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FIXED POINT THEOREM FOR WEAKLY CONTRACTIVE MAPS IN METRICALLY CONVEX SPACES

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Abstract. A fixed point theorem for weakly contractive mappings is proved which satisfy a generalized contraction condition on a complete metrically convex metric space. The result in this paper generalizes the relevant results due to Rhoades [13], Alber and Guerre-Delabriere [2] and others. An illustrative example is also furnished in support of our result.

1. Introduction

There are many results on fixed point theorems for self mapping of a Banach space. In this direction Banach Contraction Principle is one of the celebrated theorem. This theorem is widely considered as a source of metric fixed point theory. Significantly, it has huge applications not only in metric fixed point theory but in different areas of mathematical research. For the sake of completeness here we mentioned this celebrated theorem.

Let (X, d) be a complete metric space and let $T : X \to X$ satisfy the contraction condition:

$$d(Tx, Ty) \leq kd(x, y), \tag{1.1}$$

for all $x, y \in X$, where $0 \leq k < 1$. Then T has a unique fixed point.

There exists a various extention and generalisation of the above said theorem, here we mentioned a few, we cite [5]-[9].

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Recently, Alber and Guerre-Delabriere [2] coined the concept of weakly contractive maps and obtained fixed point results in Hilbert space for self mappings. Rhoades [13] extended some of their works to Banach spaces for the same setting.

However, in many applications the mappings involved is not always a self map. So, it is interested to find sufficient conditions for which such mappings will guarantee the existence of a fixed point. Assad and Kirk [4] initiated the study of fixed point of nonself mappings in metrically convex spaces. The technique due to Assad and Kirk [4] has been utilized by many researchers and there exists considerable literature on this topic. To mentioned a few, we cite [3,4,10,11,12,14].

In this paper, we prove a fixed point theorem for single valued nonself mappings by utilizing the idea of Rhoades [13] which either partially or completely generalize the results due to Rhoades [13], Alber and Guerre-Delabriere [2] and others. Here, we state the result of Rhoades [13] which runs as follows:

Theorem 1.1. *Let (X, d) be a complete metric space, T a weakly contractive map. Then T has a unique fixed point p in X.*

Definition 1.2. Let K be a nonempty subset of a metric space (X, d). A mapping $T : K \to X$ is said to be weakly contractive if $Tx \in K, Tx \cap K$ is nonempty and $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$ where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function with $\phi(t) = 0$ iff $t = 0$.

Definition 1.3. ([4]) A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

2. Result

Our result is proved for single valued nonself mappings for weakly contractive maps.

Theorem 2.1. *Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T : K \to X$ satisfying:*

(i) *for each $x \in \delta K, Tx \in K$, and*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \tag{2.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function with $\phi(t) = 0$ iff $t = 0$.

Then T has a unique fixed point in K.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x_0 \in K$. Define $y_1 = Tx_0$. If $y_1 \in K$ set $y_1 = x_1$. If $y_1 \notin K$, then choose $x_1 \in \delta K$ so that

$$d(x_0, x_1) + d(x_1, y_1) = d(x_0, y_1).$$

If $y_2 \in K$ then set $y_2 = x_2$. If $y_2 \notin K$, then choose $x_2 \in \delta K$ so that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(ii) $y_{n+1} = Tx_n$,
(iii) $y_n = x_n$ if $y_n \in K$,
(iv) If $x_n \in \delta K$, then $d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n)$, where $y_n \notin K$.

Here, one obtains two types of sets we denote as follows:

$$P = \{x_i \in \{x_n\} : x_i = y_i\} \text{ and } Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

One can note that if $x_n \in Q$ then x_{n-1} and $x_{n+1} \in P$.

We wish to estimate $d(x_n, x_{n+1})$. Now we distinguish the following three cases.

Case 1. If x_n and $x_{n+1} \in P$, then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) - \phi(d(x_{n-1}, x_n)) \\ &\leq d(x_{n-1}, x_n), \text{ (using monotone property of } \phi \text{ function).} \end{aligned}$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1}).$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, y_{n+1}) = d(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) - \phi(d(x_{n-1}, x_n)) \\ &\leq d(x_{n-1}, x_n), \text{ (as in Case 1).} \end{aligned}$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$. Since $x_n \in Q$ and is a convex linear combination of x_{n-1} and y_n, it follows that

$$d(x_n, x_{n+1}) \leq \max\{d(x_{n-1}, x_{n+1}), d(y_n, x_{n+1})\}.$$

If $d(x_{n-1}, x_{n+1}) \leq d(x_{n+1}, y_n)$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_{n+1}, y_n) = d(Tx_{n-1}, Tx_n) \\ &\leq d(x_{n-1}, x_n) - \phi(d(x_{n-1}, x_n)) \\ &\leq d(x_{n-1}, x_n), \text{ (using monotone property of } \phi \text{ function).} \end{aligned}$$

Otherwise if $d(x_{n+1}, y_n) \leq d(x_{n-1}, x_{n+1})$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_{n-1}, x_{n+1}) = d(Tx_{n-2}, Tx_n) \\ &\leq d(x_{n-2}, x_n) - \phi(d(x_{n-2}, x_n)) \\ &\leq d(x_{n-2}, x_n). \end{aligned}$$

Notice that

$$\begin{aligned} d(x_{n-2}, x_n) &\leq d(x_{n-2}, x_{n-1}) \ + \ d(x_{n-1}, x_n) \\ &\leq \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Here, if $d(x_{n-2}, x_{n-1}) \leq d(x_{n-1}, x_n)$, then $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. Otherwise, if $d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1})$, then $d(x_n, x_{n+1}) \leq d(x_{n-2}, x_{n-1})$. Thus in all the cases, we have

$$d(x_n, x_{n+1}) \leq \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}.$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing. Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Let on contrary that the sequence $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$.

Further, corresponding to each $m(k)$, we can find $n(k)$ in such a way that the smallest positive integer $n(k) > m(k)$ satisfying $d(x_{m_k}, x_{n_{k-1}}) < \epsilon$. Now we have

$$\epsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) < \epsilon + d(x_{n_{k-1}}, x_{n_k}).$$

On letting $k \to \infty$, we have $d(x_{m_k}, x_{n_k}) = \epsilon$. Again,

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k})$$

whereas

$$d(x_{n_{k-1}}, x_{m_{k-1}}) \leq d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_{k-1}}).$$

Now on letting $k \to \infty$ in the above inequalities, one obtains,

$$\lim_{k \to \infty} d(x_{n_{k-1}}, x_{m_{k-1}}) = \epsilon.$$

By setting $x = x_{m_{k-1}}$ and $y = x_{n_{k-1}}$ in (2.1), we obtain,

$$\epsilon \leq d(x_{m_{k-1}}, x_{n_{k-1}}) - \phi(d(x_{m_{k-1}}, x_{n_{k-1}})).$$

On letting $k \to \infty$, we have $\epsilon > 0$, which is a contradiction. Thus the sequence $\{x_0, x_1, x_2, x_3, \cdots, x_{n-1}, x_n, x_{n+1}, \cdots\}$ is Cauchy and hence convergent.

Let $x_n \to z$ as $n \to \infty$. Substituting $x = x_{n-1}$ and $y = z$ in equation (2.1) we obtain

$$d(Tz, x_n) \leq d(z, x_{n-1}) - \phi(d(x_{n-1}, z)).$$

Letting $n \to \infty$ and using continuity of ϕ, we have,

$$d(Tz, z) \leq 0 - \phi(0),$$

implying thereby $Tz = z$. This shows that z is a fixed point of T. To prove that the uniqueness of fixed points.

Let us suppose that z_1 and z_2 are two fixed points of T. Then

$$\begin{aligned} & d(Tz_1, Tz_2) \leq d(z_1, z_2) - \phi(d(z_1, z_2)) \\ \Rightarrow\ & d(z_1, z_2) \leq d(z_1, z_2) - \phi(d(z_1, z_2)) \\ \Rightarrow\ & \phi(d(z_1, z_2)) \leq 0, \end{aligned}$$

implying thereby $z_1 = z_2$. This shows the uniqueness of fixed point. □

Remark 2.2. By setting $K = X$ in Theorem 2.1, one deduces a theorem due to Rhoades [13].

Remark 2.3. By setting $K = X$ in Theorem 2.1, one deduces a partial generalization of theorem due to Alber and Guerre-Delabriere [2].

Corollary 2.4. *Let X is a Banach space. Let B be a nonempty closed convex subset of X and K a nonempty closed subset of B. Let $T : K \to B$ satisfying the contraction condition* (2.1) *with the property that $x \in \delta K$ relative to B implies that $Tx \in K$. Then T has a unique fixed point.*

By setting $K = X,\ \phi(t) = kt,\ 0 \leq k < 1$ in Theorem 2.1, one deduces a corollary in the form of Banach Contraction Principle.

Corollary 2.5. *Let (X, d) be a complete metric space and let $T : X \to X$ satisfying contraction condition $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, where $0 \leq k < 1$. Then T has a unique fixed point.*

3. An illustrative example

Finally, we furnish an example to establish the utility of our result.

Example 3.1. Let $X = R$ with Euclidean metric and $K = [0, 1]$. Define $T : K \to X$ and $\phi : [0, \infty) \to [0, \infty)$ as

$$Tx = \begin{cases} (x - x^2)\ , & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1\ , & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \qquad \text{and} \quad \phi(t) = \begin{cases} t^2\ , & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1\ , & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

Since δK(boundary of K) $= \{0, 1\}$, where $0 \in \delta K \Rightarrow T0 = 0 \in K$ and $1 \in \delta K \Rightarrow T1 = 1 \in K$.

Moreover, for the verification of contraction condition (2.1), the following cases arise:

Case 1. If $0 \leq x, y \leq \frac{1}{2}$, then

$$\begin{aligned} d(Tx, Ty) &= (x - x^2) - (y - y^2) = (x - y) - (x^2 - y^2) \\ &= (x - y) - (x - y)(x + y) \\ &\leq (x - y) - (x - y)(x - y) = (x - y) - (x - y)^2 \\ &\leq d(x, y) - \phi(d(x, y)). \end{aligned}$$

Case 2. If $0 \leq x \leq \frac{1}{2}$ and $\frac{1}{2} < y \leq 1$, then

$$d(Tx, Ty) = (x - x^2) - 1 \leq (x - y) - 1 \leq d(x, y) - \phi(d(x, y)).$$

Case 3. If $\frac{1}{2} < x, y \leq 1$, then

$$d(Tx, Ty) = 0 \leq d(x, y) - \phi(d(x, y)).$$

Thus the contraction condition (2.1) and all other conditions of the Theorem 2.1 are satisfied. Note that '0' is the fixed point of T.

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