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EXPLICIT ITERATIVE ALGORITHM FOR SOLVING SPLIT VARIATIONAL INCLUSION AND FIXED POINT PROBLEM FOR THE INFINITE FAMILY OF NONEXPANSIVE OPERATORS

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Abstract. In this paper, we introduce a new explicit iterative algorithm for finding a solution of split variational inclusion problem over the common fixed points set of a infinite family of nonexpansive mappings in Hilbert spaces. To reach this goal, the iterative algorithms which combine Tian's method with some fixed point technically proving methods are utilized for solving the problem. Under suitable assumptions, we prove that the sequence generated by the iterative algorithm converges strongly to the unique solution of the considered problem. Our result improves and extends the corresponding results announced by many others.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Moudafi [8] introduced the following Split Monotone Variational Inclusion Problem (SMVIP): find $x^* \in H_1$ such that

$$0 \in f(x^*) + B_1(x^*), \tag{1.1}$$

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and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*), \tag{1.2}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given single-valued operators. Moudafi proposed the following iterative method for solving (1.1)-(1.2): let $\gamma > 0$ and $x_0 \in H_1$ be arbitrary,

$$x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k), \quad k \in N, \tag{1.3}$$

where $\gamma \in (0, 1/L)$ with L being the spectral radius of the operator A^*A, the operator $U := J_\lambda^{B_1}(I - \lambda f)$ and $T := J_\lambda^{B_2}(I - \lambda g)$. He showed that the sequence generated by (1.3) weakly converges to a solution of SMVIP.

If $f \equiv 0$ and $g \equiv 0$ then SMVIP (1.1)-(1.2) reduces to Split Variational Inclusion Problem (SVIP): find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.4}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in B_2(y^*). \tag{1.5}$$

We denote the solution set of SVIP(1.4) and SVIP(1.5) by SOLVIP(B_1) and SOLVIP(B_2), respectively. The solution of set of SVIP(1.4)-(1.5) is denoted by $\Gamma = \{x^* \in H_1 : x^* \in \text{SOLVIP}(B_1) \text{ and } Ax^* \in \text{SOLVIP}(B_2)\}$.

On the other hand, let us recall some iterative methods for solving the fixed point problems of nonexpansive mappings. In 2000, the viscosity approximation method is proposed by Moudafi [9], which is done by considering the approximatae well-posed problem and combing the nonexpansive mapping of T with a contraction of a given mapping f over the nonempty closed convex subset. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$. Under this iterative procedure, the strong convergent result was successfully obtained.

Motivated by Moudafi's viscosity approximation and the method of finding solutions of SVIP, Nimana and Petrot [10] presented the following iterative algorithm:

$$u_n = J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n);$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_n u_n, \quad n \geq 0,$$

where $\{T_n, T\}$ satisfy AKTT-condition [1] if for each subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

Under other given conditions, they proved that the above iterative algorithm converges strongly to $z \in \cap_{i=1}^{\infty} Fix(T_i) \cap \Gamma$, where $z = P_{\cap_{i=1}^{\infty} Fix(T_i) \cap \Gamma} f(z)$. We also know that AKTT-condition is so rigorous, and most of the operators cann't satisfy this condition. This motivates the development of new algorithmic schemes.

Combing Marino and Xu [7] and Yamada [14], Tian [11] considered the following general viscosity type iterative method

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F) T x_n, \tag{1.8}$$

where F is a k-Lipschitzian continuous operator and η-strongly monotone operator with $k > 0, \eta > 0$. He proved that such sequences converges strongly to a common solution of split variational inclusion problem and fixed point problem.

In the present paper, inspired by the above cited works, we consider the problem (1.4)-(1.5) and combine the iterative method (1.8) with Moudafi's SMVIP and constitute a new iterative algorithm. Without the AKTT-condition, we prove that such sequence converges strongly to $\tilde{x} \in \bigcap_{i=1}^{\infty} Fix(T_i) \cap \Gamma$, where $\tilde{x} = P_{\bigcap_{i=1}^{\infty} Fix(T_i) \cap \Gamma} f(\tilde{x})$.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges weakly to x and converges strongly to x, respectively.

In order to prove our results, we collect some necessary conception and lemmas in this section.

Definition 2.1. A mapping $T : H \to H$ is said to be

(i) *nonexpansive*, if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$.
(ii) *firmly nonexpansive*, if $2T - I$ is nonexpansive, or equivalently for all $x, y \in H$, $\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2$.
(iii) *strongly positive*, if there exists a constant $\gamma > 0$ such that $\langle Tx, x\rangle \geq \gamma\|x\|^2$ for all $x \in H$.
(iv) *monotone*, if $\langle Tx - Ty, x - y\rangle \geq 0$ for all $x, y \in H$.
(v) *η-strongly monotone*, if there exists a constant $\eta > 0$ such that $\langle x - y, Tx - Ty\rangle \geq \eta\|x - y\|^2$, for all $x, y \in H$.

Definition 2.2. A multi-valued mapping $B : H \to 2^H$ is called

(i) *monotone*, if $\langle u - v, x - y\rangle \geq 0$ whenever $u \in B(x)$, $v \in B(y)$.
(ii) *maximal*, if, in addition, its graph $gphB := \{(x, y) \in H \times H : y \in B(x)\}$ is not properly contained in the graph of any other monotone operator.

It is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, the inequality

$$\langle (x - Tx) - (y - Ty), Ty - Tx\rangle \leq \frac{1}{2}\|(Tx - x) - (Ty - y)\|^2 \tag{2.1}$$

and therefore, we get, for all $(x, y) \in H_1 \times Fix(T)$,

$$\langle x - Tx, y - Tx\rangle \leq \frac{1}{2}\|Tx - x\|^2, \tag{2.2}$$

see e.g., [[2], Theorem 3.1] and [[3], Theorem 2.1].

Lemma 2.3. ([12]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} \mid \delta_n \mid < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. ([4]) *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and $\{(I - T)x_n\}$ converges strongly to $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.5. *In a Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \ x, y \in H.$$

Lemma 2.6. ([5]) *Let $B : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the resolvent mapping $J_\lambda^B : H \to H$ is defined by*

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x), \ \ \forall\, x \in H,$$

for some $\lambda > 0$. The resolvent operator J_λ^B is single-valued and firmly nonexpansive. It is easy deduced that J_λ^B is nonexpansive and $\frac{1}{2}$-averaged.

Lemma 2.7. ([6]) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then both T_1T_2 and T_2T_1 are α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

3. Main results

Now we state and prove our main result in this paper.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $\{T_n\}$ be a countable family of nonexpansive mappings of a real Hilbert space H, F be a k-Lipschitzian continuous and η-strongly monotone operator on H with $k > 0$ and $\eta > 0$, V be a α-Lipschitzian on H with $\alpha > 0$. Assume that $\Omega = \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap \Gamma \neq \emptyset$. Suppose $x_1 \in H$ and $0 < \mu < \frac{2\eta}{k^2}$. Define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n); \\ y_n = \beta_n u_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i u_n; \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \mu\alpha_n F)y_n, \end{cases}$$

where $\beta \in (0, \frac{1}{L})$, $0 < \gamma < \frac{\tau}{\alpha}$ with $\tau = \mu(\eta - \frac{1}{2}\mu k^2)$. Suppose $\alpha_n \in (0, 1]$ and $\{\beta_n\}$ be a strictly decreasing sequence in $(0, 1]$. If the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iv) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality:

$$\langle(\mu F - \gamma V)x^*, x - x^*\rangle \geq 0, \quad \forall\, x \in \Omega. \tag{3.1}$$

Equivalently, we have $P_\Omega(I - \mu F + \gamma V)x^* = x^*$.

Proof. We proceed with the following steps:

Step 1. We show that $\{x_n\}$ is bounded. In fact, for some point $p \in \Omega$, then we have $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2}(Ap)$ and $T_i p = p$, for all $i \in N$. Since $J_\lambda^{B_1}$ is firmly-nonexpansive, so we have

$$\begin{aligned} \|u_n - p\|^2 &= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - p\|^2 \\ &= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - J_\lambda^{B_1}p\|^2 \\ &\leq \|x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \beta^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\quad + 2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|u_n - p\|^2 \leq \|x_n - p\|^2 &+ \beta^2\langle(J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &+ 2\beta\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \end{aligned} \tag{3.2}$$

Now, we have

$$\beta^2\langle (J_\lambda^{B_2}-I)Ax_n, AA^*(J_\lambda^{B_2}-I)Ax_n\rangle \le L\beta^2\|(J_\lambda^{B_2}-I)Ax_n\|^2. \tag{3.3}$$

Using (2.2) and $J_\lambda^{B_2}Ap = Ap$, we have

$$\begin{aligned}
&2\beta\langle x_n - p, A^*(J_\lambda^{B_2}-I)Ax_n\rangle\\
&= 2\beta\langle A(x_n-p), (J_\lambda^{B_2}-I)Ax_n\rangle\\
&= 2\beta\langle A(x_n-p) + (J_\lambda^{B_2}-I)Ax_n - (J_\lambda^{B_2}-I)Ax_n, (J_\lambda^{B_2}-I)Ax_n\rangle\\
&= 2\beta\{\langle J_\lambda^{B_2}Ax_n - Ap, (J_\lambda^{B_2}-I)Ax_n\rangle - \|(J_\lambda^{B_2}-I)Ax_n\|^2\}\\
&\le 2\beta\left\{\frac{1}{2}\|(J_\lambda^{B_2}-I)Ax_n\|^2 - \|(J_\lambda^{B_2}-I)Ax_n\|^2\right\}\\
&\le -\beta\|(J_\lambda^{B_2}-I)Ax_n\|^2.
\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), it follows that

$$\begin{aligned}
\|u_n - p\|^2 &\le \|x_n + \beta A^*(J_\lambda^{B_2}-I)Ax_n - p\|^2\\
&\le \|x_n - p\|^2 + \beta(L\beta - 1)\|(J_\lambda^{B_2}-I)Ax_n\|^2.
\end{aligned} \tag{3.5}$$

Since $\beta \in (0, \frac{1}{L})$, we have $\|u_n - p\|^2 \le \|x_n - p\|^2$.

On the other hand, noting that

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n u_n + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)T_i u_n - p\|\\
&\le \beta_n\|u_n - p\| + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|T_i u_n - p\|\\
&\le \beta_n\|u_n - p\| + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|u_n - p\|\\
&= \|u_n - p\|.
\end{aligned} \tag{3.6}$$

So, we can deduce

$$\|y_n - p\| \le \|x_n - p\|. \tag{3.7}$$

Next, we estimate

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma V x_n + (I - \mu\alpha_n F)y_n - p\|\\
&= \|\alpha_n\gamma V x_n - \alpha_n\gamma V p + \alpha_n\gamma V p + (I - \mu\alpha_n F)y_n\\
&\quad - (I - \mu\alpha_n F)p - \mu\alpha_n F p\|\\
&\le \alpha_n\gamma\alpha\|x_n - p\| + (1 - \alpha_n\tau)\|y_n - p\| + \alpha_n\|\gamma V p - \mu F p\|\\
&\le \big(1 - \alpha_n(\tau - \gamma\alpha)\big)\|x_n - p\| + \alpha_n\|\gamma V p - \mu F p\|
\end{aligned}$$

$$
\begin{aligned}
&\le \max\left\{\|x_n-p\|, \frac{\|\gamma Vp-\mu Fp\|}{\tau-\gamma\alpha}\right\}\\
&\vdots\\
&\le \max\left\{\|x_1-p\|, \frac{\|\gamma Vp-\mu Fp\|}{\tau-\gamma\alpha}\right\}.
\end{aligned}
$$

Hence, $\{x_n\}$ is bounded. Therefore we can obtain that $\{y_n\}$, $\{u_n\}$, $\{T_iu_n\}$, $\{Fy_n\}$ and $\{Vx_n\}$ are bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Noting that

$$
\begin{aligned}
&\|x_{n+1}-x_n\|\\
&=\|\alpha_n\gamma Vx_n+(I-\mu\alpha_nF)y_n-(\alpha_{n-1}\gamma Vx_{n-1}\\
&\quad+(I-\mu\alpha_{n-1}F)y_{n-1})\|\\
&=\|\alpha_n\gamma Vx_n-\alpha_n\gamma Vx_{n-1}+\alpha_n\gamma Vx_{n-1}-\alpha_{n-1}\gamma Vx_{n-1}\\
&\quad+(I-\mu\alpha_nF)y_n-(I-\mu\alpha_nF)y_{n-1}+(I-\mu\alpha_nF)y_{n-1}\\
&\quad-(I-\mu\alpha_{n-1}F)y_{n-1}\|\\
&\le\alpha_n\gamma\alpha\|x_n-x_{n-1}\|+(1-\alpha_n\tau)\|y_n-y_{n-1}\|\\
&\quad+\gamma|\alpha_n-\alpha_{n-1}|\|Vx_{n-1}\|+\mu|\alpha_n-\alpha_{n-1}|\|Fy_{n-1}\|\\
&\le\alpha_n\gamma\alpha\|x_n-x_{n-1}\|+(1-\alpha_n\tau)\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|K,
\end{aligned}
\tag{3.8}
$$

where $K:=\sup\{\gamma\|V(x_{n-1})\|+\mu\|Fy_{n-1}\|:n\in N\}$. At the same time, we observe that

$$
\begin{aligned}
&\|y_n-y_{n-1}\|\\
&=\|\beta_nu_n+\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)T_iu_n-\beta_{n-1}u_{n-1}-\sum_{i=1}^{n-1}(\beta_{i-1}-\beta_i)T_iu_{n-1}\|\\
&\le\beta_n\|u_n-u_{n-1}\|+|\beta_n-\beta_{n-1}|\|u_{n-1}\|\\
&\quad+\sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|T_iu_n-T_iu_{n-1}\|+|\beta_n-\beta_{n-1}|\|T_nu_{n-1}\|\\
&\le\|u_n-u_{n-1}\|+|\beta_n-\beta_{n-1}|(\|u_{n-1}\|+\|T_nu_{n-1}\|).
\end{aligned}
\tag{3.9}
$$

By Lemma 2.6, we know that $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ both are firmly nonexpansive, so are averaged. For $\beta\in(0,\frac{1}{L})$, the mapping $(I+\beta A^*(J_\lambda^{B_2}-I)A)$ is averaged (see [8]). Using Lemma 2.7, we know that the mapping $J_\lambda^{B_1}(I+\beta A^*(J_\lambda^{B_2}-I)A)$

is averaged and hence nonexpansive, then we have

$$\begin{aligned}
&\|u_n - u_{n-1}\| \\
&= \|J_\lambda^{B_1}(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1}(x_{n-1} + \beta A^*(J_\lambda^{B_2} - I)Ax_{n-1})\| \\
&\leq \|J_\lambda^{B_1}(I + \beta A^*(J_\lambda^{B_2} - I)A)x_n - J_\lambda^{B_1}(I + \beta A^*(J_\lambda^{B_2} - I)A)x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\|.
\end{aligned} \tag{3.10}$$

Combing (3.8), (3.9) and (3.10), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| \leq \big(1 - \alpha_n(\tau - \gamma\alpha)\big)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K \\
+ |\beta_n - \beta_{n-1}|(1 - \alpha_n\tau)(\|u_{n-1}\| + \|T_n u_{n-1}\|).
\end{aligned}$$

Noticing the conditions (ii), (iii) and (iv), by Lemma 2.3, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. We show $\lim_{n\to\infty} \|T_i u_n - u_n\| = 0$ for all $i \in N$.
Since $p \in \Omega$, we note that

$$\begin{aligned}
\|u_n - p\|^2 &\geq \|T_i u_n - T_i p\|^2 \\
&= \|T_i u_n - u_n + u_n - p\|^2 \\
&= \|T_i u_n - u_n\|^2 + \|u_n - p\|^2 + 2\langle T_i u_n - u_n, u_n - p\rangle,
\end{aligned}$$

which imply that

$$\frac{1}{2}\|T_i u_n - u_n\|^2 \leq \langle u_n - T_i u_n, u_n - p\rangle. \tag{3.11}$$

So, we can deduce

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i u_n - u_n\|^2 \\
&\leq \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\langle u_n - T_i u_n, u_n - p\rangle \\
&= \langle (1 - \beta_n)u_n - \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i u_n, u_n - p\rangle \\
&= \langle (1 - \beta_n)u_n - y_n + \beta_n u_n, u_n - p\rangle \\
&= \langle u_n - y_n, u_n - p\rangle \\
&= \langle u_n - x_n, u_n - p\rangle + \langle x_n - x_{n+1}, u_n - p\rangle \\
&\quad + \langle x_{n+1} - y_n, u_n - p\rangle.
\end{aligned} \tag{3.12}$$

Since $\alpha_n \to 0$ as $n \to \infty$, it follows that

$$\begin{aligned}\lim_{n\to\infty} \|x_{n+1} - y_n\| &= \lim_{n\to\infty} \|\alpha_n \gamma V x_n + (I - \mu\alpha_n F)y_n - y_n\| \\ &= \lim_{n\to\infty} \alpha_n \|\gamma V x_n - \mu F y_n\| \\ &= 0.\end{aligned}$$

Next, we claim that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. By (3.5), (3.6) and (3.14), we note that

$$\begin{aligned}\|x_{n+1} - p\|^2 \le\ & (1 - \alpha_n\tau)^2(\|x_n - p\|^2 + \beta(L\beta - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2) \\ & + 2\alpha_n\langle \gamma V x_n - \mu F p, x_{n+1} - p\rangle \\ \le\ & \|x_n - p\|^2 + \beta(L\beta - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ & + 2\alpha_n\langle \gamma V x_n - \mu F p, x_{n+1} - p\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\beta(1 - L\beta)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\|\gamma V x_n - \mu F p\|\|x_{n+1} - p\| \\ &\le \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n\|\gamma V x_n - \mu F p\|\|x_{n+1} - p\|.\end{aligned}$$

Since $1 - L\beta > 0$, $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} \|(J_\lambda^{B_2} - I)Ax_n\| = 0. \tag{3.13}$$

Now, the firmly nonexpansiveness of $J_\lambda^{B_1}$ implies that

$$\begin{aligned}\|u_n - p\|^2 &= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - p\|^2 \\ &= \|J_\lambda^{B_1}\big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n\big) - J_\lambda^{B_1}p\|^2 \\ &\le \langle u_n - p, x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\rangle \\ &= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\quad - \|(u_n - p) - \big(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n - p\big)\|^2\} \\ &= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 + \beta(L\beta - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\quad - \|u_n - x_n - \beta A^*(J_\lambda^{B_2} - I)Ax_n\|^2\} \\ &\le \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - \big(\|u_n - x_n\|^2 \\ &\quad + \beta^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 - 2\beta\langle u_n - x_n, A^*(J_\lambda^{B_2} - I)Ax_n\rangle\big)\} \\ &\le \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 \\ &\quad + 2\beta\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|\}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\|u_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\beta\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|. \end{aligned} \tag{3.14}$$

Subsequently, by Lemma 2.5, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n \gamma V x_n + (I - \mu\alpha_n F)y_n - p\|^2 \\ &= \|\alpha_n(\gamma V x_n - \mu F p) + (I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)p\|^2 \\ &\leq (1 - \alpha_n\tau)^2\|y_n - p\|^2 + 2\alpha_n\langle \gamma V x_n - \mu F p, x_{n+1} - p\rangle. \end{aligned} \tag{3.15}$$

It follows from (3.6), (3.14) and (3.15) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 - \alpha_n\tau)^2\|u_n - p\|^2 + 2\alpha_n\langle \gamma V x_n - \mu F p, x_{n+1} - p\rangle \\ &\leq (1 - \alpha_n\tau)^2\{\|x_n - p\|^2 - \|u_n - x_n\|^2 \\ &\quad + 2\beta\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|\} \\ &\quad + 2\alpha_n\langle \gamma V x_n - \mu F p, x_{n+1} - p\rangle \\ &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\beta\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\ &\quad + 2\alpha_n\|\gamma V x_n - \mu F p\|\|x_{n+1} - p\|, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \|u_n - x_n\|^2 \leq \|x_{n+1} - x_n\|&(\|x_n - p\| + \|x_{n+1} - p\|) \\ &+ 2\beta\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\ &+ 2\alpha_n\|\gamma V x_n - \mu F p\|\|x_{n+1} - p\|. \end{aligned} \tag{3.16}$$

Hence, by (3.16), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.17}$$

Furthermore, by (3.12), we can deduce $\lim_{n\to\infty} \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i u_n - u_n\|^2 = 0$. Since $\{\beta_n\}$ is strictly decreasing, it follows that for every $i \in N$,

$$\lim_{n\to\infty} \|T_i u_n - u_n\| = 0.$$

Step 4. We claim that $\limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, x_{n+1} - \tilde{x}\rangle \leq 0$ where $\tilde{x} = P_\Omega(I - \mu F + \gamma V)\tilde{x}$.

Since $\{u_n\}$ is bounded, so, there exist a point $u^* \in H_1$ and a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, u_n - \tilde{x}\rangle = \lim_{j\to\infty}\langle(\gamma V - \mu F)\tilde{x}, u_{n_j} - \tilde{x}\rangle$ and $u_{n_j} \rightharpoonup u^*$. Now, T_i being nonexpansive, by Lemma 2.4 and $\lim_{n\to\infty}\|T_i u_n - u_n\| = 0$, we obtain that $u^* \in Fix(T_i)$. On the other hand,

$u_{n_j} = J_\lambda^{B_1}\big(x_{n_j} + \beta A^*(J_\lambda^{B_2} - I)Ax_{n_j}\big)$ can be written as

$$\frac{x_{n_j} - u_{n_j} + \beta A^*(J_\lambda^{B_2} - I)Ax_{n_j}}{\lambda} \in B_1 u_{n_j}. \tag{3.18}$$

By passing to limit $j \to \infty$ in (3.18) and by taking into account (3.13), (3.17) and the fact that the graph of maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(u^*)$, i.e., $u^* \in \mathrm{SOLVIP}(B_1)$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behaviour, $\{Ax_{n_j}\}$ weakly converges to Au^*. Again, by (3.13) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive and Lemma 2.4, we obtain that $Au^* \in J_\lambda^{B_2}(Au^*)$, i.e., $Au^* \in \mathrm{SOLVIP}(B_2)$. Thus $u^* \in \Omega$.

By obtuse angle principle,

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, x_{n+1} - \tilde{x}\rangle &= \limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, u_n - \tilde{x}\rangle \\ &= \lim_{j\to\infty}\langle(\gamma V - \mu F)\tilde{x}, u_{n_j} - \tilde{x}\rangle \\ &= \langle(\gamma V - \mu F)\tilde{x}, u^* - \tilde{x}\rangle \\ &\le 0.\end{aligned} \tag{3.19}$$

Step 5. We show that $x_n \to \tilde{x}$.
Noting that

$$\begin{aligned}&\|x_{n+1} - \tilde{x}\|^2 \\ &= \|\alpha_n \gamma V x_n + (I - \mu\alpha_n F)y_n - \tilde{x}\|^2 \\ &= \|(I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)\tilde{x} + \alpha_n(\gamma V x_n - \mu F\tilde{x}\|^2 \\ &\le \|(I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)\tilde{x}\|^2 + 2\alpha_n\langle\gamma V x_n - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &\le (1 - \alpha_n\tau)^2\|y_n - \tilde{x}\|^2 + 2\alpha_n\langle\gamma V x_n - \gamma V\tilde{x} + \gamma V\tilde{x} - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &\le (1 - \alpha_n\tau)^2\|x_n - \tilde{x}\|^2 + \alpha_n\gamma\alpha\big(\|x_n - \tilde{x}\|^2 + \|x_{n+1} - \tilde{x}\|^2\big) \\ &\quad + 2\alpha_n\langle(\gamma V - \mu F)\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &= \big((1 - \alpha_n\tau)^2 + \alpha_n\gamma\alpha\big)\|x_n - \tilde{x}\|^2 + \alpha_n\gamma\alpha\|x_{n+1} - \tilde{x}\|^2 \\ &\quad + 2\alpha_n\langle(\gamma V - \mu F)\tilde{x}, x_{n+1} - \tilde{x}\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_{n+1} - \tilde{x}\|^2 \\ &\le \frac{(1 - \alpha_n\tau)^2 - \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}\|x_n - \tilde{x}\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle(\gamma V - \mu F)\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &\le \Big(1 - \frac{2\alpha_n(\tau - \gamma\alpha)}{1 - \alpha_n\gamma\alpha}\Big)\|x_n - \tilde{x}\|^2\end{aligned}$$

$$+\frac{2\alpha_n(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\left(\langle(\gamma V-\mu F)\tilde{x},x_{n+1}-\tilde{x}\rangle+\frac{\alpha_n\tau^2}{2(\tau-\gamma\alpha)}\|x_n-\tilde{x}\|^2\right).$$

Consequently, according to (3.19) and Lemma 2.3, we deduce that $\{x_n\}$ converges strongly to $\tilde{x}$. This completes the proof. □

4. Numerical examples

In this part, we present a numerical example to demonstrate the performance and convergence of our result as follows.

Example 4.1. Let $H_1 = R^2$, $H_2 = R^2$ and we define $B_1 : R^2 \to R^2$ and $B_2 : R^2 \to R^2$ by $B_1 = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$, $B_2 = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}$.

Let $A \in R^{2\times 2} : H_1 \to H_2$ be non-singular matrix operator in which elements are random and A^* be an adjoint of A. Let $L = \|A^*A\|_2$ and $\beta \in (0, \frac{1}{L})$ is random. It is easy known that B_1 and B_2 are linear maximal monotone operators. So, the resolvent operators of B_1 and B_2 are defined by $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ where $\lambda > 0$, respectively. Let $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{n+2}$ and $\{T_n\}$ be a class of non-singular matrix operators which $\|T_n\| = 1$ and the element of $\{T_n\}$ is random. We assume that $V = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $F = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. So, It is clear that $\alpha = 2$ and $k = \eta = 3$. Let γ and μ are random in the interval $(0, \frac{\tau}{\alpha})$ and $(0, \frac{2\eta}{k^2})$ respectively, where $\tau = \mu(\eta - \frac{1}{2}\mu k^2)$ and $\frac{2\eta}{k^2} = \frac{2}{3}$.

Next, we state our new algorithm via a numerical example.

Step 1. Choose the initial value for the iterative algorithm x_1 is random in interval $(0, 1\times 10^6) \times (0, 1 \times 10^6)$, and the others is defined as above.

Step 2. Given the iterative algorithm as follows:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \beta A^*(J_\lambda^{B_2} - I)Ax_n); \\ y_n = \frac{1}{n+2}u_n + \sum_{i=1}^n(\frac{1}{i+1} - \frac{1}{i+2})T_i u_n; \\ x_{n+1} = \frac{1}{n+1}\gamma V x_n + (I - \mu\frac{1}{n+1}F)y_n. \end{cases}$$

Step 3. Put $n = n + 1$ and go to step 2.

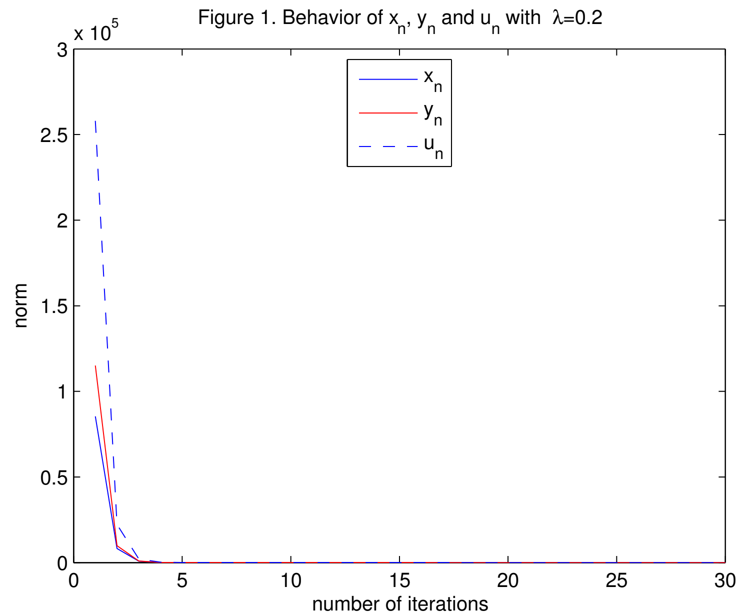


Figure 1. Behavior of x_n, y_n and u_n with λ=0.2
x 10^5
norm
number of iterations
x_n
y_n
u_n
0
0.5
1
1.5
2
2.5
3
5
10
15
20
25
30

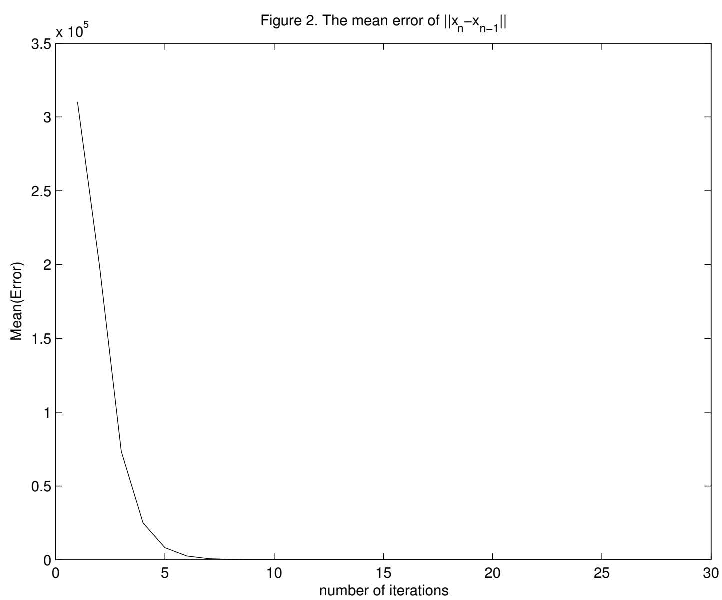


Figure 2. The mean error of ||x_n−x_{n−1}||
x 10^5
Mean(Error)
number of iterations
0
0.5
1
1.5
2
2.5
3
3.5
5
10
15
20
25
30

Figure 3. The mean error of $||y_n - y_{n-1}||$

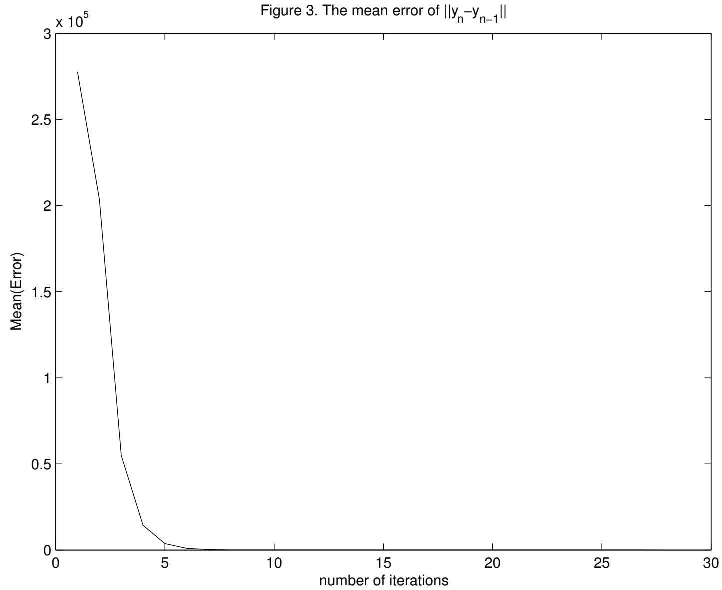
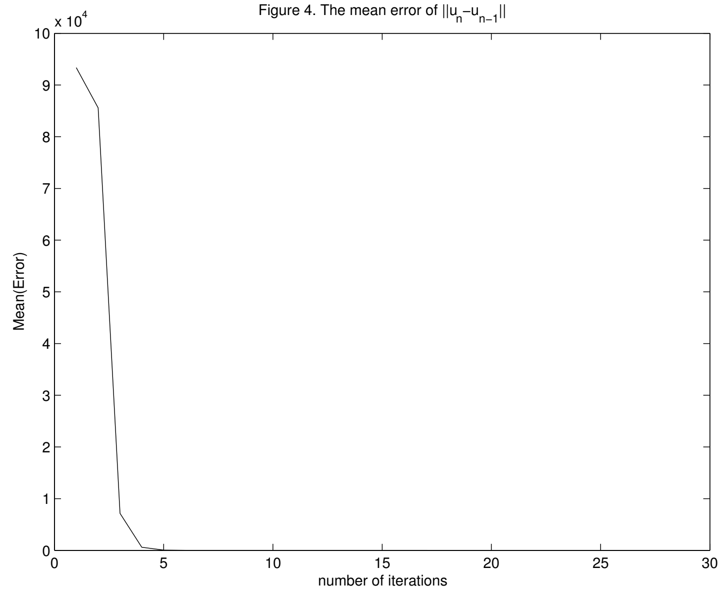


Figure 4. The mean error of $||u_n - u_{n-1}||$



By Algorithm 4.1, we present the convergence analysis of our iterative algorithm as follows:

Figure.1 shows that the norms of x_n, y_n and u_n converge to the same number 0 with a high speed. Figure.2, Figure.3 and Figure.4 show the error behaviors of $\|x_n - x_{n-1}\|$, $\|y_n - y_{n-1}\|$ and $\|u_n - u_{n-1}\|$ respectively.

According to the numerical example for the new explicit iterative algorithm for finding a solution of split variational inclusion problem over the common fixed points set of a infinite family of nonexpansive mappings in Hilbert spaces in this paper, it is clear that the convergent speed is so quickly even though the AKTT-condition dose not exist; see Figure.1.

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