= \left.\int_a^b \frac{d}{d\epsilon} F[\tilde{y}]_{k,l}^{\alpha,\beta}(t)\, dt\right|_{\epsilon=0} \\ &= \int_a^b \Big(\partial_2 F[\tilde{y}]_{k,l}^{\alpha,\beta}(t)\, \eta(t) + k\, \partial_2 F[\tilde{y}]_{k,l}^{\alpha,\beta}(t)\, {}_aI_t^{1-\alpha}\eta(t) \\ &\quad + \partial_3 F[\tilde{y}]_{k,l}^{\alpha,\beta}(t)\, \frac{d\eta(t)}{dt} + l\, \partial_3 F[\tilde{y}]_{k,l}^{\alpha,\beta}(t)\, {}_aD_t^{\beta}\eta(t) \Big)\, dt. \end{aligned} \tag{3.3}$$

Using integration by parts, we get

$$\int_a^b \partial_3 F. \frac{d\eta}{dt}\, dt = \partial_3 F.\eta\, \Big|_a^b - \int_a^b \left(\eta \frac{d}{dt} \partial F \right)\, dt$$

and

$$\int_a^b \partial_2 F\, {}_aI_t^{1-\alpha}\eta\, dt = \int_a^b \eta\, {}_tI_b^{1-\alpha}F\, dt, \tag{3.4}$$

$$\int_a^b \partial_3 F\, {}_aD_t^{\beta}\eta\, dt = \int_a^b \eta\, {}_tD_b^{\beta}F\, dt. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) with $\eta(a) = 0 = \eta(b)$, it follows that

$$\begin{aligned} \left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = & \int_a^b \eta(t)\, \Big[\partial_2 F[y]_{k,l}^{\alpha,\beta}(t) + k\, {}_tI_b^{1-\alpha} \partial_2 F[y]_{k,l}^{\alpha,\beta}(t) \\ & - \frac{d}{dt}\, \partial_3 F[y]_{k,l}^{\alpha,\beta}(t) + l\, {}_tD_b^{\beta} \partial_3 F[y]_{k,l}^{\alpha,\beta}(t) \Big]\, dt. \end{aligned}$$

We know that the necessary optimality condition is given by $\frac{dJ}{d\epsilon}\big|_{\epsilon=0}$. Hence,

$$\int_a^b \eta(t)\ \Big[\partial_2 F[y]^{\alpha,\beta}_{k,l}(t) + k\,{}_tI_b^{1-\alpha}\partial_2 F[y]^{\alpha,\beta}_{k,l}(t) - \frac{d}{dt}\ \partial_3 F[y]^{\alpha,\beta}_{k,l}(t) + l\,{}_tD_b^{\beta}\partial_3 F[y]^{\alpha,\beta}_{k,l}(t)\Big]\ dt = 0. \tag{3.6}$$

Applying the fundamental lemma of the calculus of variations to (3.6), we get

$$\partial_2 F[y]^{\alpha,\beta}_{k,l}(t) + k\,{}_tI_b^{1-\alpha}\,\partial_2 F[y]^{\alpha,\beta}_{k,l}(t) - \frac{d}{dt}\,\partial_3 F[y]^{\alpha,\beta}_{k,l}(t) + l\,{}_tD_b^{\beta}\partial_3 F[y]^{\alpha,\beta}_{k,l}(t) = 0$$

or, simply

$$\partial_2 F + k\,{}_tI_b^{1-\alpha}\partial_2 F - \frac{d}{dt}\partial_3 F + l\,{}_tD_b^{\beta}\partial_3 F = 0.$$

Thus, we arrive at the following theorem.

Theorem 3.2. (The fractional Euler-Lagrange equation) *If y is an extremizer of problem* (3.1)-(3.2)*, then y satisfies the Euler-Lagrange equation*

$$\partial_2 F[y]^{\alpha,\beta}_{k,l}(t) + k\,{}_tI_b^{1-\alpha}\partial_2 F[y]^{\alpha,\beta}_{k,l}(t) - \frac{d}{dt}\,\partial_3 F[y]^{\alpha,\beta}_{k,l}(t) + l\,{}_tD_b^{\beta}\partial_3 F[y]^{\alpha,\beta}_{k,l}(t) = 0, \tag{3.7}$$

for all $t \in [a,b]$.

Remark 3.3. For $k = l = 0$, necessary extremality condition (3.7) reduces to classical Euler-Lagrange equations [14].

Remark 3.4. For $k = 0$, the problem (3.1)-(3.2) reduces to find a function $y \in C^1[a,b]$ which extremizes the following functional

$$J(y) = \int_a^b F(t, y(t), y'(t) + l\,{}_aD_t^{\beta}y(t))\,dt,$$

subject to the boundary conditions

$$y(a) = y_a\,, \quad y(b) = y_b\,.$$

This problem was given by Odzijewicz and Torres [17].

Remark 3.5. For $\alpha = \beta$, the problem (3.1)-(3.2) reduces to find a function $y \in C^1[a,b]$ which extremizes the following functional

$$J(y) = \int_a^b F(t, y(t) + k\,{}_aI_t^{1-\alpha}y(t)\,, y'(t) + l\,{}_aD_t^{\alpha}y(t))\,dt \tag{3.8}$$

subject to the boundary conditions (3.2).

Euler-Lagrange equation (following the same procedure) for (3.8) is given in the next theorem.

Theorem 3.6. *If y is an extremizer of problem* (3.8) *subject to* (3.2)*, then y satisfies the fractional Euler-Lagrange equation given as follows*

$$\partial_2 F[\cdot](t) + k\, {}_tI_b^{1-\alpha}\partial_2 F[\cdot](t) - \frac{d}{dt}\partial_3 F[\cdot](t) + l\, {}_tD_b^{\alpha}\partial_3 F[\cdot](t) = 0,$$

for all $t \in [a,b]$.

Remark 3.7. Note that the function F considered here contains both left fractional derivatives and integrals. One may take F with both right fractional derivatives and integrals.

Remark 3.8. Let $F \equiv F(t, y(t) + k\, {}_tI_b^{1-\alpha}y(t)\,, y'(t) + l\, {}_tD_b^{\beta}y(t))$, then problem (3.8) reduces to find an extremum $y \in C^1[a,b]$ for the functional

$$J(y) = \int_a^b F(t, y(t) + k\, {}_tI_b^{1-\alpha}y(t)\,, y'(t) + l\, {}_tD_b^{\beta}y(t))\, dt, \tag{3.9}$$

subject to the boundary conditions (3.2).

Remark 3.9. Let $F \equiv F(t, y(t) + k\, {}_aI_t^{1-\alpha}y(t)\,, y'(t) + l\, {}_tD_b^{\beta}y(t))$, then problem (3.8) reduces to find an extremum $y \in C^1[a,b]$ for the functional

$$J(y) = \int_a^b F(t, y(t) + k\, {}_aI_t^{1-\alpha}y(t)\,, y'(t) + l\, {}_tD_b^{\beta}y(t))\, dt, \tag{3.10}$$

subject to the boundary conditions (3.2).

Euler-lagrange equations for (3.9) and (3.10) subject to boundary conditions (3.2) are stated in the following theorems.

Theorem 3.10. *If y is an extremizer of problem* (3.9) *subject to* (3.2)*, then y satisfies the fractional Euler-Lagrange equation given as follows*

$$\partial_2 F[\cdot](t) + k\, {}_aI_t^{1-\alpha}\partial_2 F[\cdot](t) - \frac{d}{dt}\partial_3 F[\cdot](t) + l\, {}_aD_t^{\beta}\partial_3 F[\cdot](t) = 0,$$

for all $t \in [a,b]$.

Theorem 3.11. *If y is an extremizer of problem* (3.10) *subject to* (3.2)*, then y satisfies the fractional Euler-Lagrange equation given as follows*

$$\partial_2 F[\cdot](t) + k\, {}_tI_b^{1-\alpha}\partial_2 F[\cdot](t) - \frac{d}{dt}\partial_3 F[\cdot](t) + l\, {}_aD_t^{\beta}\partial_3 F[\cdot](t) = 0,$$

for all $t \in [a,b]$.

3.2. **Generalized Fractional Isoperimetric Problem.**

Problem Statement: Let $0 < \alpha, \beta < 1$. The generalized fractional isoperimetric problem is to extremize the functional

$$J(y) = \int_a^b F(t, y(t) + k\, {}_aI_t^{1-\alpha} y(t), y'(t) + l\, {}_aD_t^{\beta} y(t))\, dt, \tag{3.11}$$

in the class $C^1[a,b]$, subject to the boundary conditions

$$y(a) = y_a\,, \quad y(b) = y_b, \tag{3.12}$$

and an isoperimetric constraint

$$I(y) = \int_a^b G(t, y(t) + k\, {}_aI_t^{1-\alpha} y(t), y'(t) + l\, {}_aD_t^{\beta} y(t))\, dt = \xi\,. \tag{3.13}$$

Here we assume that k, l, ξ are fixed real numbers and $F\,, G \in C^2([a,b] \times \mathbb{R}^2; \mathbb{R})$. $\partial_2 F\,, \partial_2 G$ have continuous (RRLFI) of order $(1-\alpha)$ and $\partial_3 F\,, \partial_3 G$ have continuous (LRLFD) of order β.

Definition 3.12. A function $y \in C^1[a,b]$ that satisfies the given boundary conditions (3.12) and isoperimetric constraint (3.13) is said to be admissible for the problem (3.11)-(3.13).

Definition 3.13. An admissible function y is an extremal for I if it satisfies the fractional Euler-Lagrange equation

$$\begin{aligned}&\partial_2 G[y]_{k,l}^{\alpha,\beta}(t) + k\, {}_tI_b^{1-\alpha} \partial_2 G[y]_{k,l}^{\alpha,\beta}(t) - \frac{d}{dt}\partial_3 G[y]_{k,l}^{\alpha,\beta}(t)\\ &+ l\, {}_tD_b^{\beta} \partial_3 G[y]_{k,l}^{\alpha,\beta}(t) = 0,\end{aligned}$$

for all $t \in [a,b]$.

The next theorem gives a necessary optimality condition for the fractional isoperimetric problem (3.11)-(3.13).

Theorem 3.14. *Let y be an extremizer for the functional* (3.11) *subject to the boundary conditions* (3.12) *and the isoperimetric constraint* (3.13)*. If y is not an extremal for I, then there exists a constant λ such that*

$$\begin{aligned}&\partial_2 H[y]_{k,l}^{\alpha,\beta}(t) + k\, {}_tI_b^{1-\alpha} \partial_2 H[y]_{k,l}^{\alpha,\beta}(t) - \frac{d}{dt}\partial_3 H[y]_{k,l}^{\alpha,\beta}(t)\\ &+ l\, {}_tD_b^{\beta} \partial_3 H[y]_{k,l}^{\alpha,\beta}(t) = 0,\end{aligned} \tag{3.14}$$

for all $t \in [a,b]$, where $H(\cdot\,,\cdot\,,\cdot) = F(\cdot\,,\cdot\,,\cdot) - \lambda\, G(\cdot\,,\cdot\,,\cdot)$.

Proof. We introduce the two parameter family

$$\tilde{y} = y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, \tag{3.15}$$

in which η_1 and $\eta_2 \in C^1[a,b]$ are such that $\eta_1(a) = \eta_1(b) = 0 = \eta_2(a) = \eta_2(b)$ and they have continuous left and right fractional order derivatives/integrals.

First we show that in the family (3.15), there are curves such that $\tilde{y}$ satisfies (3.13). Substituting y by $\tilde{y}$ in (3.13), $I(\tilde{y})$ becomes a function of two parameters ϵ_1, ϵ_2.

$$I(\epsilon_1, \epsilon_2) = \int_a^b G\left(t,\, \tilde{y}(t) + k\, {}_aI_t^{1-\alpha}\, \tilde{y}(t),\, \tilde{y}'(t) + l\, {}_aD_t^{\beta}\, \tilde{y}(t)\right) dt = \xi\,.$$

Then $I(0,0) = 0$ and

$$\left.\frac{\partial I}{\partial \epsilon_2}\right|_{(0,0)} = \int_a^b \eta_2 \left(\partial_2 G + k\, {}_tI_b^{1-\alpha} \partial_2 G - \frac{d}{dt}\partial_3 G + l\, {}_tD_b^{\beta} \partial_3 G\right) dt\,.$$

Since y is not an extremal for I,

$$\left.\frac{\partial I}{\partial \epsilon_2}\right|_{(0,0)} \neq 0.$$

By the implicit function theorem, there exists a function $\epsilon_2(\cdot)$ defined in a neighborhood of zero, such that $I(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. Let $J(\epsilon_1, \epsilon_2) = J(\tilde{y})$. Then, by the Lagrange multiplier rule, there exists a real λ such that

$$\bigtriangledown(J(0,0) - \lambda I(0,0)) = 0.$$

Because

$$\left.\frac{\partial J}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \eta_1 \left(\partial_2 F + k\, {}_tI_b^{1-\alpha} \partial_2 F - \frac{d}{dt}\partial_3 F + l\, {}_tD_b^{\beta} \partial_3 F\right) dt$$

and

$$\left.\frac{\partial I}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \eta_1 \left(\partial_2 G + k\, {}_tI_b^{1-\alpha} \partial_2 G - \frac{d}{dt}\partial_3 G + l\, {}_tD_b^{\beta} \partial_3 G\right) dt\,,$$

we have

$$\int_a^b \eta_1 \Big[(\partial_2 F + k\, {}_tI_b^{1-\alpha} \partial_2 F - \frac{d}{dt}\partial_3 F + l\, {}_tD_b^{\beta} \partial_3 F) \\ -\lambda(\partial_2 G + k\, {}_tI_b^{1-\alpha} \partial_2 G - \frac{d}{dt}\partial_3 G + l\, {}_tD_b^{\beta} \partial_3 G)\Big] = \mathbf{0}.$$

Since η_1 is an arbitrary function, (3.14) follows by Lemma 1 (fundamental lemma of the calculus of variations). □

Example 3.15. Let $\alpha, \beta \in (0,1)$ and $k\,, l,\, \xi \in \mathbb{R}$. Consider the following fractional isoperimetric problem:

$$\text{Minimize} \quad J(y) = \int_0^1 \left(k\,(y + k\,{}_0I_t^{1-\alpha} y) + (y' + l\,{}_0D_t^{\beta} y)^2 \right) dt, \tag{3.16}$$

subject to

$$y(0) = 0\,, \quad y(1) = \int_0^1 E_{1-\beta,\,1}\,(-l\,(1-\tau)^{1-\beta})\xi\, d\tau$$

and an isoperimetric constraint

$$\mathcal{I}(y) = \int_0^1 \left(\frac{k}{2\xi}(y + k\,{}_0I_t^{1-\alpha} y) + (y' + l\,{}_0D_t^{\beta} y) \right) dt = \xi\,. \tag{3.17}$$

Here $F = k\,(y + k\,{}_0I_t^{1-\alpha} y) + (y' + l\,{}_0D_t^{\beta} y)^2$ and $G = \frac{k}{2\xi}(y + k\,{}_0I_t^{1-\alpha} y) + (y' + l\,{}_0D_t^{\beta} y)^2$. Thus, the augmented Lagrangian is $H = F - \lambda G$. It can be easily checked that

$$y(t) = \int_0^t E_{1-\beta,1}(-l\,(t-\tau)^{1-\beta})\,\xi\, d\tau, \tag{3.18}$$

- $y(t)$ is not an extremal of I,
- $y(t)$ satisfies $y' + l\,{}_0D_t^{\beta} y = \xi$.

Moreover $y(t)$ satisfies (3.14) for $\lambda = 2\xi$, *i.e.*,

$$\begin{aligned}
&(\partial_2 F - \lambda\,\partial_2 G) + k\,{}_tI_1^{1-\alpha}(\partial_2 F - \lambda\,\partial_2 G) - \frac{d}{dt}(\partial_3 F - \lambda \partial_3 G) + l\,{}_tD_1^{\beta}(\partial_3 F - \lambda\,\partial_3 G)\\
&= (k - \frac{\lambda\,k}{2\xi}) + k\,{}_tI_1^{1-\alpha}(k - \frac{\lambda\,k}{2\xi}) \;-\; \frac{d}{dt}(2(y' + l_0 D_t^{\beta} y) - \lambda)\\
&\quad + \; l\,{}_tD_1^{\beta}(2(y' + l_0 D_t^{\beta} y) - \lambda) = \mathbf{0}.
\end{aligned}$$

Thus, we conclude that (3.18) is an extremal for functional $J(y)$ subject to the prescribed boundary conditions and given isoperimetric constant.

Example 3.16. Choose $k = l = 0$. In this case the isoperimetric constraint is trivially satisfied and the problem (3.16)-(3.17) reduces to the classical problem of the calculus of variations:

$$\text{Minimize} \quad J(y) = \int_0^1 (y'(t))^2\, dt,$$

subject to

$$y(0) = 0 \quad \text{and} \quad y(1) = \xi$$

and extremal (3.18) simplifies to the minimizer $y(t) = \xi\, t$ of $J(y)$.

Example 3.17. Choose $k = 0$, $\beta \to 1$. The problem (3.16)-(3.17) reduces to the classical variational problem

$$\text{Minimize} \quad J(y) = (l+1)^2 \int_0^1 (y'(t))^2 \, dt,$$

subject to

$$y(0) = 0 \quad \text{and} \quad y(1) = \frac{\xi}{l+1}$$

and extremal (3.18) simplifies to $y(t) = \frac{\xi}{(l+1)} t$

Example 3.18. Choose $k = 0$, $l = \xi = 1$. If $\beta \to 0$, the problem (3.16)-(3.17) reduces to the classical isoperimetric problem

$$\text{Minimize} \quad J(y) = \int_0^1 (y'(t) + y(t))^2 \, dt,$$

subject to

$$y(0) = 0 \quad \text{and} \quad y(1) = 1 - \frac{1}{e},$$

and an isoperimetric constant

$$\mathcal{I}(y) = \int_0^1 (y'(t) + y(t)) \, dt = 1.$$

Extremal (3.18) reduces to the classical extremal $y(t) = 1 - e^{-t}$.

3.3. **Solution Scheme.** In this section, we define a numerical formulation for approximation of the composition of fractional derivatives. One can clearly observe that the composition of left and right fractional derivatives often occurs while dealing with fractional Euler-Lagrange equations.

We present here an approximation of the following composition of fractional derivatives for $t \in [0, b]$, and $\alpha > 0$.

(a) ${}_t^cD_b^\alpha \, {}_0^cD_t^\alpha f(t)$ (b) ${}_tD_b^\alpha \, {}_0D_t^\alpha f(t)$ (c) ${}_tD_b^\alpha \, {}_0^cD_t^\alpha f(t)$

(It may be noted that these are not the only composition of fractional derivatives, other possible combinations also exist and can be approximated in the similar manner.)

(a) We first consider the following composition of fractional differential operator of order α, for $t \in [0, b]$ and $0 < \alpha < 1$,

$$ {}_t^cD_b^\alpha \, {}_0^cD_t^\alpha f(t), \tag{3.19}$$

where ${}_t^cD_b^\alpha$, ${}_0^cD_t^\alpha$ are the right and left Caputo fractional derivative. Equation (3.19) is supplemented with the boundary conditions

$$f(0) = f_0 \quad \text{and} \quad f(b) = f_b.$$

Consider a partition $\{0 = t_0 < t_1 < \cdots < t_{i-1} < t_i < t_{i+1} < \cdots < t_N = b\}$ of $[0, b]$ by introducing N homogeneous grid of nodes: $t_i = i\Delta t, \Delta t = b/N$. The value of the function f at the point t_i is denoted as $f_i = f(t_i)$.

The value of left Caputo derivative occurring in (3.19) at $t = t_i$ can be approximated as

$$\begin{aligned}
{}_0^cD_t^\alpha f(t)\,|_{t=t_i} &= \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t_i}\frac{f'(\tau)}{(t_i-\tau)^\alpha}d\tau \\
&\approxeq \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{i-1}\frac{f_{j+1}-f_j}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{1}{(t_i-\tau)^\alpha}d\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{i-1}\frac{f_{j+1}-f_j}{\Delta t}.\frac{(t_i-t_j)^{1-\alpha}-(t_i-t_{j+1})^{1-\alpha}}{1-\alpha} \\
&= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{i-1}(f_{j+1}-f_j)[(i-j)^{1-\alpha}-(i-j-1)^{1-\alpha}] \\
&= (\Delta t)^{-\alpha}\sum_{j=0}^{i} f_j\, v_1(i,j)\,,
\end{aligned}$$

where

$$v_1(i,j) = \frac{1}{\Gamma(2-\alpha)}\begin{cases} (i-1)^{1-\alpha}-i^{1-\alpha} & ; j=0, \\ (i-j+1)^{1-\alpha}-2(i-j)^{1-\alpha}+(i-j-1)^{1-\alpha} & ; j=1,\cdots,i-1, \\ 1 & ; j=i. \end{cases}$$

We denote $g(t) = {}_0^cD_t^\alpha f(t)$, the discrete form of composition of operators ${}_t^cD_b^\alpha$ and ${}_0^cD_t^\alpha$ can be given as

$$\begin{aligned}
{}_t^cD_b^\alpha\, {}_0^cD_t^\alpha f(t)\,|_{t=t_i} &= {}_t^cD_b^\alpha g(t)\,|_{t=t_i} \\
&= g_N\frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_{t_i}^{t_N}\frac{g'(\tau)}{(\tau-t_i)^\alpha}d\tau \\
&\approxeq g_N\frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\sum_{j=i}^{N-1}\frac{g_{j+1}-g_j}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{1}{(\tau-t_i)^\alpha}d\tau
\end{aligned}$$

$$= g_N \frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{g_{j+1}-g_j}{\Delta t} . \frac{(t_{j+1}-t_i)^{1-\alpha} - (t_j - t_i)^{1-\alpha}}{1-\alpha}$$

$$= g_N \frac{(1-\alpha)((N-i)\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}$$

$$+ \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=i}^{N-1} (g_{j+1}-g_j)[(j+1-i)^{1-\alpha} - (j-i)^{1-\alpha}]$$

$$= (\Delta t)^{-\alpha} \sum_{j=i}^{N} g_j \, w_1(i,j) \, ,$$

where

$$w_1(i,j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} 1 & ; j = i, \\ (j-i+1)^{1-\alpha} - 2(j-i)^{1-\alpha} + (j-i-1)^{1-\alpha} & ; j = i+1, \cdots, N-1, \\ (N-i-1)^{1-\alpha} - (N-i)^{1-\alpha} & ; j = N. \end{cases}$$

Using $v_1(i,j)$ and $w_1(i,j)$, we describe the discrete form of the fractional operator in (3.19),

$${}^cD^\alpha_{b^-} \, {}^cD^\alpha_{0^+} f(t) \mid_{t=t_i} \quad \approxeq \quad (\Delta t)^{-2\alpha} \sum_{j=i}^{N} \left[w(i,j) \sum_{k=0}^{j} v(j,k) \, f_k \right]$$

with given boundary conditions $f(0) = f_0$ and $f(b) = f_b$.

Example 3.19. Let us consider the equation ${}^c_tD^\alpha_1 \, {}^c_0D^\alpha_t f(t) = 0$, for $0 < \alpha < 1$ and $t \in [0,1]$. Clearly $f(t) = t^\alpha$ is the analytical solution of given fractional differential equation with boundary condition $f(0) = 0$ and $f(b) = f(1) = 1$. Calculating the numerical values for $N = 3, 4$, we determine the values of f_i for $i = 1, 2, \cdots, N-1$. Table 1, 2 presents the numerical values of f_i and their respective numerical errors for $N = 3, 4$. $ERR\, f_i = |(f_i - f'_i)|/f'_i$. f'_i is the actual value of f at the point $t = t_i$.

(b) Consider the following composition of fractional derivatives, for $t \in [0,b]$ and $0 < \alpha < 1$

$${}_tD^\alpha_b \, {}_0D^\alpha_t f(t), \tag{3.20}$$

where ${}_tD^\alpha_b$, ${}_0D^\alpha_t$ are the right and left Riemann-Liouville fractional order derivative. (3.20) is supplemented with the boundary conditions

$$f(0) = f_0 \quad \text{and} \quad f(b) = f_b.$$

TABLE 1. Numerical values and relative errors of f_i, $(i = 1, 2)$ for different values of α in example (3.19)

$N = 3$	$\alpha = .001$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = .999$
f_1	0.998097	0.836628	0.621955	0.493751	0.411201	0.354956	0.333533
f_2	0.999472	0.948006	0.855493	0.782983	0.727355	0.684433	0.666833
errf_1	$8.058758 * 10^{-4}$	0.066220	0.135239	0.144798	0.112764	0.045922	$4.996667 * 10^{-4}$
errf_2	$1.226668 * 10^{-4}$	0.012766	0.033851	0.041046	0.033926	0.014145	$1.559841 * 10^{-4}$

TABLE 2. Numerical values and relative errors of f_i, $(i = 1, 2, 3)$ for different values of α in example (3.19)

$N = 4$	$\alpha = 0.001$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 0.999$
f_1	0.99775	0.80784	0.56142	0.41856	0.32993	0.27176	0.25136
f_2	0.99914	0.91603	0.77223	0.66375	0.58361	0.52401	0.50256
f_3	1.00129	0.96628	0.90267	0.84771	0.80237	0.76561	0.75365
errf_1	$8.61860 * 10^{-4}$	0.07203	0.14904	0.16287	0.12929	0.05367	$4.07908 * 10^{-3}$
errf_2	$1.67208 * 10^{-4}$	0.01821	0.04926	0.06131	0.05192	0.02214	$4.42754 * 10^{-3}$
errf_3	$1.58709 * 10^{-3}$	$5.51402 * 10^{-3}$	0.01596	0.02454	0.01863	$8.12358 * 10^{-3}$	$4.58562 * 10^{-3}$

(All the calculations are performed with the help of Mathematica.)

By considering a partition $\{0 = t_0 < t_1 < \cdots < t_{i-1} < t_i < t_{i+1} < \cdots < t_N = b\}$ of $[0, b]$ (following the same procedure as in **(a)**), the value of the left Riemann-Liouville derivative occurring in (3.20) at $t = t_i$ can be approximated as:

$$
\begin{aligned}
{}_0D_t^\alpha f(t)\mid_{t=t_i} &= f_0 \frac{t_i^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_i} \frac{f'(\tau)}{(t_i-\tau)^\alpha} d\tau \\
&\approxeq f_0 \frac{t_i^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{f_{j+1}-f_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(t_i-\tau)^\alpha} d\tau \\
&= f_0 \frac{t_i^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{f_{j+1}-f_j}{\Delta t} . \frac{(t_i-t_j)^{1-\alpha} - (t_i-t_{j+1})^{1-\alpha}}{1-\alpha} \\
&= f_0 \frac{(i\Delta t)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} (f_{j+1}-f_j)[(i-j)^{1-\alpha} - (i-j-1)^{1-\alpha}] \\
&= f_0 \frac{(1-\alpha)(i\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \\
&\quad + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} (f_{j+1}-f_j)[(i-j)^{1-\alpha} - (i-j-1)^{1-\alpha}] \\
&= (\Delta t)^{-\alpha} \sum_{j=0}^{i} f_j \, v_2(i,j) \,,
\end{aligned}
$$

where

$$v_2(i,j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} (1-\alpha)i^{-\alpha} + (i-1)^{1-\alpha} - i^{1-\alpha} & ; j = 0, \\ (i-j+1)^{1-\alpha} - 2(i-j)^{1-\alpha} + (i-j-1)^{1-\alpha} & ; j = 1, \cdots, i-1, \\ 1 & ; j = i. \end{cases}$$

Denote $g(t) = {}_0D_t^\alpha f(t)$, the discrete form of composition of operators ${}_tD_b^\alpha$ and ${}_0D_t^\alpha$ is

$$\begin{aligned} {}_tD_b^\alpha\, {}_0D_t^\alpha f(t) \mid_{t=t_i} &= D_{b^-}^\alpha g(t) \mid_{t=t_i} \\ &= g_N \frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_{t_i}^{t_N} \frac{g'(\tau)}{(\tau - t_i)^\alpha} d\tau \\ &\approxeq g_N \frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{g_{j+1} - g_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(\tau - t_i)^\alpha} d\tau \\ &= g_N \frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{g_{j+1} - g_j}{\Delta t} . \frac{(t_{j+1} - t_i)^{1-\alpha} - (t_j - t_i)^{1-\alpha}}{1-\alpha} \\ &= g_N \frac{((N-i)\Delta t)^{-\alpha}}{\Gamma(1-\alpha)} \\ &\quad + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=i}^{N-1} (g_{j+1} - g_j)[(j+1-i)^{1-\alpha} - (j-i)^{1-\alpha}] \\ &= g_N \frac{(1-\alpha)((N-i)\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=i}^{N-1} (g_{j+1} - g_j)[(j+1-i)^{1-\alpha} - (j-i)^{1-\alpha}] \\ &= (\Delta t)^{-\alpha} \sum_{j=i}^{N} g_j\, w_2(i,j), \end{aligned}$$

where

$$w_2(i,j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} 1 & ; j = i, \\ (j-i+1)^{1-\alpha} - 2(j-i)^{1-\alpha} + (j-i-1)^{1-\alpha} & ; j = i+1, \cdots, N-1, \\ (N-i-1)^{1-\alpha} - (N-i)^{1-\alpha} + (1-\alpha)(N-i)^{-\alpha} & ; j = N. \end{cases}$$

Using $v_2(i,j)$ and $w_2(i,j)$, we describe the discrete form of the fractional operator in equation (3.20),

$$ {}_tD_b^{\alpha}\, {}_0D_t^{\alpha} f(t)\mid_{t=t_i} \;\; \approxeq \;\; (\Delta t)^{-2\alpha} \sum_{j=i}^{N} \left[w(i,j) \sum_{k=0}^{j} v(j,k) f_k \right] $$

with given boundary conditions $f(0) = f_0$ and $f(b) = f_b$.

(c) We now consider the following composition of fractional derivatives of order α, for $t \in [0,b]$ and $0 < \alpha < 1$

$$ {}_tD_b^{\alpha}\, {}_0^cD_t^{\alpha} f(t), \tag{3.21} $$

where ${}_tD_b^{\alpha}$ is the right Riemann-Liouville fractional order derivative and ${}_0^cD_t^{\alpha}$ is left Caputo's fractional order derivative. (3.21) is supplemented with the boundary conditions

$$ f(0) = f_0 \quad \text{and} \quad f(b) = f_b $$

Following the same procedure as in **(a)** and **(b)**, consider a partition $\{0 = t_0 < t_1 < \cdots < t_{i-1} < t_i < t_{i+1} < \cdots < t_N = b\}$ of $[0,b]$ by introducing N homogeneous grid of nodes: $t_i = i\Delta t, \Delta t = b/N$. The value of the function f at the point t_i is denoted as $f_i = f(t_i)$.
The value of the left Caputo derivative occurring in (3.21) at $t = t_i$ can be approximated as

$$ \begin{aligned} {}_0^cD_t^{\alpha} f(t)\mid_{t=t_i} &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_i} \frac{f'(\tau)}{(t_i-\tau)^{\alpha}} d\tau \\ &\approxeq \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{f_{j+1}-f_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(t_i-\tau)^{\alpha}} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{f_{j+1}-f_j}{\Delta t} . \frac{(t_i-t_j)^{1-\alpha} - (t_i-t_{j+1})^{1-\alpha}}{1-\alpha} \\ &= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} (f_{j+1}-f_j)[(i-j)^{1-\alpha} - (i-j-1)^{1-\alpha}] \\ &= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} (f_{j+1}-f_j)[(i-j)^{1-\alpha} - (i-j-1)^{1-\alpha}] \\ &= (\Delta t)^{-\alpha} \sum_{j=0}^{i} f_j\, v_3(i,j)\,, \end{aligned} $$

where

$$v_3(i,j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} (i-1)^{1-\alpha} - i^{1-\alpha} & ; j = 0, \\ (i-j+1)^{1-\alpha} - 2(i-j)^{1-\alpha} + (i-j-1)^{1-\alpha} & ; j = 1, \cdots, i-1, \\ 1 & ; j = i. \end{cases}$$

Denote $g(t) = {}_0^cD_t^\alpha f(t)$, the discrete form of the composition of operators ${}_tD_b^\alpha$ and ${}_0^cD_t^\alpha$ is given by

$$\begin{aligned} {}_tD_b^\alpha\, {}_0^cD_t^\alpha f(t)\,|_{t=t_i} &= {}_tD_b^\alpha g(t)\,|_{t=t_i} \\ &= g_N \frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_{t_i}^{t_N} \frac{g'(\tau)}{(\tau - t_i)^\alpha} d\tau \\ &\approxeq g_N \frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{g_{j+1} - g_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(\tau - t_i)^\alpha} d\tau \\ &= g_N \frac{(b-t_i)^{-\alpha}}{\Gamma(1-\alpha)} \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{g_{j+1} - g_j}{\Delta t} . \frac{(t_{j+1} - t_i)^{1-\alpha} - (t_j - t_i)^{1-\alpha}}{1-\alpha} \\ &= g_N \frac{(1-\alpha)((N-i)\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \\ &\quad + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=i}^{N-1} (g_{j+1} - g_j)[(j+1-i)^{1-\alpha} - (j-i)^{1-\alpha}] \\ &= (\Delta t)^{-\alpha} \sum_{j=i}^{N} g_j\, w_3(i,j)\,, \end{aligned}$$

where

$$w_3(i,j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} 1 & ; j = i, \\ (j-i+1)^{1-\alpha} - 2(j-i)^{1-\alpha} + (j-i-1)^{1-\alpha} & ; j = i+1, \cdots, N-1, \\ (N-i-1)^{1-\alpha} - (N-i)^{1-\alpha} + (1-\alpha)(N-i)^{-\alpha} & ; j = N. \end{cases}$$

Using formula $v_3(i,j)$ and $w_3(i,j)$, we describe a discrete form of the fractional operator in (3.21),

$$D_{b^-}^\alpha\, {}^cD_{0^+}^\alpha f(t)\,|_{t=t_i} \approxeq (\Delta t)^{-2\alpha} \sum_{j=i}^{N} \left[w(i,j) \sum_{k=0}^{j} v(j,k)\, f_k \right]$$

with given boundary conditions $f(0) = f_0$ and $f(b) = f_b$.

We can rewrite the above discussion in a well-organized form as follows:

$$\textbf{(a)}\quad {}_t^cD_b^\alpha\, {}_0^cD_t^\alpha f(t)\,|_{t=t_i} \;\approxeq\; (\Delta t)^{-2\alpha}\sum_{j=i}^{N}\left[w_1(i,j)\sum_{k=0}^{j} v_1(j,k)\, f_k\right],$$

$$\textbf{(b)}\quad {}_tD_b^\alpha\, {}_0D_t^\alpha f(t)\,|_{t=t_i} \;\approxeq\; (\Delta t)^{-2\alpha}\sum_{j=i}^{N}\left[w_2(i,j)\sum_{k=0}^{j} v_2(j,k)\, f_k\right],$$

$$\textbf{(c)}\quad {}_tD_b^\alpha\, {}_0^cD_t^\alpha f(t)\,|_{t=t_i} \;\approxeq\; (\Delta t)^{-2\alpha}\sum_{j=i}^{N}\left[w_3(i,j)\sum_{k=0}^{j} v_3(j,k)\, f_k\right],$$

with given boundary conditions $f(0)=f_0$ and $f(b)=f_b$. Here,

$$v_1(i,j) = \frac{1}{\Gamma(2-\alpha)}\begin{cases}(i-1)^{1-\alpha}-i^{1-\alpha} & ; j=0\\ (i-j+1)^{1-\alpha}-2(i-j)^{1-\alpha}+(i-j-1)^{1-\alpha} & ; j=1,...,i-1\\ 1 & ; j=i.\end{cases}$$

$$w_1(i,j) = \frac{1}{\Gamma(2-\alpha)}\begin{cases}1 & ; j=i\\ (j-i+1)^{1-\alpha}-2(j-i)^{1-\alpha}+(j-i-1)^{1-\alpha} & ; j=i+1,...,N-1\\ (N-i-1)^{1-\alpha}-(N-i)^{1-\alpha} & ; j=N.\end{cases}$$

$$v_2(i,j) = \frac{1}{\Gamma(2-\alpha)}\begin{cases}(1-\alpha)i^{-\alpha}+(i-1)^{1-\alpha}-i^{1-\alpha} & ; j=0\\ (i-j+1)^{1-\alpha}-2(i-j)^{1-\alpha}+(i-j-1)^{1-\alpha} & ; j=1,...,i-1\\ 1 & ; j=i.\end{cases}$$

$$w_2(i,j) = \frac{1}{\Gamma(2-\alpha)}\begin{cases}1 & ; j=i\\ (j-i+1)^{1-\alpha}-2(j-i)^{1-\alpha}+(j-i-1)^{1-\alpha} & ; j=i+1,...,N-1\\ (N-i-1)^{1-\alpha}-(N-i)^{1-\alpha}+(1-\alpha)(N-i)^{-\alpha} & ; j=N.\end{cases}$$

$$v_3(i,j) = \frac{1}{\Gamma(2-\alpha)}\begin{cases}(i-1)^{1-\alpha}-i^{1-\alpha} & ; j=0\\ (i-j+1)^{1-\alpha}-2(i-j)^{1-\alpha}+(i-j-1)^{1-\alpha} & ; j=1,...,i-1\\ 1 & ; j=i.\end{cases}$$

$$w_3(i,j) = \frac{1}{\Gamma(2-\alpha)}\begin{cases}1 & ; j=i\\ (j-i+1)^{1-\alpha}-2(j-i)^{1-\alpha}+(j-i-1)^{1-\alpha} & ; j=i+1,...,N-1\\ (N-i-1)^{1-\alpha}-(N-i)^{1-\alpha}+(1-\alpha)(N-i)^{-\alpha} & ; j=N.\end{cases}$$

The next section presents the implementation of the solution scheme to solve the problem of fractional calculus of variations.

4. Implementation of solution scheme

Consider the problem of fractional calculus of variations, for $t \in [a,b]$ and $0 < \alpha < 1$:

$$\text{minimize} \quad J(y) = \int_0^1 F(t, y(t), {}_0D_t^\alpha y(t))dt \tag{4.1}$$

subject to

$$y(0) = 0 \quad \text{and} \quad y(1) = 1.$$

Take

$$F = \frac{1}{2}({}_0D_t^\alpha y)^2 - \frac{\Gamma(1+\alpha)(1-t)^{-\alpha}}{\Gamma(1-\alpha)} y.$$

Euler-Lagrange equation (necessary condition) for (4.1) is given by

$$\partial_2 F + {}_tD_1^\alpha \partial_3 F = 0, \tag{4.2}$$

$$i.e. \quad -\frac{\Gamma(1+\alpha)(1-t)^{-\alpha}}{\Gamma(1-\alpha)} + {}_tD_1^\alpha\, {}_0D_t^\alpha y(t) = 0. \tag{4.3}$$

One can clearly observe that $y(t) = t^\alpha$ is an analytical solution of (4.3) subject to the prescribed boundary conditions $y(0) = 0$ and $y(1) = 1$.

Applying the formulation (b) (Section 3) for composition of fractional derivatives, (4.1) together with given boundary conditions can be written as

$$y(0) = 0, \quad i = 0, \tag{4.4}$$

$$A_i + (\Delta t)^{-2\alpha} \sum_{j=i}^{N} \left[w(i,j) \sum_{k=0}^{j} v(j,k)\, y_k \right] = 0, \quad i = 1, \cdots, N-1, \tag{4.5}$$

$$y(1) = 1, \quad i = N. \tag{4.6}$$

where $A_i = -\frac{\Gamma(1+\alpha)(1-t_i)^{-\alpha}}{\Gamma(1-\alpha)}$. In order to solve (4.1) with given boundary conditions, we need to solve the system of algebraic equation (4.4)-(4.6). We have used Mathematica to perform the numerical segments.

Take $N = 3$ (number of homogeneous grid of nodes), $t_i = i\Delta t$, $\Delta t = 1/3$ for $i = 0, 1, 2, 3$. We have $y_0 = 0$ and $y(1) = y(t_3) = 1$. Thus, now we find the values of y_i for $i = 1, 2$. Table 3 presents the numerical values of y_i and their respective numerical errors for $N = 3$. $ERR\, y_i = |(y_i - y_i^*)|/y_i^*$, where y_i^* is the actual value of y at the point $t = t_i$.

5. Conclusions

Calculus of variations involving fractional operators is currently a rich research topic, where the results are firm but very rare. Results of Euler-Lagrange equations for a variety of classes of fractional variational problem has already been given. These include fractional variational problem with the

TABLE 3. Numerical values and relative errors of y_i, $(i = 1, 2)$ for different values of α

$N = 3$	$\alpha = .001$	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = .999$
y_1	0.998479	0.820545	0.594847	0.473378	0.402498	0.353894	0.333533
y_2	0.999288	0.931596	0.824715	0.758929	0.716804	0.683121	0.666833
errf_1	$4.234559 * 10^{-4}$	0.084171	0.172931	0.180085	0.131542	0.048777	$4.996667 * 10^{-4}$
errf_2	$3.067414 * 10^{-4}$	0.029854	0.068611	0.070505	0.047940	0.016034	$1.559841 * 10^{-4}$

Riemann-Liouville operators, Caputo derivatives, Riesz derivatives, a combination of classical and fractional derivatives, etc. In this paper, we go a step further with some additional term, *i.e.*, we consider the variational problem with a combination of classical and fractional derivatives/integrals and prove the optimality conditions. Furthermore, necessary extremality conditions for generalized fractional isoperimetric problem for the same class is proved.
A generalized numerical scheme has been presented for solving fractional Euler-Lagrange equations. The approach is to approximate composition of fractional derivatives occurring all the while in necessary conditions. This numerical scheme can further be extended for the composition of variety of fractional operators.

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