Nonlinear Functional Analysis and Applications
Vol. 21, No. 4 (2016), pp. 717-725

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ON BANAS-HAJNOSZ-WEDRYCHOWICZ TYPE MODULUS OF CONVEXITY AND FIXED POINT PROPERTY

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Abstract. Let X be a Banach space with the unit sphere $S_X = \{x \in X : \|x\| = 1\}$. In this paper, inspired by Banaś *et. al.*, in [1], the new parameter $SY_X(\varepsilon) = \sup\{\langle x - y, f_x\rangle : x, y \in S_X, \frac{\|x+y\|}{2} \geq 1 - \varepsilon \text{ for some } f_x \in \nabla_x\}$, where $\nabla_x \subseteq S_{X^*}$ is the norm 1 supporting functionals at x, is introduced. Several properties of this parameter are investigated. The main result are that if $SY_X(t) < 2$, for some $t \in (0, 1]$ then X is uniformly non-square; and if $SY_X(\varepsilon) < 1 + 2\varepsilon$ for some $0 < \varepsilon < \frac{1}{2}$, then both X and X^* have uniform normal structure. In particular, if $\varepsilon_U = \lim_{\varepsilon\to 0} SY_X(\varepsilon) < 1$, then X is uniformly non-square and both X and X^* have uniform normal structure. We have an example to show this condition is the best possible.

[0]Received May 13, 2016. Revised July 20, 2016.
[0]2010 Mathematics Subject Classification: 46B20, 47H09, 47H10.
[0]Keywords: Non-expansive mapping, normal structure, uniformly non-square space, uniform normal structure.
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1. Introduction

Let X be a Banach space with the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ and the closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$. For $x \in S_X$, let $\nabla_x \subset S_{X^*}$ be the set of norm 1 supporting functionals of S_X at x, that is, $f \in \nabla_x \iff \langle x, f \rangle = 1$. For a nonempty subset C of X, the *diameter* of C is denoted by $\operatorname{diam} C$, that is, $\operatorname{diam} C = \sup\{\|x - y\| : x, y \in C\}$.

Definition 1.1. ([3]) Let X be a Banach space. A nonempty bounded and convex subset K of X is said to have *normal structure* if for every convex subset C of K that contains more than one point there is a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\| : y \in C\} < \operatorname{diam} C.$$

A Banach space X is said to have

- *normal structure* if every bounded convex subset of X has normal structure;
- *weak normal structure* if every weakly compact convex set K of X has normal structure;
- *uniform normal structure* if there exists $0 < c < 1$ such that for every bounded closed convex subset C of K that contains more than one point there is a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\| : y \in C\} < c \cdot \operatorname{diam} C.$$

Remark 1.2. The following facts are known.

- uniform normal structure $\Longrightarrow$ normal structure $\Longrightarrow$ weak normal structure.
- In the setting of reflexive spaces, normal structure $\iff$ weak normal structure.

Let C be a nonempty subset of a Banach space$X$. A mapping $T : C \to C$ is called to be non-expensive whenever $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A Banach space has fixed point property if for every bounded closed and convex subset C of X and for each non-expansive mapping $T : C \to C$, there is a point $x \in C$ such that $x = Tx$ (see [14]). It was proved by Kirk [13] that if X has normal structure, then it has fixed point property. Since then many mathematician have investigated various sufficient conditions for normal structure.

In [4], Clarkson introduced the following modulus of convexity: $\delta_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in S_X, \|x - y\| \geq \varepsilon\}$, where $0 \leq \varepsilon \leq 2$. It was proved that if there exists $\varepsilon > 0$ such that $\delta_X(1 + \varepsilon) > \frac{\varepsilon}{2}$, then X has uniform normal structure [8].

In [7], Gao introduced the modulus of U-convexity which is a generalization of $\delta_X(\varepsilon)$: $U_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : x, y \in S_X, \langle x-y, f_x\rangle \geq \varepsilon$ for some $f_x \in \nabla_x\}$, where $0 \leq \varepsilon \leq 2$. This measures a certain geometric property of the unit sphere S_X. It was also proved that if there exists $\delta > 0$ such that $U_X(\frac{1}{2}-\delta) > 0$, then X has uniform normal structure. Mazcuñán-Navarro [14] proved that a Banach space X has fixed point property if there exists $\delta > 0$ such that $U_X(1-\delta) > 0$. This was strengthened by Saejung [15]. In fact, it was proved that if a Banach space X is super-reflexive, then the moduli of U-convexity of the ultra-power $X_{\mathcal{U}}$ of X and of X itself coincide. By using ultra-power method he showed that a Banach space X and its dual X^* have uniform normal structure whenever $U_X(1) > 0$.

On the other hand, Banas-Hajnosz-Wedrychowicz [1] introduced the following parameter $\beta_X(\varepsilon) = \sup\{\|x-y\| : x, y \in S_X, \frac{1}{2}\|x+y\| \geq 1-\varepsilon\}$. The properties of $\beta_X(\varepsilon)$ were obtained, and a relationship between this new modulus and $\delta_X(\varepsilon)$ were studied.

In this paper, inspired by the modulus $\beta_X(\varepsilon)$ of Banas et al. [1], the new parameter $SY_X(\varepsilon)$, an inverse function of $U_X(\varepsilon)$ in a certain sense, is introduced and its properties are investigated. Some sufficient conditions of $SY_X(\varepsilon)$ for uniform non-squareness and uniform normal structure are given in terms of this parameter. More precisely, if $SY_X(t) < 2$, for some $t \in (0,1]$ then X is uniformly non-square; if $SY_X(\varepsilon) < 1 + 2\varepsilon$ for some $0 < \varepsilon < \frac{1}{2}$, then both X and X^* have uniform normal structure, and if $SY_X(\varepsilon) < 2$ for some $\frac{1}{2} \leq \varepsilon < 1$, then X has normal structure. In particular, if $\varepsilon_U = \lim_{\varepsilon\to 0} SY_X(\varepsilon) < 1$, then X is uniformly non-square and both X and X^* have uniform normal structure. We have an example to show this condition is the best possible.

2. Main results

First let us recall the concept of the ultra-power technique. Let $\mathcal{U}$ be a ultrafilter on the set $\mathbb{N}$ of natural numbers, that is, it is a filter on $\mathbb{N}$ which is maximal with respect to set inclusion. We also assume that it is *nontrivial*, that is, it is not of the form $\{A \subset \mathbb{N} : i_0 \in A\}$ for some $i_0 \in \mathbb{N}$. From now on, we assume that $\mathcal{U}$ is a nontrivial ultrafilter. A sequence $\{x_n\}$ in a Banach space X *converges to an element* $x \in X$ *with respect to* $\mathcal{U}$ if $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{U}$ for each neighborhood U of x. In this case, we write $\lim_{\mathcal{U}} x_n = x$. Let $\ell^\infty(X)$ denote the set all bounded sequences $\{x_n\}$ in X, that is, $\sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$ and let $\mathcal{N}_{\mathcal{U}} := \{\{x_n\} \in \ell^\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$. The *ultra-power of* X *with respect to* $\mathcal{U}$ is the quotient space $\ell^\infty(X)/\mathcal{N}_{\mathcal{U}}$ equipped with the quotient norm $|\cdot|_{\mathcal{U}}$. We write $\{x_n\}_{\mathcal{U}}$ to denote the elements of $\ell^\infty(X)/\mathcal{N}_{\mathcal{U}}$ and it is not hard to see that $|\{x_n\}_{\mathcal{U}}|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_n\|$. Note that, since $\mathcal{U}$ is nontrivial, X can be embedded isometrically into $X_{\mathcal{U}}$ (see [5], [17].)

Definition 2.1. Let X be a Banach space. The function $SY_X : [0,1] \to [0,2]$ defined by

$$SY_X(\varepsilon) = \sup\left\{\langle x - y, f_x\rangle : x, y \in S_X, \frac{\|x+y\|}{2} \geq 1 - \varepsilon, f_x \in \nabla_x\right\}$$

is called the *modulus of SY-convexity* of X.

Proposition 2.2. *Suppose that X is a Banach space and $0 \leq \varepsilon \leq 1$. Then*

$$SY_X(\varepsilon) = \sup\left\{\langle x - y, f_x\rangle : x \in S_X, y \in B_X, \frac{\|x+y\|}{2} \geq 1 - \varepsilon \text{ for some } f_x \in \nabla_x\right\}.$$

Proof. Let $\widehat{SY}_X(t)$ denote the right hand of the expression of the proposition. Obviously, $SY_X(t) \leq \widehat{SY}_X(t)$. To see the reverse inequality, let $\eta > 0$. Then there exist elements $x \in S_X$, $y \in B_X$ and $x^* \in \nabla_x$ such that

$$\frac{\|x+y\|}{2} \geq 1 - \varepsilon \quad \text{and} \quad \langle x - y, x^*\rangle > \widehat{SY}_X(t) - \eta.$$

Let $y', y'' \in S_X$ be such that $\langle y', x^*\rangle = \langle y'', x^*\rangle$ and $y = (1-\alpha)y' + \alpha y''$ for some $\alpha \in [0,1]$. This implies that $(1-\alpha)\|x + y'\| + \alpha\|x + y''\| \geq 2(1-\varepsilon)$. So we may assume that $\|x + y'\| \geq 2(1-\varepsilon)$. Note that $\langle x - y', x^*\rangle = \langle x - y, x^*\rangle$. This implies that

$$SY_X(\varepsilon) \geq \langle x - y', x^*\rangle > \widehat{SY}_X(\varepsilon) - \eta.$$

Since $\eta > 0$ is arbitrary, we have $SY_X(\varepsilon) \geq \widehat{SY}_X(\varepsilon)$. This completes the proof. $\square$

Theorem 2.3. *The following are some basic properties of $SY_X(\varepsilon)$:*

- $SY_X(\varepsilon) = \sup\left\{\langle x - y, f_x\rangle : x, y \in S_X,\ \frac{\|x+y\|}{2} = 1 - \varepsilon,\ f_x \in \nabla_x\right\}$.
- *$SY_X(\varepsilon)$ is an increasing function of ε.*
- *For any Banach space X and $0 \leq \varepsilon \leq 1$, $SY_X(\varepsilon) \leq \beta_X(\varepsilon)$.*
- *For any Banach space X, $SY_X(1) = 2$.*
- *For a Hilbert space H,*

$$SY_H(\varepsilon) = \begin{cases} 1 - \sqrt{1 - 4\varepsilon(2-\varepsilon)(1-\varepsilon)^2} & \text{if } 0 \leq \varepsilon \leq 1 - \frac{1}{\sqrt{2}}, \\ 1 + \sqrt{1 - 4\varepsilon(2-\varepsilon)(1-\varepsilon)^2} & \text{if } 1 - \frac{1}{\sqrt{2}} \leq \varepsilon \leq 1. \end{cases}$$

Lemma 2.4. ([2], Bishop-Phelps-Bollobás) *Let X be a Banach space and $\varepsilon \in (0,1)$. Then for each $z \in B_X$ and $h \in S_{X^*}$ with $\langle z, h\rangle > 1 - \frac{\varepsilon^2}{4}$ there exist $x \in S_X$ and $x^* \in \nabla_x$ such that $\|x - z\| < \varepsilon$ and $\|x^* - h\| < \varepsilon$.*

Proposition 2.5. *Suppose that X is a superreflexive space and $\mathcal{U}$ is a non-trivial ultrafilter on $\mathbb{N}$. Then $SY_{X_\mathcal{U}}(t) = SY_X(t)$ for all $t \in [0,1]$.*

Proof. Since X can be embedded into $X_\mathcal{U}$ isometrically, we may consider X as a subspace of $X_\mathcal{U}$. It then follows from the definition that $SY_X(\varepsilon) \le SY_{X_\mathcal{U}}(\varepsilon)$.

We now prove the reverse inequality. For any small $\eta > 0$, from the definition of $SY_{X_\mathcal{U}}(\varepsilon)$, let $\{x_n\}_\mathcal{U} \in S_{X_\mathcal{U}}$, $\{y_n\}_\mathcal{U} \in S_{X_\mathcal{U}}$, and let $\{f_n\}_\mathcal{U} \in \nabla_{\{x_n\}_\mathcal{U}}$ be such that

$$\frac{1}{2}|\{x_n\}_\mathcal{U} + \{y_n\}_\mathcal{U}|_\mathcal{U} \ge 1 - \varepsilon$$

and

$$\langle \{x_n\}_\mathcal{U} - \{y_n\}_\mathcal{U}, \{f_n\}_\mathcal{U}\rangle \ge SY_{X_\mathcal{U}}(\varepsilon) - \eta.$$

Set

$$\begin{aligned} P &= \{n : 1 - \eta < \|x_n\| < 1 + \eta\}; \\ Q &= \{n : 1 - \eta < \|f_n\| < 1 + \eta\}; \\ W &= \{n : \langle x_n - y_n, f_n\rangle \ge SY_{X_\mathcal{U}}(\varepsilon) - \eta\}; \\ T &= \{n : \frac{1}{2}\|x_n + y_n\| \ge 1 - \varepsilon\}. \end{aligned}$$

Note that all the sets P, Q, W and T belong to $\mathcal{U}$ and so does the intersection $P \cap Q \cap W \cap T$. In particular, $P \cap Q \cap W \cap T \neq \varnothing$.

Let $n \in P \cap Q \cap W \cap T$ be fixed. This implies that

$$\begin{gathered} 1 - \eta < \|x_n\| < 1 + \eta; \\ 1 - \eta < \|f_n\| < 1 + \eta; \\ 1 - \eta < \langle x_n, f_n\rangle < 1 + \eta; \\ \langle x_n - y_n, f_n\rangle \ge SY_{X_\mathcal{U}}(\varepsilon) - \eta; \\ \frac{1}{2}\|x_n + y_n\| \ge 1 - \varepsilon. \end{gathered}$$

If necessary, we normalize vectors x_n and f_n to use Lemma 2.4 because η can be arbitrarily small. Then there are $z_n \in S_X$ and $g_{z_n} \in S_{X^*}$ such that

- $g_{z_n} \in \nabla_{z_n}$;
- $\frac{1}{2}\|z_n - y_n\| \ge 1 - \varepsilon$;
- $\langle z_n - y_n, g_{z_n}\rangle > SY_{X_\mathcal{U}}(\varepsilon) - \eta$.

This implies that $SY_X(\varepsilon) \geq SY_{X_{\mathcal{U}}}(\varepsilon) - \eta$. Since $\eta > 0$ can be arbitrarily small, we have

$$SY_{X_{\mathcal{U}}}(\varepsilon) \leq SY_X(\varepsilon).$$

This complites that proof. □

Definition 2.6. ([6, 9, 10]) A Banach space X is said to be *uniformly non-square* if there exists a $\delta > 0$ such that either $\frac{1}{2}\|x+y\| \leq 1-\delta$ or $\frac{1}{2}\|x-y\| \leq 1-\delta$ for any $x, y \in S_X$.

Theorem 2.7. *If $SY_X(\varepsilon) < 2$ for some $\varepsilon \in (0,1]$ then X is uniformly non-square.*

Proof. Suppose that X is not uniformly non-square. Then there are sequences $\{y_n\}$ and $\{z_n\}$ in S_X such that $\lim_n \|y_n + z_n\| = \lim_n \|y_n - z_n\| = 2$. For each $n \in \mathbb{N}$, let $h_n \in S_{X^*}$ be such that $\langle z_n - y_n, h_n\rangle = \|z_n - y_n\|$. Note that $\lim_n\langle z_n, h_n\rangle = -\lim_n\langle y_n, h_n\rangle = 1$. It follows from Lemma 2.4 that there are sequences $\{x_n\}$ in S_X and $\{x_n^*\}$ in S_{X^*} such that

- $\langle x_n, x_n^*\rangle = 1$ for all $n \in \mathbb{N}$;
- $\lim_n \|x_n - z_n\| = \lim_n \|x_n^* - h_n\| = 0$.

It follows that

$$\lim_n \|x_n + y_n\| \geq \lim_n(\|z_n + y_n\| - \|z_n - x_n\|) = \lim_n \|z_n + y_n\| = 2.$$

Moreover, $\lim_n\langle y_n, x_n^*\rangle = \lim_n\langle y_n, x_n^* - h_n\rangle + \langle y_n, h_n\rangle = -1$. The last equality follows since $\lim_n |\langle y_n, x_n^* - h_n\rangle| \leq \lim_n \|y_n\|\|x_n^* - h_n\| = 0$. Therefore, we have $\lim_n\langle x_n - y_n, x_n^*\rangle = 2$. Consequently, $SY_X(\varepsilon) = 2$ for all $\varepsilon \in (0,1]$. □

Lemma 2.8. ([16]) *If X is a superreflexive Banach space and fails to have normal structure, then there are elements $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{X_{\mathcal{U}}}$ and $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in S_{(X_{\mathcal{U}})^*}$ such that*

- $|\widetilde{x}_i - \widetilde{x}_j|_{\mathcal{U}} = 1$ *and* $\langle \widetilde{x}_i, \widetilde{f}_j\rangle = 0$ *for all* $i \neq j$;
- $\langle \widetilde{x}_i, \widetilde{f}_i\rangle = 1$ *for all* $i = 1, 2, 3$.

Theorem 2.9. *If $SY_X(\varepsilon) < 1+2\varepsilon$ for some $\varepsilon \in (0,1/2]$, then X and X^* have uniform normal structure.*

Proof. We assume that $SY_X(\varepsilon) < 1 + 2\varepsilon$ for some $\varepsilon \in (0,1/2]$. This implies that X is uniformly non-square and hence super-reflexive. Suppose first that X does not have normal structure. So there are elements $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{X_{\mathcal{U}}}$ and $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in S_{(X_{\mathcal{U}})^*}$ satisfying all the conditions in Lemma 2.8. Set $\mathbf{x} := \widetilde{x}_1 - \widetilde{x}_2$, $\mathbf{y} := (1-2\varepsilon)\widetilde{x}_1 + 2\varepsilon\widetilde{x}_2$ and $\mathbf{x}^* = -\widetilde{f}_2$. Note that

- $\langle \mathbf{x}, \mathbf{x}^* \rangle = 1$;
- $|\mathbf{x}+\mathbf{y}|_{\mathcal{U}} = |(2-2\varepsilon)\widetilde{x}_1 - (1-2\varepsilon)\widetilde{x}_2|_{\mathcal{U}}$
  $\geq \langle (2-2\varepsilon)\widetilde{x}_1 - (1-2\varepsilon)\widetilde{x}_2, \widetilde{f}_1 \rangle = 2-2\varepsilon$;
- $\langle \mathbf{x}-\mathbf{y}, \mathbf{x}^* \rangle = 1 - \langle (1-2\varepsilon)\widetilde{x}_1 + 2\varepsilon\widetilde{x}_2, -\widetilde{f}_2 \rangle = 1+2\varepsilon$.

This implies that

$$SY_X(\varepsilon) = SY_{X_{\mathcal{U}}}(\varepsilon) \geq 1+2\varepsilon.$$

Consequently, if $SY_X(\varepsilon) < 1+2\varepsilon$ for some $\varepsilon \in (0,1/2]$, then X has normal structure.

Secondly, we show that if X does not have normal structure, then $SY_{X^*}(\varepsilon) \geq 1+2\varepsilon$. To see this, we set $\mathbf{x}^* := \widetilde{f}_1$, $\mathbf{y}^* := 2\varepsilon\widetilde{f}_2 - (1-2\varepsilon)\widetilde{f}_3$ and $\mathbf{x}^{**} := \widetilde{x}_1 - \widetilde{x}_2$. Note that

- $\langle \mathbf{x}^*, \mathbf{x}^{**} \rangle = 1$;
- $|\mathbf{x}^*+\mathbf{y}^*|_{\mathcal{U}} = |\widetilde{f}_1 + 2\varepsilon\widetilde{f}_2 - (1-2\varepsilon)\widetilde{f}_3|_{\mathcal{U}}$
  $\geq \langle \widetilde{f}_1 + 2\varepsilon\widetilde{f}_2 - (1-2\varepsilon)\widetilde{f}_3, \widetilde{x}_1 - \widetilde{x}_3 \rangle = 2-2\varepsilon$;
- $\langle \mathbf{x}^*-\mathbf{y}^*, \mathbf{x}^{**} \rangle = \langle \widetilde{x}_1 - \widetilde{x}_2, \widetilde{f}_1 - 2\varepsilon\widetilde{f}_2 + (1-2\varepsilon)\widetilde{f}_3 \rangle = 1+2\varepsilon$.

This implies that

$$SY_{X^*}(t) = SY_{(X^*)_{\mathcal{U}}}(\varepsilon) = SY_{(X_{\mathcal{U}})^*}(t) \geq 1+2\varepsilon.$$

In this second part, we can conclude that if $SY_X(\varepsilon) < 1+2\varepsilon$ for some $\varepsilon \in (0,1/2]$, then X^* has normal structure.

Finally, to conclude the uniform normal structure of X and X^*, we just invoke the fact that uniform normal structure and super-normal structure are the same whenever the space is super-reflexive [12]. □

We now discuss the situation $\varepsilon \in [1/2,1)$. Here we give a partial result. We do not known that if the following result can conclude the uniform normal structure of the dual space.

Theorem 2.10. *If $SY_X(\varepsilon) < 2$ for some $\varepsilon \in [1/2,1)$, then X has uniform normal structure.*

Proof. We assume that $SY_X(\varepsilon) < 2$ for some $\varepsilon \in [1/2,1)$. We then follow the first part of the proof of the preceding theorem but we set $\mathbf{y} := \widetilde{x}_2 - (2\varepsilon-1)\widetilde{x}_1$. Note that

- $|\mathbf{y}|_{\mathcal{U}} = |(2\varepsilon-1)(\widetilde{x}_2 - \widetilde{x}_1) - (2-2\varepsilon)\widetilde{x}_2)|_{\mathcal{U}}$
  $\leq (2\varepsilon-1)|\widetilde{x}_2 - \widetilde{x}_1|_{\mathcal{U}} + (2-2\varepsilon)|\widetilde{x}_2|_{\mathcal{U}} = 1$;
- $|\mathbf{x}+\mathbf{y}|_{\mathcal{U}} = 2(1-\varepsilon)$;
- $\langle \mathbf{x}-\mathbf{y}, \mathbf{x}^* \rangle = 2$.

This implies that

$$SY_X(\varepsilon) = SY_{X_{\mathcal{U}}}(\varepsilon) = 2.$$

□

We now present a sufficient condition of $\beta_X(t)$ for uniform normal structure.

Corollary 2.11. *If $\beta_X(t) < 1 + 2t$ for some $t \in (0, 1/2]$, then X is uniformly non-square and both X and X^* have uniform normal structure.*

Corollary 2.12. *If $\lim_{\varepsilon\to 0} SY_X(\varepsilon) < 1$, then X is uniformly non-square and both X and X^* have uniform normal structure.*

Proof. This is a direct result of Theorem 2.7 and Theorem 2.9. □

For a Banach space X, we define $\varepsilon_U = \sup\{\varepsilon \geq 0 : U_X(\varepsilon) = 0\}$ to be the characteristic of U-convexity of a space X.

Proposition 2.13. $\lim_{\varepsilon\to 0} SY_X(\varepsilon) = \varepsilon_U$.

Proof. The idea of the proof is similar to the proof of Theorem 3.1 in [1]. If $\varepsilon_U = 0$, we have $\lim_{\varepsilon\to 0} SY_X(\varepsilon) \geq \varepsilon_U$. Otherwise, fix $t < \varepsilon_U$. From definition of $U_X(\varepsilon)$ and ε_U, for an arbitrary $\varepsilon > 0$ there exist $x, y \in S(X)$ and $f_x \in \nabla_x$ such that $\langle x - y, f_x\rangle = t$, and $1 - \frac{\|x+y\|}{2} \leq \varepsilon$.

From definition of $SY_X(\varepsilon)$, this implies that $SY_X(\varepsilon) \geq t$, therefore

$$\lim_{\varepsilon\to 0} SY_X(\varepsilon) \geq t.$$

Since t can be arbitrarily close to ε_U, we have that $\lim_{\varepsilon\to 0} SY_X(\varepsilon) \geq \varepsilon_U$. We prove the inverse direction. Let $t > 0$ be arbitrarily small. Since $SY_X(\varepsilon)$ be a nondecresing function, from definition of $SY_X(\varepsilon)$ there exist $x, y \in S(X)$, $f_x \in \nabla_x$ such that $1 - \frac{\|x+y\|}{2} = t$, and $\langle x - y, f_x\rangle = SY_X(t) \geq \lim_{\varepsilon\to 0} SY_X(\varepsilon)$.

From definition of $U_X(\varepsilon)$, this implies $U_X(\lim_{\varepsilon\to 0} SY_X(\varepsilon)) \leq t$. Since t can be arbitrarily small, $U_X(\lim_{\varepsilon\to 0} SY_X(\varepsilon)) = 0$. Therefore from definition of ε_U, we have $\lim_{\varepsilon\to 0} SY_X(\varepsilon) \leq \varepsilon_U$. □

We borrow the following example from [15] to show that the condition of Corollary 2.12 is the best possible.

Example 2.14. For $p \in (1, \infty)$, we denote by $l_{p,\infty}$ the l_p space with the norm

$$\|x\| = \max\{\|x^+\|_p, \|x^-\|_p\},$$

where x^+ and x^- are the positive and negative part of x respectively. It is known that $l_{p,\infty}$ is a superreflexive space and fails to have normal structure [11]. Moreover, $U_{l_{p,\infty}}(1) = 0$ and $U_{l_{p,\infty}}(\varepsilon) > 0$ for all $\varepsilon > 1$ [15]. It follows from Proposition 2.13 that $\varepsilon_U = \lim_{\varepsilon\to 0} SY_X(\varepsilon) = 1$.

Acknowledgments: The research of the first author is supported by the Thailand Research Fund and Khon Kaen University (RSA5980006).

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