Nonlinear Functional Analysis and Applications
Vol. 21, No. 4 (2016), pp. 693-715

http://nfaa.kyungnam.ac.kr/journal-nfaa

A CLASS OF SEMILINEAR INFINITE DIMENSIONAL UNCERTAIN SYSTEMS ON BANACH SPACES CONTROLLED BY VECTOR MEASURES CONTAINING DIRAC MEASURES

N. U. Ahmed

Department of EECS
University of Ottawa, Ottawa, Canada
e-mail: ahmed@site.uottawa.ca

Abstract. In this paper we consider a class of semilinear uncertain dynamic systems on Banach spaces subject to dynamic and additive uncertainty without any probabilistic structure. The problem is to find a control law that minimizes the maximum risk. We present two distinct results on the existence of optimal controls in the presence of system uncertainty and additive unstructured disturbance. The first result is concerned with the question of existence of optimal controls from the class of general vector measures, and the second result is concerned with the question of existence of optimal policies from the class of purely impulsive controls.

1. Introduction

In this paper we consider optimal control problems in the presence of uncertainty in the system dynamics and additive perturbation without assuming any particular probabilistic structure. The problem is formulated as min-max problem minimizing the maximum loss or equivalently maximizing minimum payoff. These problems are substantially more difficult compared to the problems of optimal control of deterministic finite or infinite dimensional systems [13,16,19]. In recent years substantial interest in impulse driven systems has been noted. In particular, important and interesting applications in physics,

[0]Received March 12, 2016. Revised June 16, 2016.
[0]2010 Mathematics Subject Classification: 49J24, 49J27, 49K24, 49K35, 93E20, 93B52, 35R60, 35R70, 34G25, 34H05, 47A62.
[0]Keywords: Uncertain systems on Banach spaces, optimal control, vector measures, Dirac measures, Radon Nikodym Property (RNP), optimal controls.

engineering, biology and medicine [17,21,22,24,25,26] have been recorded. Impulse driven systems are particular cases of measure driven systems, particularly measures taking values in Banach spaces. In this area substantial progress has been made as seen in the work of the author [2]-[14] including the references therein. Most of the systems in this series of papers are governed by semilinear evolution equations with the principal operator being the infinitesimal generator of C_0-semigroups except the reference [8]. In [8] the principal operator is nonlinear, monotone, hemicontinuous, and driven by signed measure. In most of these cases, the system is infinite dimensional offering the possibility of further applications in distributed parameter systems (for example, systems governed by partial differential equations) arising in physical and biological sciences. Here in this paper we consider optimal control by use of vector measures as well as Dirac measures which include the impulse controls as special cases. We present a unified approach not only for control of deterministic systems but also for uncertain systems. This is very important because most of the systems used in physical and biological sciences are incomplete in the sense that the values of the fundamental parameters determining the system are very often only the best estimates thereby introducing uncertainty. Further, there are unknown external forces perturbing the systems. These are systems [1] subject to bounded parametric uncertainty as well as unstructured external forces and therefore do not fall into the class of stochastic differential equations driven by Brownian motion or Poisson random measures. It is expected that more theoretical development will be inspired by applications.

The rest of the paper is organized as follows. In section 2, we present some typical notations. In section 3, we present the mathematical model describing the system including the uncertainties and formulate the problem considered in the paper. After introducing the basic assumptions the question of existence of solutions and their regularity properties are presented. In section 4, we present results on continuous dependence of solutions on the operators representing perturbation of the semigroup (generator), and the process representing additive noise. In section 5, these results are used to prove the existence of optimal control policies from the class of vector measures. In section 6, purely impulsive controls are considered and existence of optimal policies are proved.

2. Some notations

Let $\{X, Y, U\}$ denote a triple of real separable Banach spaces representing the state space, the space of additive uncertainty, and the control space respectively. Let $I = [0, T]$ denote any closed bounded interval. For any separable reflexive Banach space Z, we let $L_1(I, Z)$ denote the space of Bochner integrable functions with values in Z, and its dual by $L_\infty(I, Z^*)$. Let Z_1, Z_2 be any pair of real Banach spaces and $\mathcal{L}(Z_1, Z_2)$ the Banach space of bounded linear operators from Z_1 to Z_2.

In any Banach space Z, and for any $s \geq 0$, let $B_s(Z)$ denote the closed ball of radius $s > 0$ centered at the origin. Let $B_\infty(I, \mathcal{L}(Z_1, Z_2))$ denote the space of operator valued functions which are measurable in the uniform operator topology and uniformly bounded on the interval I in the sense that

$$\sup\{\| T(t) \|_{\mathcal{L}(Z_1,Z_2)}, t \in I\} < \infty$$

for $T \in B_\infty(I, \mathcal{L}(Z_1, Z_2))$. Suppose this is furnished with the topology of strong convergence (convergence in the strong operator topology) uniformly on I in the sense that, given $T_n, T \in B_\infty(I, \mathcal{L}(Z_1, Z_2))$, $T_n \to T$ in this topology iff for every $z \in Z_1$,

$$\sup\{|T_n(t)z - T(t)z|_{Z_2}, t \in I\} \to 0$$

as $n \to \infty$. In particular, for any normed space Z, $B_\infty(I, Z)$ denotes the class of bounded measurable functions defined on I and taking values from Z.

3. System with uncertainties and problem formulation

Let X, Y, U be real Banach spaces with topological duals denoted by $\{X^*, Y^*, U^*\}$. The space X denotes the state space where x takes its values from, Y is the space where additive noise (uncertainty) ξ takes its values from, and U is the space where the controls u take their values from. The system is governed by the following semi-linear evolution equation in the Banach space X,

$$\begin{aligned} &dx = Axdt + R(t)xdt + F(t, x(t))dt + G(t)\xi(t)dt + B(t)u(dt), \\ &x(0) = x_0, \ t \in I \equiv [0, T], \ \ T < \infty, \end{aligned} \tag{3.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, of bounded linear operators on X. The operator valued process R, perturbing the semigroup generator A, takes values from the Banach space $\mathcal{L}(X)$ of bounded linear operators in X.

This represents the uncertainty in the system dynamics (modeling uncertainty), in the sense that the exact value of R at any given time is not known, but it is known that it takes values from a bounded set in $\mathcal{L}(X)$, for example, the closed ball $B_\gamma(\mathcal{L}(X))$ of radius $\gamma > 0$ around the origin. We denote this class of operator valued functions by $\mathcal{V}_\gamma \equiv B_\infty(I, B_\gamma(\mathcal{L}(X)))$.

The function $F : I \times X \longrightarrow X$ represents nonlinearity in the system. For most practical situations, it is reasonable to assume that the disturbance process $\{\xi\}$ is bounded. So without any loss of generality we may assume that the process $\{\xi\}$ is strongly measurable taking values from the closed ball $B_\delta(Y)$ of radius $\delta > 0$ and centered at the origin. We denote this class of disturbance processes by $\mathcal{D}_\delta \equiv B_\infty(I, B_\delta(Y))$. The operator $G \in L_\infty(I, \mathcal{L}(Y, X))$, the space of essentially bounded uniformly measurable operator valued functions while the operator $B \in B_\infty(I, \mathcal{L}(U, X))$, the space of bounded uniformly measurable operator valued functions.

Let Σ_I denote the sigma algebra of Borel subsets of the set I and $\mathcal{M}_{cabv}(\Sigma_I, U)$ the space of countably additive bounded (in total variation norm) U valued vector measures. For any $\mu \in \mathcal{M}_{cabv}(\Sigma_I, U)$, its total variation norm is given by

$$\parallel \mu \parallel_v \equiv \sup_\pi \sum_{\sigma \in \pi} |\mu(\sigma)|_U,$$

where π denotes any finite disjoint measurable partition of the interval $I \equiv [0, T]$ and the supremum is taken over all such partitions. Furnished with the total variation norm, $\mathcal{M}_{cabv}(\Sigma_I, U)$ is a Banach space. We denote the admissible controls by $\mathcal{U}_{ad} \subset \mathcal{M}_{cabv}(\Sigma_I, U)$. Exact choice of the set of admissible controls is specified later.

The performance of the system over the time horizon $I \equiv [0, T]$ is measured by the following functional (called cost functional)

$$J(u, R, \xi) \equiv \int_I \ell(t, x(t))dt + \Phi(x(T)) + \Psi(u) \equiv \hat{J}(u, R, \xi) + \Psi(u), \quad (3.2)$$

where the first term represents the running cost, the second term gives the terminal cost and the last term given by $\Psi(u)$ represents the cost of control. In general ℓ is a Borel measurable map from $I \times X$ to $\overline{R}$ and continuous in the second argument; $\Phi : X \longrightarrow R$ is continuous and $\Psi : M_{cabv}(\Sigma_I, U) \longrightarrow \overline{R}_0 \equiv [0, \infty]$. The cost functional depends on control u and the dynamic uncertainty R and the additive noise (uncertainty) ξ in force during the period I. Our objective is to find a control $u \in \mathcal{U}_{ad}$ that minimizes the maximum risk (maximum possible cost). This problem can be formulated as min-max problem:

$$\inf_{u \in \mathcal{U}_{ad}} \sup_{(R,\xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta} J(u, R, \xi).$$

Given this pessimistic view, an element $u^o \in \mathcal{U}_{ad}$ is said to be optimal if and only if

$$\begin{aligned} J_o(u^o) &\equiv \sup_{(R,\xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta} J(u^o, R, \xi) \\ &\leq \sup_{(R,\xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta} J(u, R, \xi) \equiv J_o(u), \quad \forall\, u \in \mathcal{U}_{ad}. \end{aligned} \quad (3.3)$$

BASIC ASSUMPTIONS: To consider the above problem, we introduce the following basic assumptions:

(A1) The Banach spaces $\{X, Y\}$ are reflexive and U is any real Banach space.

(A2) The operator A is the infinitesimal generator of a C_0-semigroup of compact operators $S(t), t \geq 0$, on X satisfying $\sup\{\| S(t) \|_{\mathcal{L}(X)}, t \in I\} \equiv M \in [1, \infty)$.

(A3) The operator valued process R perturbing the semigroup is any uniformly measurable operator valued function defined on I and taking values from the closed ball $B_\gamma(\mathcal{L}(X))$ of radius $\gamma > 0$. This represents uncertainty in the system model and it is denoted by $\mathcal{V}_\gamma \equiv B_\infty(I, B_\gamma(\mathcal{L}(X)))$.

(A4) The nonlinear operator F is a Borel measurable map from $I \times X$ to X and uniformly Lipschitz in the second argument with Lipschitz constant K.

(A5) The disturbance (noise) process $\xi : I \longrightarrow Y$, is any strongly measurable function taking values from the closed ball $B_\delta(Y)$ of radius $\delta > 0$. We denote this family by $\mathcal{D}_\delta \equiv B_\infty(I, B_\delta(Y))$. This represents the uncertainty without any probabilistic structure.

(A6) $G \in L_\infty(I, \mathcal{L}(Y, X))$ and $B \in B_\infty(I, \mathcal{L}(U, X)) \cap C(I, \mathcal{L}(U, X))$ where $C(I, \mathcal{L}(U, X))$ denotes the Banach space of bounded and strongly continuous operator valued functions.

It is important to mention that we do not assume any probabilistic structure for the uncertainties $\{R, \xi\}$. They are simply bounded strongly measurable process.

Before we conclude this section we present the following result on the existence and regularity of solutions of the system (3.1). This is used later in the paper.

Lemma 3.1. *Consider the uncertain system given by* (3.1) *over any finite time horizon* $I \equiv [0, T]$, *and suppose the assumptions* **(A1)**-**(A6)** *hold. Then, for every initial state* $x(0) = x_0 \in X$, *and any control* $u \in \mathcal{U}_{ad}$ *and disturbance* $(R, \xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta$, *the system* (3.1) *has a unique mild solution* $x \in B_\infty(I, X)$. *Further, the solution set*

$$\mathcal{X} \equiv \left\{ x(u, R, \xi)(\cdot) \in B_\infty(I, X) : u \in \mathcal{U}_{ad}, R \in \mathcal{V}_\gamma, \xi \in \mathcal{D}_\delta \right\}$$

is a bounded subset of $B_\infty(I, X)$.

Proof. By definition, the mild solution of the system (3.1) is given by the solution of the following integral equation

$$\begin{aligned} x(t) \equiv\; & S(t)x_0 + \int_0^t S(t-r)R(r)x(r)dr + \int_0^t S(t-r)F(r,x(r))dr \\ & + \int_0^t S(t-r)G(r)\xi(r)dr + \int_0^t S(t-r)B(r)u(dr) \end{aligned} \tag{3.4}$$

on the Banach space X. Define the integral operator $\mathcal{F}$ as follows,

$$\begin{aligned} (\mathcal{F}x)(t) \equiv\; & S(t)x_0 + \int_0^t S(t-r)R(r)x(r)dr + \int_0^t S(t-r)F(r,x(r))dr \\ & + \int_0^t S(t-r)G(r)\xi(r)dr + \int_0^t S(t-r)B(r)u(dr),\ t \in I. \end{aligned} \tag{3.5}$$

Clearly it suffices to verify that $\mathcal{F}$ has a fixed point. Since $R \in \mathcal{V}_\gamma \subset B_\infty(I, \mathcal{L}(X))$, and $u \in \mathcal{U}_{ad}$ and $\xi \in \mathcal{D}_\delta$ are given, it is easy to verify that $\mathcal{F}$ maps $B_\infty(I,X)$ into itself and that for each $n \in N$ (the set of positive integers), the n-th iterate of the operator $\mathcal{F}$ satisfies the following inequality

$$\parallel \mathcal{F}^n x - \mathcal{F}^n y \parallel_{B_\infty(I,X)} \leq \alpha_n \parallel x - y \parallel_{B_\infty(I,X)} \tag{3.6}$$

where $\alpha_n = (M(\gamma + K)T)^n/n!$. Clearly for sufficiently large n, $\alpha_n < 1$ and therefore the corresponding iterate $\mathcal{F}^n$ is a contraction and by Banach fixed point theorem it has a unique fixed point, say, $x^* \in B_\infty(I,X)$. It follows from this that x^* is also a unique fixed point of the operator $\mathcal{F}$ itself. Hence follows existence as well as uniqueness of solution for the integral equation (3.4) and therefore, the existence and uniqueness of a mild solution for the evolution equation (3.1). That the set $\mathcal{X}$ is bounded follows from Gronwall lemma applied to the following inequality,

$$|x(t)|_X \leq C_1 + C_2 \int_0^t |x(r)|_X dr, \quad t \in I,$$

where $C_2 = M(\gamma + K)$ and

$$C_1 = M\big(|x_0|_X + \delta \parallel G \parallel_{L_\infty(I,\mathcal{L}(Y,X))} + \parallel B \parallel_{B_\infty(I,\mathcal{L}(U,X))} \sup\{\parallel u \parallel_v, u \in \mathcal{U}_{ad}\}\big),$$

with $\parallel u \parallel_v$ denoting the total variation norm as defined above. Since $\mathcal{U}_{ad}$ is a bounded subset of $\mathcal{M}_{cabv}(\Sigma_I, U)$ it is clear that $C_1 < \infty$. This completes the brief outline of the proof. □

4. Continuous dependence of solutions

For proof of existence of optimal controls we need continuity of solutions with respect to the triple $\{u, R, \xi\} \in \mathcal{U}_{ad} \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$. Since continuity is crucially dependent on the topology of both the domain and the target spaces, it is necessary to specify the admissible topologies. For the target space $B_\infty(I,X)$, we have already the natural sup-norm topology. So we must specify the topologies on the domain spaces

$$\mathcal{U}_{ad} \subset \mathcal{M}_{cabv}(\Sigma_I, U),$$
$$\mathcal{V}_\gamma \equiv B_\infty(I, B_\gamma(\mathcal{L}(X))) \subset B_\infty(I, \mathcal{L}(X))$$

and

$$\mathcal{D}_\delta \equiv B_\infty(I, B_\delta(Y)) \subset B_\infty(I, Y).$$

(A7) (Admissible Controls $\mathcal{U}_{ad}$) Let $C(I, U^*)$, furnished with the standard sup norm topology, denote the Banach space of continuous functions on I with values in the Banach space U^*, the topological dual of U. Let $\mathcal{M}_{cabv}(\Sigma_I, U)$ denote the space of countably additive U valued vector measures having bounded total variation. Furnished with the total variation norm it is a Banach space. It is clear that the embedding

$$C(I, U^*) \hookrightarrow (M_{cabv}(\Sigma_I, U))^*$$

is continuous but not surjective. We assume that $\mathcal{M}_{cabv}(\Sigma_I, U)$ is equipped with the weak topology τ_w, and that the admissible controls $\mathcal{U}_{ad}$ is a weakly sequentially compact subset of $\mathcal{M}_{cabv}(\Sigma_I, U)$.

The necessary and sufficient conditions for weak compactness is given by the celebrated Bartle-Dunford-Schwartz theorem [Diestel and Uhl.Jr. 18, Theorem 5, p105]. For convenience of the reader we present this below.

Theorem 4.1. (Bartle-Dunford-Schwartz) *Suppose both the Banach spaces $\{U, U^*\}$ satisfy Radon-Nikodym property (RNP). Then a set $M_0 \subset M_{cabv}(\Sigma_I, U)$ is weakly conditionally compact if and if the following three conditions hold:*

(c1) *M_0 is bounded,*

(c2) *there exists $\nu \in M^+_{cabv}(\Sigma_I)$ such that $\lim_{\nu(\sigma)\to 0} |u|(\sigma) = 0$ uniformly with respect to $u \in M_0$.*

(c3) *for each $\sigma \in \Sigma_I$, the set $M_0(\sigma) \equiv \{u(\sigma), u \in M_0\}$ is a conditionally weakly compact subset of U.*

Now we can characterize the system uncertainty which consists of model uncertainty and uncertainty due to additive noise without satisfying any probabilistic structure.

(A8) (System Uncertainty $\mathcal{V}_\gamma$) We consider the set $\mathcal{V}_\gamma$ representing uncertainty in the system model. Since X is a reflexive Banach space, it is well known that the closed ball $B_\gamma(\mathcal{L}(X))$ is compact with respect to the weak operator topology τ_{wo}. Using this fact we may now equip $\mathcal{V}_\gamma \equiv B_\infty(I, B_\gamma(\mathcal{L}(X)))$ with the Tychonoff product topology and denote this by τ_{Two}. With respect to this topology $\mathcal{V}_\gamma$ is a compact Hausdorff space.

(A9) (Additive Uncertainty $\mathcal{D}_\delta$) We consider the set $\mathcal{D}_\delta \equiv B_\infty(I, B_\delta(Y))$ with $B_\delta(Y)$ denoting the closed ball of radius $\delta > 0$ (centered at the origin) representing the measurement uncertainty. Reflexivity of Y implies that $B_\delta(Y)$ is weakly compact. The set $\mathcal{D}_\delta$ is endowed with the Tychonoff product topology τ_{Tw}. With respect to this topology $\mathcal{D}_\delta$ is a compact Hausdorff space.

Now we are prepared to consider the question of continuity. This is given in the following theorem.

Theorem 4.2. *Consider the system* (3.1) *and suppose the assumptions* **(A7)-(A9)** *and those of Lemma* 3.1 *and Theorem* 4.1 *hold and that the operator* A *is the infinitesimal generator of a compact* C_0*-semigroup* $S(t), t > 0$. *Then the map* $(u, R, \xi) \longrightarrow x(u, R, \xi)$ *is jointly continuous from* $\mathcal{U}_{ad} \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ *to* $B_\infty(I, X)$ *with respect to their respective topologies.*

Proof. Let $\{u^n, R^n, \xi^n\} \in \mathcal{U}_{ad} \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ be a sequence and suppose $u^n \xrightarrow{\tau_w} u^o$ in $\mathcal{U}_{ad}$, $R^n \xrightarrow{\tau_{Two}} R^o$ in $\mathcal{V}_\gamma$ and $\xi^n \xrightarrow{\tau_{Tw}} \xi^o$ in $\mathcal{D}_\delta$. Let $\{x^n, x^o\}$ denote the mild solutions of equation (3.1) corresponding to the triples $\{(u^n, R^n, \xi^n)\}$ and $\{(u^o, R^o, \xi^o)\}$ respectively. Then by the definition of mild solutions, $\{x^n, x^o\}$ are the solutions of the following integral equations,

$$
\begin{aligned}
x^n(t) &= S(t)x_0 + \int_0^t S(t-s)R^n(s)x^n(s)ds + \int_0^t S(t-s)F(s, x^n(s))ds \\
&\quad + \int_0^t S(t-s)G(s)\xi^n(s)ds + \int_0^t S(t-s)B(s)u^n(ds), \quad t \in I, (4.1)
\end{aligned}
$$

$$
\begin{aligned}
x^o(t) &= S(t)x_0 + \int_0^t S(t-s)R^o(s)x^o(s)ds + \int_0^t S(t-s)F(s, x^o(s))ds \\
&\quad + \int_0^t S(t-s)G(s)\xi^o(s)ds + \int_0^t S(t-s)B(s)u^o(ds), \quad t \in I, (4.2)
\end{aligned}
$$

respectively and they belong to $B_\infty(I, X)$. Taking the difference and rearranging the terms suitably, we have the following expression

$$
\begin{aligned}
& x^o(t) - x^n(t) \\
&= \int_0^t S(t-r)R^n(r)(x^o(r) - x^n(r))dr + \int_0^t S(t-s)F(s, x^o(s)) - F(s, x^n(s))ds \\
&\quad + \int_0^t S(t-r)(R^o(r) - R^n(r))x^o(r)dr + \int_0^t S(t-r)G(r)(\xi^o - \xi^n))(r)dr \\
&\quad + \int_0^t S(t-r)B(r)(u^o(dr) - u^n(dr)), \quad t \in I. \qquad (4.3)
\end{aligned}
$$

Introduce the sequence of functions $\{e_{i,n}, i = 1,2,3\}_{n\in N}$ as follows:

$$e_{1,n}(t) \equiv \int_0^t S(t-r)(R^o(r) - R^n(r))x_o(r)dr, \;\; t \in I,$$
$$e_{2,n}(t) \equiv \int_0^t S(t-r)G(r)(\xi^o - \xi^n)(r)dr, \;\; t \in I,$$
$$e_{3,n}(t) \equiv \int_0^t S(t-r)B(r)(u^o(dr) - u^n(dr)), \;\; t \in I.$$

Clearly, these are elements of $B_\infty(I, X)$. Using these expressions in equation (4.3) and evaluating the norms on either side of the identity, it follows from assumptions (A2)-(A4) that

$$|x_o(t) - x_n(t)|_X \le \eta_n(t) + M(\gamma + K)\int_0^t |x_o(r) - x_n(r)|_X dr, \;\; t \in I, \quad (4.4)$$

where $\eta_n(t)$ is given by

$$\eta_n(t) \equiv |e_{1,n}(t)|_X + |e_{2,n}(t)|_X + |e_{3,n}(t)|_X.$$

Hence it follows from Gronwall inequality applied to (4.4) that

$$|x_o(t) - x_n(t)|_X$$
$$\le \eta_n(t) + M(\gamma + K)\exp[MT(\gamma + K)]\int_0^t \eta_n(r)dr, \;\; t \in I. \quad (4.5)$$

We show that the expression on righthand side of the above inequality converges to zero uniformly on I. Note that the integrand, defining $e_{1,n}$, is dominated by $2M\gamma|x_o(t)|_X \le 2M\gamma \parallel x_0 \parallel_{B_\infty(I,X)}, t \in I$. Since $R^n \to R^o$ in the Tychonoff product topology τ_{Two}, it is clear that $(R^o(t) - R^n(t))x_o(t)$ converges weakly to zero in X for each $t \in I$. Thus by the compactness of the semigroup $S(t), t > 0$, we have $e_{1,n}(t) \to 0$ strongly in X uniformly on I. Consider the second term $e_{2,n}$. Since $\xi^n \to \xi^o$ in the Tychonoff product topology τ_{Tw} on $\mathcal{D}_\delta$ and $G \in B_\infty(I, \mathcal{L}(Y,X))$ and so for each $r \in I$, $G(r) \in \mathcal{L}(Y,X)$ (so norm bounded), it is clear that $G(r)(\xi^o(r) - \xi^n(r))$ converges weakly to zero in X for each $r \in I$, and the integrand is dominated by the integrable function $2M\delta \parallel G(r) \parallel_{\mathcal{L}(Y,X)}$. Thus again by virtue of compactness of the semigroup, we conclude that $e_{2,n}(t)$ converges strongly in X uniformly on I. We use Bartle-Dunford-Schwartz theorem 4.1 and compactness of the semigroup to prove that $e_{3,n}(t)$ converges strongly to zero uniformly on I. By our assumption (A7), the set of admissible controls $\mathcal{U}_{ad} \subset M_{cabv}(\Sigma_I, U)$ is weakly compact and hence it follows from Bartle-Dunford-Schwartz theorem that there exists a measure $\nu \in M^+_{cabv}(\Sigma_I)$ such that the elements of $\mathcal{U}_{ad}$ are uniformly ν continuous. Since U satisfies RNP, for each sequence $\{u^n\} \in \mathcal{U}_{ad}$, there exists a sequence $g^n \in L_1(\nu, U)$ such that for every $\sigma \in \Sigma_I$

$$u^n(\sigma) = \int_\sigma g^n(s)\nu(ds).$$

Since the set $\mathcal{U}_{ad}$ is bounded it is clear that the sequence $\{g^n\}$ is contained in a bounded subset of $L_1(\nu, U)$ and since the elements of $\mathcal{U}_{ad}$ are uniformly ν-continuous, the sequence $\{g^n\}$ is uniformly integrable. By hypothesis, both U and U^* satisfy RNP and therefore it follows from Dunford theorem [Diestel & Uhl.Jr, 18, Theorem 1, p101] that the sequence $\{g^n\}$ is weakly sequentially compact and since $u^n \xrightarrow{w} u^o$, there exists a $g^o \in L_1(\nu, U)$ such that $g^n \xrightarrow{w} g^o$ in $L_1(\nu, U)$. Hence $e_{3,n}$ can be written as

$$e_{3,n}(t) \equiv \int_0^t S(t-r)B(r)(g^o(r) - g^n(r))\nu(dr).$$

For any $\varepsilon > 0$ and $t \in I$, we can split this integral into two parts giving

$$\begin{aligned} e_{3,n}(t) &\equiv \int_0^t S(t-r)B(r)(g^o(r) - g^n(r))\nu(dr) \\ &= S(\varepsilon)\int_0^{t-\varepsilon} S(t-\varepsilon-r)B(r)(g^o(r) - g^n(r))\nu(dr) \\ &\quad + \int_{t-\varepsilon}^t S(t-r)B(r)(g^o(r) - g^n(r))\nu(dr). \end{aligned} \tag{4.6}$$

By virtue of compactness of the semigroup, it is clear that the first term converges to zero uniformly on the interval $(\varepsilon, T]$. For the second term we use a well known result that states that a Banach space satisfies RNP if and only if it satisfies RNP with respect to Lebesgue measure. Since U satisfies RNP with respect to $\nu \in M^+_{cabv}(\Sigma_I)$, it satisfies RNP also with respect to Lebesgue measure. Thus there exists an $h \in L_1(\nu)$ such that $\nu(ds) = hds$ and the second term of the expression (4.6) can be rewritten as

$$\begin{aligned} E_{n,\varepsilon}(t) &\equiv \int_{t-\varepsilon}^t S(t-r)B(r)(g^o(r) - g^n(r))\nu(dr) \\ &= \int_{t-\varepsilon}^t S(t-r)B(r)(g^o(r) - g^n(r))h(r)dr \\ &= \int_{t-\varepsilon}^t S(t-r)B(r)(\tilde{g}^o(r) - \tilde{g}^n(r))dr, \end{aligned} \tag{4.7}$$

where $\tilde{g} = gh \in L_1(I, U)$ for $g = \{g^o, g^n\}$. Since the operator valued function B is uniformly bounded we have $\sup\{\| B(t) \|_{\mathcal{L}(U,X)}, t \in I\} \equiv b < \infty$, and consequently it follows from (4.7) that

$$|E_{n,\varepsilon}(t)|_X \leq Mb \int_{t-\varepsilon}^{t} |\tilde{g}^o - \tilde{g}^n|_U dr.$$

We conclude from the above inequality and uniform (Bochner) integrability of the family $\{\tilde{g}^o, \tilde{g}^n\} \subset L_1(I, X)$, that

$$\lim_{\varepsilon \downarrow 0} E_{n,\varepsilon}(t) = 0$$

for all $n \in N$, uniformly on $(\varepsilon, T]$. Since $\varepsilon > 0$ is arbitrary, it follows from these facts that $\eta_n(t) \equiv |e_{1,n}(t)|_X + |e_{2,n}(t)|_X + |e_{3,n}(t)|_X \to 0$ uniformly on I. Consequently, it follows from the inequality (4.5) that $x_n(t) \xrightarrow{s} x_o(t)$ in X uniformly on I, that is, $x^n \xrightarrow{s} x^o$ in $B_\infty(I, X)$. This completes the proof. □

Note that the cost functional has two parts $\hat{J}$ and Ψ. First we prove that the first component $\hat{J}$, appearing in the expression (3.2), is jointly continuous on $\mathcal{U}_{ad} \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$.

Corollary 4.3. *Suppose the assumptions of Theorem* 4.2 *hold and the functions* ℓ *and* Φ *satisfy the following assumptions:*

(A10) (a1) *The integrand* ℓ *is measurable in the first variable and continuous in the second and there exists a* $p \in [1, \infty)$ *and* $g \in L_1^+(I)$ *and* $c_1 \geq 0$, *such that*

$$|\ell(t, x)| \leq g(t) + c_1 |x|_X^p, \quad x \in X, \quad t \geq 0.$$

(a2) *The function* Φ *is continuous on* X *and there exist constants* $c_2, c_3 \geq 0$ *such that*

$$|\Phi(x)| \leq c_2 + c_3 |x|_X^p$$

for the same p.

Then, the functional $(u, R, \xi) \longrightarrow \hat{J}(u, R, \xi)$ *is jointly continuous on* $\mathcal{U}_{ad} \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ *with respect to the topology* $\tau_w \times \tau_{Two} \times \tau_{Tw}$.

Proof. Let $\{u^n, R^n, \xi^n\}$ be a sequence from the set $\mathcal{U}_{ad} \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ converging to $\{u^o, R^o, \xi^o\}$. Let $x^n \in B_\infty(I, X)$, $x^o \in B_\infty(I, X)$, denote the corresponding mild solutions of the evolution equation (3.1). Then by assumption **(A10)**, it follows from Theorem 4.2 that, along a subsequence if necessary, $\ell(t, x^n(t)) \to \ell(t, x^o(t))$ *a.e*; and that it is dominated by an integrable function since the sequence of solutions $\{x^n\}$ are uniformly bounded [see Lemma 3.1]. Since $\{x^n, x^o\} \in B_\infty(I, X)$ and by our assumption the Banach U has RNP, it is easy to verify that, for each $t \in I$, the states $\{x(t), x_n(t)\}$ are well defined as elements of X. Thus it follows from continuity of the function Φ

that $\Phi(x^n(T)) \to \Phi(x^o(T))$. Hence, letting $n \to \infty$, it follows from dominated convergence theorem that

$$\begin{aligned}\lim_{n\to\infty} \hat{J}(u^n, R^n, \xi^n) &= \lim_{n\to\infty}\left\{\int_I \ell(t, x^n(t))dt + \Phi(x^n(T))\right\}\\ &= \int_I \ell(t, x^o(t))dt + \Phi(x^o(T))\\ &\equiv J(u^o, R^o, \xi^o).\end{aligned}$$

This proves the joint continuity of the functional $\hat{J}$ as stated. □

Now we are prepared to prove the existence of an optimal control $u^o \in \mathcal{U}_{ad}$ that solves the min-max problem in the sense of (3.3). For this we need the notions of upper and lower semi-continuity of multi functions.

Definition 4.4. Let Z_1, Z_2 be any pair of topological spaces. A multi function $G : Z_1 \longrightarrow 2^{Z_2} \setminus \emptyset$ is upper semi-continuous if for every closed set $C \subset Z_2$, the preimage $G^{-1}(C) \equiv \{x \in Z_1 : G(x) \cap C \neq \emptyset\}$ is closed. And it is lower semi-continuous if for every open set $D \subset Z_2$ the preimage $G^{-1}(D) \equiv \{x \in Z_1 : G(x) \cap D \neq \emptyset\}$ is open.

For details on multi-functions see the Handbook by Hu and Papageorgiou [16].

5. Existence of optimal control

In this section we consider the question of existence of optimal controls. We prove this following a similar approach as in [1, Theorem 6.1].

Theorem 5.1. *Consider the control system* (3.1) *with the min-max problem* (3.3). *Suppose the assumptions of Theorem* 4.2 *and Corollary* 4.3 *hold and the functional* Ψ *is weakly lower semicontinuous on* $M_{cabv}(\Sigma_I, U)$ *satisfying* $\Psi(u) \geq 0$. *Then there exists an optimal control* $u^o \in \mathcal{U}_{ad}$ *in the sense that*

$$\begin{aligned}J_o(u^o) \;&\equiv\; \sup_{R\in\mathcal{V}_\gamma, \xi\in\mathcal{D}_\delta} J(u^o, R, \xi)\\ &\leq\; \sup_{R\in\mathcal{V}_\gamma, \xi\in\mathcal{D}_\delta} J(u, R, \xi) \;\equiv\; J_o(u), \quad \forall\, u \in \mathcal{U}_{ad}. \qquad (5.1)\end{aligned}$$

Proof. We prove this in two steps. First we prove that, under the given assumptions, the functional $\hat{J}_o$ given by

$$\hat{J}_o(u) \equiv \sup\left\{\hat{J}(u, R, \xi), (R, \xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta\right\} \qquad (5.2)$$

is continuous and then we show that $J_o \equiv \hat{J}_o + \Psi$ is weakly lower semicontinuous. For each $u \in \mathcal{U}_{ad}$, define the set

$$\Pi(u) \equiv \{(R,\xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta : \hat{J}(u,R,\xi) = \sup\{\hat{J}(u,Q,\eta) : (Q,\eta) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta\}\}.$$

By virtue of joint continuity of $\hat{J}$ in all its arguments (Corollary 4.3) it is clear that, for each fixed $u \in \mathcal{U}_{ad}$, the functional $(R,\xi) \longrightarrow \hat{J}(u,R,\xi)$ is $\tau_{Two} \times \tau_{Tw}$-continuous on $\mathcal{V}_\gamma \times \mathcal{D}_\delta$. Since the set $\mathcal{V}_\gamma \times \mathcal{D}_\delta$ is $\tau_{Two} \times \tau_{Tw}$ compact, the set $\Pi(u)$ is well defined and so $\Pi(u) \neq \emptyset$. In general, this is a multifunction $\Pi : \mathcal{U}_{ad} \longrightarrow 2^{\mathcal{V}_\gamma \times \mathcal{D}_\delta} \setminus \emptyset$. Define, for each $u \in \mathcal{U}_{ad}$,

$$\hat{J}_o(u) \equiv \hat{J}(u, \Pi(u)).$$

We show that this functional is τ_w continuous on $\mathcal{U}_{ad}$. Consider the sequence $\{u^n\} \subset \mathcal{U}_{ad}$ and $u^o \in \mathcal{U}_{ad}$ such that $u^n \xrightarrow{\tau_w} u^o$. By definition of $\hat{J}_o$, it is clear that $\hat{J}_o(u^n) = \hat{J}(u^n, \Pi(u^n))$ for $n \in N$. Thus there exists a sequence $(R^n, \xi^n) \in \Pi(u^n)$ such that $\hat{J}_o(u^n) = \hat{J}(u^n, R^n, \xi^n)$. Since $\mathcal{U}_{ad} \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ is $\tau_w \times \tau_{Two} \times \tau_{Tw}$ compact, there exists a subsequence, relabeled as the original sequence, and a triple $(u^o, R^o, \xi^o) \in \mathcal{U}_{ad} \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ such that

$$(u^n, R^n, \xi^n) \longrightarrow (u^o, R^o, \xi^o)$$

with respect to the product topology $\tau_w \times \tau_{Two} \times \tau_{Tw}$. Again, by virtue of joint continuity of $\hat{J}$ (Corollary 4.3), we have $\lim_{n\to\infty} \hat{J}(u^n, R^n, \xi^n) = \hat{J}(u^o, R^o, \xi^o)$. Thus, to complete the proof of continuity of the functional $u \longrightarrow \hat{J}_o(u)$, we must show that $(R^o, \xi^o) \in \Pi(u^o)$. For this it suffices to verify that the graph $\mathcal{G}_r(\Pi)$ of the multifunction Π is closed. It is well known that an upper semicontinuous multifunction from a Hausdorff topological space to a regular topological space has closed graph [20, Proposition 2.17]. Thus, as both $\mathcal{U}_{ad}$ and $\mathcal{V}_\gamma \times \mathcal{D}_\delta$ are Hausdorff regular, it suffices to show that the multifunction Π is upper semi-continuous. More precisely, we show that $u \longrightarrow \Pi(u)$ is upper semi continuous (usc) with respect to the given topologies on the domain space $\mathcal{U}_{ad}$ and the target space $\mathcal{V}_\gamma \times \mathcal{D}_\delta$. According to the Definition 4.4, we must verify that, for any closed set $\mathcal{C} \subset \mathcal{V}_\gamma \times \mathcal{D}_\delta$, the preimage

$$\Pi^{-1}(\mathcal{C}) \equiv \{u \in \mathcal{U}_{ad} : \Pi(u) \cap \mathcal{C} \neq \emptyset\}$$

is closed. Let $\{u^n\} \in \Pi^{-1}(\mathcal{C}) \subset \mathcal{U}_{ad}$ be any sequence and note that it follows from the definition of Π that

$$\hat{J}(u^n, \Pi(u^n)) \geq \hat{J}(u^n, R, \xi), \quad \forall\, (R,\xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta. \tag{5.3}$$

Hence, for any sequence $\{R^n, \xi^n\} \in \Pi(u^n) \cap \mathcal{C}$, we have

$$\hat{J}(u^n, R^n, \xi^n) \geq \hat{J}(u^n, R, \xi), \quad \forall\, (R,\xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta, \quad n \in N. \tag{5.4}$$

Since $\mathcal{U}_{ad}\times\mathcal{V}_\gamma\times\mathcal{D}_\delta$ is compact in the product topology (because of compactness in their respective topologies), there exists a subsequence, relabeled as the original sequence, (u^n,R^n,ξ^n) and an element $(u^o,R^o,\xi^o)\in\mathcal{U}_{ad}\times\mathcal{V}_\gamma\times\mathcal{D}_\delta$ such that

$$(u^n,R^n,\xi^n)\longrightarrow(u^o,R^o,\xi^o)$$

in the product topology. Since the topologies are Hausdorff the limit is unique. Taking the limit in (5.4), it follows from Corollary 4.3, asserting joint continuity of $\hat{J}$, that

$$\hat{J}(u^o,R^o,\xi^o)\geq\hat{J}(u^o,R,\xi)\ \forall\ (R,\xi)\in\mathcal{V}_\gamma\times\mathcal{D}_\delta. \tag{5.5}$$

Since this holds for all $(R,\xi)\in\mathcal{V}_\gamma\times\mathcal{D}_\delta$, it follows from this inequality and the definition of the multi function Π that $(R^o,\xi^o)\in\Pi(u^o)$. On the other hand, since $\mathcal{C}$ is closed, the limit of any sequence from it must belong to it and hence $(R^o,\xi^o)\in\mathcal{C}$ and therefore $(R^o,\xi^o)\in\Pi(u^o)\cap\mathcal{C}$. Thus $u^o\in\Pi^{-1}(\mathcal{C})$ proving the closure as required. Hence $u\longrightarrow\Pi(u)\subset 2^{\mathcal{V}_\gamma\times\mathcal{D}_\delta}\setminus\emptyset$ is an upper semicontinuous multi function and therefore, by Proposition 2.17 [14], the graph $\mathcal{G}_r(\Pi)$ is closed. Thus we conclude that $\hat{J}(u^o,R^o,\xi^o)=\hat{J}(u^o,\Pi(u^o))=\hat{J}_o(u^o)$ proving the continuity, $\lim_{n\to\infty}\hat{J}_o(u^n)\longrightarrow\hat{J}_o(u^o)$, as required. This completes the first part of the proof. By assumption, Ψ is weakly lower semicontinuous and therefore the functional J_o given by the sum

$$J_o(u)\equiv\hat{J}_o(u)+\Psi(u)$$

is also weakly lower semicontinuous. Since $\mathcal{U}_{ad}$ is τ_w compact it is clear that J_o attains its minimum on $\mathcal{U}_{ad}$. This proves that the min-max problem (5.1), equivalently (3.3), has a solution and hence an optimal control exists. □

Remark 5.2. An important example of the control cost $\Psi(u)$ is given by the total variation norm $\Psi(u)\equiv\parallel u\parallel_v$ as defined in section 3. It is well known that the norm in any Banach space is weakly lower semi continuous and hence this functional is weakly lower semi continuous. Another related functional is given by $\Psi(u)\equiv\varphi(\parallel u\parallel_v)$ where φ is a continuous nonnegative nondecreasing extended real valued function satisfying $\varphi(0)=0$.

In case of linear-quadratic-regulator (LQR) problems, the control cost is quadratic. In this case

$$\Psi(u)\equiv(\mathcal{K}u,u)=\int_{I\times I}\langle K(t,s)u(ds),u(dt)\rangle_{U^*,U}$$

where K is the kernel corresponding to the operator

$$\mathcal{K}\in\mathcal{L}(\mathcal{M}_{cabv}(\Sigma_I,U),C(I,U^*)).$$

For example, the kernel may be chosen as the tensor product of linearly independent elements from $C(I, U^*) \subset \mathcal{M}_{cabv}(\Sigma_I, U)$ giving

$$K(t,s) \equiv \sum_{i\geq 1} \varphi_i(t) \otimes \varphi_i(s).$$

Clearly, in this case

$$0 \leq \Psi(u) = \sum_{i\geq 1} \left(\int_I \langle \varphi_i(t), u(dt) \rangle_{U^*,U} \right)^2 \leq \left(\sum_{i=1}^{\infty} \| \varphi_i \|_\infty^2 \right) \| u \|_v^2,$$

where $\| \varphi \|_\infty \equiv \sup\{|\varphi_i(t)|_{U^*}, t \in I\}$ and $\| u \|_v$ is the total variation norm of the vector measure u. Clearly, the operator $\mathcal{K}$ is positive and it is also bounded if $\sum_{i=1}^{\infty} \| \varphi_i \|_\infty^2 < \infty$. Thus Ψ, as defined above, is lower semicontinuous in the weak star topology, which is weaker than the weak topology. It is known that if U has the RNP (Radon Nikodym Property) then, under certain additional technical assumptions such as continuum hypothesis and that the cardinality of $\mathcal{M}_{cabv}(\Sigma_I)$ is $\leq 2^{\aleph_0}$ holds [23], the dual of $\mathcal{M}_{cabv}(\Sigma_I, U)$ is given by the space $B_\infty(I, U^*)$ of bounded measurable functions with values in U^*. In this case $\{\varphi_i\} \subset B_\infty(I, U^*)$ and $u \longrightarrow \Psi(u)$ is weakly lower semicontinuous.

6. Purely impulsive controls

For practical applications, it is often preferable to consider control policies which consist of a finite number of impulsive forces delivered at a discrete set of appropriate time instants to steer the system along a desirable path and reach the goal with minimum possible cost.

Let $I \equiv [0, T]$ be any closed bounded interval and D any countable subset of I and Σ_D the power set of D. The class of countably additive discrete vector measures defined on Σ_D and taking values in the Banach space U having bounded variation is denoted by $\mathcal{M}_{cabv}(\Sigma_D, U)$. By definition, for any set $J \in \Sigma_D$, an element $\mu \in \mathcal{M}_{cabv}(\Sigma_D, U)$ has the following representation

$$\mu(J) = \sum_{t\in J} u_t \; \delta_t$$

where $u_t \in U$ and δ_t denotes the (unit) Dirac measure concentrated at the one point set $\{t\}$. The variation of μ on J is given by $|\mu|(J) = \sum_{t\in J} |u_t|_U$ and hence the total variation norm is given by $\| \mu \|_v \equiv |\mu|(D)$. With respect to this norm topology $\mathcal{M}_{cabv}(\Sigma_D, U)$ is a Banach space. Since μ is of bounded variation, it is clear that for any $\varepsilon > 0$, the cardinality of the set $\{t \in J : |u_t|_U > \varepsilon\}$ is finite. In case U is the real line, $\mathcal{M}_{cabv}(\Sigma_D, R) \equiv \mathcal{M}_{cabv}(\Sigma_D)$ denotes the space of signed measures supported on the set D. For any set $J \in \Sigma_D$, an element $\nu \in \mathcal{M}_{cabv}(\Sigma_D)$ has the representation

$$\nu(J) \equiv \sum_{t\in J} \alpha_t \delta_t$$

with $\alpha_t \in R$ satisfying $\sum_{t\in J} |\alpha_t| < \infty$. Let $\nu \in \mathcal{M}^+_{cabv}(\Sigma_D)$ having the above representation with $\alpha_t \geq 0$ and define the Lebesgue-Bochner space $L_p(\nu, U)$, $1 \leq p < \infty$, of U valued ν-measurable functions which are Bochner integrable with respect to the measure ν, that is, for each $\Gamma \in \Sigma_D$, $\mu(\Gamma) \equiv \int_\Gamma u(t)\nu(dt) \in U$ and

$$\int_I |u(t)|_U^p \nu(dt) = \sum_{t\in D} \alpha_t |u_t|^p < \infty.$$

Let N denote the set of natural numbers and let $\{t_i \in R : t_i \in I \cap D, i \in N\}$ be any enumeration of the set D. This makes D isomorphic to N. Then it is clear that $L_p(\nu, U)$ is actually equivalent to the weighted sequence space $\ell_p(\nu, U)$ which consists of U valued sequences $\{u_i\} \subset U$ such that $\sum_i \alpha_i |u_i|_U^p < \infty$. In general, for $1 \leq p < \infty$ with $1/p + 1/q = 1$, and any finite positive measure ν, the dual of $L_p(\nu, U)$ equals $L_q(\nu, U^*)$ if and only if U^* has the Radon-Nikodym Property (RNP). In the case of sequence spaces it turns out that this is not necessary. The topological dual of $\ell_p(\nu, U)$ (denoted by $\ell_p(\nu, U)^*$) is given by $\ell_q(\nu, U^*)$ (without requiring U^* to have RNP). Thus the dual of $\ell_1(\nu, U)$ is given by $\ell_\infty(\nu, U^*)$. For any $u \in \ell_p(\nu, U)$ and $v \in \ell_q(\nu, U^*)$ we have the duality pairing

$$(u, v)_{\ell_p(\nu,U),\ell_q(\nu,U^*)} = \int_I (u(t), v(t))_{U,U^*} \nu(dt) = \sum_{t\in D} \alpha_t (u_t, v_t)_{U,U^*}.$$

Clearly, it follows from Hölder inequality that

$$\begin{aligned} \left| \int_I (u(t), v(t))_{U,U^*} \nu(dt) \right| &= \left| \sum_{t\in D} \alpha_t (u_t, v_t)_{U,U^*} \right| \\ &\leq \left(\sum_{t\in D} \alpha_t |u_t|_U^p \right)^{1/p} \left(\sum_{t\in D} \alpha_t |v_t|_{U^*}^q \right)^{1/q} \\ &= \| u \|_{\ell_p(\nu,U)} \| v \|_{\ell_q(\nu,U^*)} . \end{aligned}$$

We state the following Lemma characterizing conditionally (sequentially) weakly compact sets in the Banach space $\ell_1(\nu, U)$. Let $\mathcal{P}_t$ denote the projection map of $\ell_1(\nu, U)$ into its t-section (or $t-$th coordinate), that is, for each $u \in \ell_1(\nu, U)$, $\mathcal{P}_t(u) = u_t \in U$.

Lemma 6.1. *A set $K \subset \ell_1(\nu, U)$ is conditionally weakly compact if and only if the following two conditions are satisfied:*

(C1) $\sum_{i\geq k} \alpha_{t_i} |u_{t_i}| \to 0$ *as* $k \to \infty$ *, uniformly for* $u \in K$.

(C2) $\mathcal{P}_{t_i}(K) \equiv \{u_{t_i}, u \in K\}$ *is conditionally weakly compact in* U *for each* $i \in N$.

Proof. See [15, Theorem 2, p.183]. □

We are interested in the characterization of weakly compact subsets of the space of vector measures $\mathcal{M}_{cabv}(\Sigma_D, U)$. This is where our controls take values from. The following result on the characterization of weak compactness is essentially the sequential version of the celebrated Bartle-Dunford-Schwartz theorem [18, Theorem 5, p105] for the continuous case.

Theorem 6.2. *A set* $M_0 \subset \mathcal{M}_{cabv}(\Sigma_D, U)$ *is conditionally weakly compact if and only if the following conditions hold:*

(1) *The set* M_0 *is bounded.*
(2) *There exists a* $\nu \in \mathcal{M}^+_{cabv}(\Sigma_D)$ *such that* $\lim_{\nu(\sigma)\to 0} |u|(\sigma) = 0$ *uniformly with respect to* $u \in M_0$.
(3) *For each* $\sigma \in \Sigma_D$, *the set* $\{u(\sigma) : u \in M_0\}$ *is a conditionally weakly compact subset of* U.

Proof. Fundamentally the proof is similar to that of Bartle-Dunford-Schwartz theorem given by Diestel and Uhl. Jr [18, Theorem 5, p.105] as stated here in Theorem 4.1. In fact it is simpler. For convenience of the reader we give a short proof. To prove the necessity, suppose M_0 is conditionally weakly compact. Clearly, the condition (1) is obvious. For each $\sigma \in \Sigma_D$, define the linear operator $L_\sigma : \mathcal{M}_{cabv}(\Sigma_D, U) \longrightarrow U$ by $L_\sigma(u) = u(\sigma)$. Since M_0 is conditionally weakly compact and L_σ is a bounded linear operator, $L_\sigma(M_0)$ is conditionally weakly compact proving the necessity of condition (3). We prove condition (2) by contradiction. Suppose it is false. Then there exists a sequence $\{u^n\} \subset M_0$ such that the sequence of scalar measures $\{|u^n|(\cdot)\}$ is not uniformly countably additive. Define the measure $\mu(\cdot) \equiv \sum_n (1/2^n)|u^n|(\cdot)$ on Σ_D and note that it is a bounded positive measure. Consider the space $\ell_1(\mu, U)$, and the linear operator $T_\mu : \ell_1(\mu, U) \longrightarrow \mathcal{M}_{cabv}(\Sigma_D, U)$ given by

$$(T_\mu g)(\sigma) = \int_\sigma g(s)\mu(ds), \quad g \in \ell_1(\mu, U).$$

We show that T_μ is an isometric-isomorphism of $\ell_1(\mu, U)$ on to a closed subspace of $\mathcal{M}_{cabv}(\Sigma_D, U)$. Define $m_g(E) \equiv (T_\mu g)(E)$, for $E \in \Sigma_D$. The variation of the measure m_g on any set $E \in \Sigma_D$ is given by

$$|m_g|(E) \equiv \sup_\pi \sum_{\sigma\in\pi} |m_g(\sigma)|_U,$$

where π is any partition of the set E into a finite number of disjoint Σ_D-measurable sets with the supremum taken over all such partitions. It is clear from this that $|m_g|(E) \leq \int_E |g(s)|_U \mu(ds)$. Since $E(\in \Sigma_D)$ is otherwise arbitrary, it follows from this that

$$\| m_g \|_v \leq \int_D |g(s)|_U \mu(ds). \tag{6.1}$$

Now we verify the reverse inequality. For any $g \in \ell_1(\mu, U)$ it follows from Hahn-Banach theorem that there exists an $h \in \ell_\infty(\mu, U^*)$ with $\| h \|_{\ell_\infty(\mu,U^*)} = 1$ such that

$$\int_D |g(s)|_U \mu(ds) = \int_D \langle g(s), h(s) \rangle_{U,U^*} \mu(ds).$$

Since $h \in \ell_\infty(\mu, U^*)$ there exists a sequence of (simple) functions $\{h_n\} \subset \ell_\infty(\mu, U^*)$ converging in μ-measure to h such that

$$h_n(t) \equiv \sum_{i=1}^{n} u^*_{i,n} \chi_{\sigma_{i,n}}(t), \quad t \in D,$$

where $\bigcup_{i=1}^n \sigma_{i,n} = D$ for all $n \in N$ and $u^*_{i,n} \in \partial B_1(U^*)$ for $1 \leq i \leq n \in N$. Thus

$$\begin{aligned}
&\int_D |g(s)|_U \mu(ds) = \int_D \langle g(s), h(s) \rangle_{U,U^*} \mu(ds) \\
&= \lim_{n\to\infty} \int_D \langle g(s), h_n(s) \rangle_{U,U^*} \mu(ds) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left\langle m_g(\sigma_{i,n}), u^*_{i,n} \right\rangle_{U,U^*} \\
&\leq \lim_{n\to\infty} \sum_{i=1}^{n} |m_g(\sigma_{i,n})|_U \ \leq \| m_g \|_v .
\end{aligned} \tag{6.2}$$

It follows from the inequalities (6.1) and (6.2) that T_μ is an isometric isomorphism of $\ell_1(\mu, U)$ on to a closed subspace of $\mathcal{M}_{cabv}(\Sigma_D, U)$. Clearly the range of T_μ contains the set $\{u^n\}$, that is, $R(T_\mu) \supset \{u^n\}$. Since $\{u^n(\cdot)\}$ is not uniformly countably additive, the set $T_\mu^{-1}(\{u^n\}) \subset \ell_1(\mu, U)$ is not uniformly integrable, and so it follows from Lemma 6.1, condition (C1), that the set $T_\mu^{-1}(\{u^n\})$ can not be contained in a weakly conditionally compact subset of $\ell_1(\mu, U)$, and therefore $T_\mu(T_\mu^{-1}(\{u^n\})) = \{u^n\}$ can not be contained in a conditionally weakly compact set M_0 leading to a contradiction. Thus condition (2) is necessary. Now we prove that the conditions (1)-(3) are sufficient for the set M_0 to be conditionally weakly compact. Let $M(\nu, U) \subset \mathcal{M}_{cabv}(\Sigma_D, U)$ denote the class of ν-continuous vector measures. It follows from condition (2) that $M_0 \subset M(\nu, U)$. For the given ν, define the operator $T_\nu : \ell_1(\nu, U) \longrightarrow M(\nu, U) \subset \mathcal{M}_{cabv}(\Sigma_D, U)$ by

$$T_\nu(h)(E) \equiv \int_E h(s)\nu(ds), \text{ for } h \in \ell_1(\nu, U) \quad \text{and} \quad E \in \Sigma_D.$$

It is clear from the preceding analysis that T_ν is an isometric isomorphism of $\ell_1(\nu, U)$ on to a closed subspace of $\mathcal{M}_{cabv}(\Sigma_D, U)$ and that the range of T_ν contains M_0, that is, $R(T_\nu) \supset M_0$. Clearly, $T_\nu^{-1}(M_0) \subset \ell_1(\nu, U)$. Since M_0 satisfies conditions (1) and (2), $T_\nu^{-1}(M_0)$ is a bounded subset of $\ell_1(\nu, U)$ satisfying condition (C1) of Lemma 6.1. It follows from assumption (3) that for each $\{t_i\} \in \Sigma_D$,

$$\mathcal{P}_{t_i}\big(T_\nu^{-1}(M_0)\big) \equiv \big\{h_{t_i}, h \in T_\nu^{-1}(M_0)\big\}$$

is conditionally weakly compact. Thus condition (C2) of Lemma 6.1 is satisfied for $K \equiv T_\nu^{-1}(M_0)$. Hence, by Lemma 6.1, the set $T_\nu^{-1}(M_0) \subset \ell_1(\nu, U)$ is conditionally weakly compact. Since compactness is preserved under isomorphism, we conclude that the set $T_\nu(T_\nu^{-1}(M_0)) = M_0$ is conditionally weakly compact. This completes the proof. □

Now we are prepared to consider control problems for the system (3.1) with purely impulsive controls as the admissible controls. First we prove a result on continuity of solution with respect to the triple $\{u, R, \xi\} \longrightarrow x(u, R, \xi)$. Here we consider admissible controls given by the weak closure of the conditionally weakly compact set $M_0 \subset \mathcal{M}_{cabv}(\Sigma_D, U)$ of Theorem 6.2. That is $\mathcal{U}_{ad} = \mathcal{U}_o \equiv \overline{M}_0^w$.

Theorem 6.3. *Consider the system* (3.1) *with admissible controls* $\mathcal{U}_o = \overline{M}_0^w$ *and suppose the assumptions of Lemma* 3.1 *including* (A8)-(A9) *hold and that the operator A is the infinitesimal generator of a compact C_0-semigroup $S(t), t > 0$, and the operator valued function $B \in C(I, \mathcal{L}(U, X))$ and that it is compact for each $t \in I$. Then the map $(u, R, \xi) \longrightarrow x(u, R, \xi)$ is jointly continuous with respect to product topology $\tau_w \times \tau_{Two} \times \tau_{Tw}$ on $\mathcal{U}_o \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ and the uniform norm topology on $B_\infty(I, X)$.*

Proof. The proof is similar to that of Theorem 4.2 with slight modification required because of the choice of impulsive controls $\mathcal{U}_o$. It suffices to show that under the given assumptions the function $e_{3,n}$ (see Theorem 4.2), given by

$$e_{3,n}(t) \equiv \int_0^t S(t-r)B(r)(u^o(dr) - u^n(dr)), \;\; t \in I,$$

converges to zero strongly in X uniformly in $t \in I$. Since $\{u^o, u^n\} \subset \mathcal{U}_o$ it follows from Theorem 6.2 that there exists a sequence $\{g^o, g^n\} \subset \ell_1(\nu, U)$ such that

$$\begin{aligned} e_{3,n}(t) &\equiv \int_0^t S(t-r)B(r)(u^o(dr)-u^n(dr)), \\ &= \int_0^t S(t-r)B(r)(g^o(r)-g^n(r))\nu(dr), \;\; t \in I. \end{aligned} \tag{6.3}$$

Since $\nu \in \mathcal{M}^+_{cabv}(\Sigma_D)$, it has the representation $\nu(dr) = \sum \alpha_{t_i}\delta_{t_i}(dr)$ where $\alpha_{t_i} \geq 0$ and $\delta_{t_i}(\cdot)$ is the Dirac measure with support $\{t_i\} \subset D$. Hence

$$e_{3,n}(t) \equiv \sum_{t_i \subset D\cap[0,t]} S(t-t_i)B(t_i)(g^o(t_i)-g^n(t_i))\alpha_{t_i}, \;\; t \in I. \tag{6.4}$$

It is clear that weak convergence of u^n to u^o in $\mathcal{U}_o$ implies weak convergence of g^n to g^o in $\ell_1(\nu, U)$ which, in turn, leads to weak convergence of $(g^o(t_i) - g^n(t_i))\alpha_{t_i}$ to zero in U for each $\{t_i\} \subset D$. For $t = t_k \in D$, the expression (6.4) reduces to

$$\begin{aligned} e_{3,n}(t_k) &\equiv \sum_{i\leq k-1} S(t_k-t_i)B(t_i)(g^o(t_i)-g^n(t_i))\alpha_{t_i} \\ &\quad +B(t_k)(g^o(t_k)-g^n(t_k))\alpha_{t_k}. \end{aligned} \tag{6.5}$$

Since for each $t \in I$, $B(t) \in \mathcal{L}(U,X)$ is compact, both the terms on right hand side of the above identity converge to zero strongly in X. For any $t \in [t_k, t_{k+1})$, we have

$$e_{3,n}(t) = S(t-t_k)e_{3,n}(t_k), \;\; t_k \leq t < t_{k+1} \tag{6.6}$$

and $e_{3,n}(\cdot) \in C([t_k, t_{k+1}), X)$ followed by a jump at t_{k+1} of intensity

$$B(t_{k+1})(g^o(t_{k+1}) - g^n(t_{k+1}))\alpha_{t_{k+1}}$$

and they both converge to zero strongly in X. Thus $e_{3,n}(t) \to 0$ strongly in X uniformly in $t \in I$. In view of Theorem 4.2 and the necessary modifications as described above, we conclude that the map $(u, R, \xi) \longrightarrow x(u, R, \xi)$ is jointly continuous with respect to the product topology $\tau_w \times \tau_{Two} \times \tau_{Tw}$ on $\mathcal{U}_o \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ and the supnorm topology on $B_\infty(I, X)$. This completes the proof. □

Remark 6.4.

(a) It is clear from the expression (6.5) why the assumption on compactness of the semigroup alone does not guarantee strong convergence in X.

(b) It is also clear from the above result that the mild solutions of equation (3.1) corresponding to controls from the set $\mathcal{U}_o$ are elements of $B_\infty(I, X)$ and that each solution x admits a decomposition into a continuous and a discontinuous part: $x = x_c + x_d$ where $x_c \in C(I \setminus D, X)$ and $x_d \in B_\infty(D, X)$.

Now we are prepared to prove the existence of an optimal control from the class $\mathcal{U}_o$. Recall that $J(u,R,\xi) \equiv \hat{J}(u,R,\xi) + \Psi(u)$ and

$$J_o(u) \equiv \sup\{J(u,R,\xi),(R,\xi) \in \mathcal{V}_\gamma \times \mathcal{D}_\delta\}.$$

Theorem 6.5. *Consider the system* (3.1) *with admissible controls* $\mathcal{U}_o = \overline{M}_0^w$ *and the min-max problem* (3.3)*. Suppose the functional* $\Psi(\geq 0)$ *is weakly lower semicontinuous on* $\mathcal{M}_{cabv}(\Sigma_D, U)$ *and the assumption* **(A10)** (*related to the cost functional, see Corollary* 4.3) *and those of Theorem* 6.3 *hold. Then there exists a control* $u^* \in \mathcal{U}_o$ *such that* $J_o(u^*) \leq J_o(u)$ *for all* $u \in \mathcal{U}_o$.

Proof. It follows from Theorem 6.3 that the map $(u,R,\xi) \longrightarrow x(u,R,\xi)$ is jointly continuous with respect to the topology $\tau_w \times \tau_{Two} \times \tau_{Tw}$ on $\mathcal{U}_o \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ and the supnorm topology on $B_\infty(I,X)$. Thus in the case of sequence spaces as we have here, the conclusions of Theorem 4.2 and Corollary 4.3 remain valid (without requiring the pair (U,U^*) to satisfy RNP). Therefore, under the assumption **(A10)**, related to the functions $\{\ell,\Phi\}$ determining the functional $\hat{J}$, it follows from Corollary 4.3 that the functional $(u,R,\xi) \longrightarrow \hat{J}(u,R,\xi)$ is jointly continuous on $\mathcal{U}_o \times \mathcal{V}_\gamma \times \mathcal{D}_\delta$ with respect to the topology $\tau_w \times \tau_{Two} \times \tau_{Tw}$. Hence it follows from Theorem 5.1 that $u \longrightarrow \hat{J}_o(u)$ is weakly continuous on $\mathcal{U}_o$ and therefore it follows from weak lower semi continuity of Ψ that $u \longrightarrow J_o(u) \equiv \hat{J}_o(u) + \Psi(u)$ is weakly lower semi continuous. Since $\mathcal{U}_o$ is weakly compact there exists a control policy $u^* \in \mathcal{U}_o$ such that

$$J_o(u^*) \leq J_o(u), \quad \forall\, u \in \mathcal{U}_o.$$

This completes the proof. □

Remark 6.6. In the case of vector measures with values in general Banach spaces, one can use Lebesgue decomposition theorem [18, Theorem 9, p31] to conclude that any vector measure $u \in \mathcal{M}_{cabv}(\Sigma_I, U)$ admits Lebesgue decomposition (with respect to Lebesgue measure λ) giving $u = u_c + u_s$ where u_c is λ-continuous and u_s is singular with respect to λ. Then the set of admissible controls can be decomposed as $\mathcal{U}_{ad} = \mathcal{U}_c \oplus \mathcal{U}_s$ and so for $\mathcal{U}_{ad}$ to be weakly compact, it is necessary and sufficient that each member of the decomposition has this property. We believe our results from section 5 and section 6 can be combined to deal with this case.

An Open Problem in Vector Measure Theory: An interesting open problem in the general case (where $\{U,U^*\}$ do not posses RNP) is the problem of characterization of conditionally (or relatively) weakly compact sets in $\mathcal{M}_{cabv}(\Sigma_I, U)$ with respect to a topology possibly weaker than the weak topology used in Bartle-Dunford-Schwartz theorem.

Interesting Potential Extensions:

(1) Necessary conditions of optimality for impulsive systems without uncertainties can be found in [5],[6],[7]. Necessary conditions of optimality of output feedback control law for uncertain stochastic systems can be found in [10]. It will be interesting to develop necessary conditions of optimality for uncertain systems driven and controlled by vector measures considered here.

(2) An interesting problem is to extend the results of this paper to a larger class of systems where the operator valued function B appearing in equation (3.1) is state dependent as in [11]. Also it is of interest to extend the results of this paper to infinite dimensional stochastic systems subject to both the system and measurement uncertainty as in [1].

References

[1] N.U. Ahmed, *Infinite Dimensional Uncertain Dynamic Systems on Banach spaces and Their optimal Output Feedback Control*, Discussiones Mathematicae Differential Inclusions, Control and Optimization, DICO, **35** (2015), 65–87.

[2] N.U. Ahmed, *Controllability of Evolution Equations and Inclusions Driven by Vector Measures*, Discussiones Mathematicae Differential Inclusions, Control and Optimization, **24** (2004), 49–72.

[3] N.U. Ahmed, *Optimal Relaxed Controls for Systems Governed by Impulsive Differential Inclusions*, Nonlinear Funct. Anal.& Appl., **10**(3) (2005), 427–460.

[4] N.U. Ahmed, *Vector and operator Valued Measures as Controls for infinite Dimensional Systems: Optimal Control*, Discussiones Mathematicae, Differential Inclusions, Control and Optimization, **28** (2008), 95–131.

[5] N.U. Ahmed, *Optimal impulse Control for Impulsive systems in Banach spaces*, International Journal of Differential Equations and Applications, **1**(1) 2000, 37–52.

[6] N.U. Ahmed, *Necessary Conditions of optimality for Impulsive Systems on Banach Spaces*, Nonlinear Analysis, **51** (2002), 409–424.

[7] N.U. Ahmed, *Existence of Optimal Controls for a General Class of Impulsive Systems on Banach Spaces*, SIAM J. Control and Optim., **42**(2) (2003), 669–685.

[8] N.U. Ahmed, K.L. Teo and S.H. Hou, *Nonlinear Impulsive Systems on Infinite Dimensional Spaces*, Nonlinear Analysis, **54** (2003), 907–925.

[9] N.U. Ahmed, *Optimal Feedback Control for Impulsive Systems on the Space of Finitely Additive Measures*, Publ. Math. Debrecen, **70**(4-4) (2007), 371–393.

[10] N.U. Ahmed, *Necessary Conditions of Optimality for Output Feedback Control Law for a Class of Uncertain Infinite Dimensional Stochastic Systems*, Nonli. Func. Anal. and Appl., **18**(1) (2013), 103–126. http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

[11] N.U. Ahmed, *State Dependent Vector Measures as feedback Control for Impulsive Systems in Banach Spaces*, Dynamics of Continuous Discrete and Impulsive Systems, **8** (2001), 251–261.

[12] N.U. Ahmed, *Optimal Control of Impulsive Stochastic Evolution Inclusions*, Discussiones Mathematicae, Differential Inclusions, Control and Optimization, **22** (2002), 155–184.

[13] N.U. Ahmed and K.L. Teo, *Optimal Control of Distributed Parameter Systems*, Elsevier, North Holland, New York, Oxford, (1981).
[14] N.U. Ahmed, *Dynamics of Hybrid Systems Induced by operator Valued Measures*, Nonlinear Analysis: Hybrid Systems, **2**(2) (2008), 359–367.
[15] J.K. Brooks and N. Dinculeanu, *Weak Compactness in Spaces of Bochner Integrable Functions and Applications*, Advances in Mathematics, **24** (1977), 172–188.
[16] L. Cesari, *Optimization Theory and Applications*, Springer-Verlag, (1983).
[17] G. Chen and X. Dong, *From Chaos to Order-Perspectives and Methodologies in Controlling Chaotic Nonlinear Dynamical Systems*, Int. J. Bifurcation and Chaos, **03** 1363, (1993). DOI: 10.1142/SO218127493001112.
[18] J. Diestel and J.J. Uhl,Jr., Vector Measures, Mathematical Surveys and Monographs, AMS, Vol.15. (1977).
[19] H.O. Fattorini, *Infinite Dimensional optimization and Control Theory*, Encyclopedia of Mathematics and its Applications, **62**, Cambridge University Press, (1999).
[20] S. Hu and N.S. Papageorgiou, Handbook of Multivalued Analysis,(Theory), Kluwer Academic Publishers, Dordrecht, Boston, London, (1997).
[21] M. Itoh, T. Yang and L.O. Chua, *Conditions for Impulsive Synchronization of Chaotic and Hyperchaotic Systems*, Int. J. Bifurcation Chaos, **11** 551, (2001). DOI:10.1142/SO218127401002262.
[22] Z.H. Lu, X.B. Chi and L.S. Chen, *Impulsive Control Strategies in Biological Control of Pesticide*, Theoretical Population Biology, **64**(1) (2003), 39–47.
[23] R.D. Mauldin, *The Continuum Hypothesis,Integrationand Duals of Measure Spaces*, Illinois J. Math., **19** (1975), 33–40.
[24] O.J. Sul, M.R. Falvo, R.M. Taylor II, S. Washburn and R. Superfine, *Thermally Actuated Untethered Impact-driven Locomotive Microdevices*, Applied Physics Letters, 89203512, (2006).
[25] W. Verhelst, *On the Quality of Speech Produced by Impulse Driven Linear Systems*, ICASSP, 1991, IEEE International Conference, Toronto, ISBN:O-7803-0003-3, 501–504.
[26] X. Wang and S.N. Balakrishnan, *Optimal Neuro-Controller Synthesis for Variable-time Impulse Driven Systems*, American Control Conference, 2008, June 11-13, 3817–3822.