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INEXACT PROJECTION METHOD FOR THE VARIATIONAL INEQUALITIES OVER THE FIXED POINT SET AND ITS APPLICATION TO CDMA DATA NETWORKS

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Abstract. We propose a new iteration method for solving variational inequalities over the fixed point set of a firmly nonexpansive mapping, where the cost functions are continuous and not necessarily monone, which is called the *inexact subgradient method*. One application of the problem is a power control for a direct-sequence code-division multiple-access data network. Finally, we present some numerical experiments to illustrate the behavior of the proposed algorithms.

1. Introduction

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a firmly nonexpansive mapping, i.e., $\|T(x) - T(y)\|^2 \leq \langle T(x) - T(y), x - y\rangle$ for all $x, y \in \mathbb{R}^n$, and mapping $F : \mathbb{R}^n \to \mathbb{R}$. We consider the following *variational inequalities over the fixed point set* $\big($shortly, $VI(F, Fix(T))\big)$:

$$\text{Find } x^* \in Fix(T) \text{ such that } \langle F(x^*), y - x^*\rangle \geq 0, \quad \forall\, y \in Fix(T),$$

where $Fix(T) := \{x \in C : \ Tx = x\}$. Problem $VI(F, Fix(T))$ is a special class of equilibrium problems on the nonempty closed convex constraint set. Many

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iterative methods for solving such problems have been presented in [3, 4, 5, 8, 9, 17, 18]. A well-known application of this problem is the power-control problem for code-division multiple-access (shortly, CDMA) systems (see [10, 11, 12, 19, 22]). Power control is needed for efficient resource allocation and interference management in the uplink and downlink of CDMA systems. For a power-control, user terminal has to be able to quickly transmit at an ideal power level such that it can get a sufficient signal-to-interference-plus-noise ratio (shortly, SINR) and achieve the required quality of service. In the uplink, each user must transmit just enough power to achieve the required quality of service without causing unnecessary interference to other users in the system. One method of doing so is a game-theoretic approach, the preferences of users are represented by utility functions. We consider a utility function in [11, 13, 15, 19] defined as the follows: Let $I := \{1, \cdots, s\}$ be the set of users and $C_k := [a_k, b_k]$ be the strategy set for the kth user, where $a_k \geq 0$ and $b_k > 0$ are the minimum power and the maximum allowed power for transmission of the kth user. Then, the common power set of all users is

$$C := C_1 \times \cdots \times C_s.$$

Recently, Tse *et al.*, in [19], for each $x = (x_1, \cdots, x_s) \in C$, introduced the utility function as follows

$$f_i(x) := \frac{Lc_i(1 - e^{-t_i(x)})^K}{K},$$

where

- L is the number of information bits;
- K is the total number of bits in a packet;
- c_i is the transmission rate for the ith user, which is also the ratio of the bandwidth to the processing gain;
- Denote the channel gain for the ith user by h_i, the processing gain by $N > 0$, and the noise power by σ^2. Then, the signal-to-interference-plus-noise ratio SINR is defined as

$$t_i(x) := \frac{x_i h_i^2}{\sigma^2 + \frac{1}{N} \sum_{j \neq i} x_j h_j^2}. \tag{1.1}$$

For each $t \in \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$, the expresion $(1 - e^{-t})^K$ is the approximate packet success rate (shortly, PSR). Thus, users choose their transmit powers in order to maximize their utility functions $f_i(x)$ for all $i = 1, \cdots, s$. The mapping $F : C \to \mathbb{R}^n$ is defined by

$$F(x) = \left(-\frac{\partial f_1}{\partial x_1}(x), \cdots, -\frac{\partial f_s}{\partial x_s}(x) \right)^T,$$

and

$$\frac{\partial f_i}{\partial x_i}(x) = \frac{LNc_i h_i^2 e^{-t_i(x)}\left(1 - e^{-t_i(x)}\right)^{K-1}}{N\sigma^2 + \sum_{j\neq i} x_j h_j^2}, \quad \forall i = 1, \cdots, s.$$

Then, a power-control point x^* for CDMA systems is and only if it is a solution to Problem $VI(F, Fix(T))$. When each users strategy set depends on only the transmit power, some users far from the base station might not be able to get a sufficient SINR. Moreover, in point of fact that each users strategy set depends on the other users strategies. To solve these problems, Iiduka and Yamada in [12] replaced the strategy set C by a polyhedral set:

$$\bigcap_{i\in I}\{x \in \mathbb{R}^n : \ t_i(x) \geq \delta_i\} \cap C,$$

where $\delta_i > 0$ for all $i \in I$ is the required SINR for the ith user in the network. However, Yamada in [21] showed that, in general, when the noise power is enough large, this strategy set may be empty set. Then, the author produced a generalized convex feasible set as follows:

$$\hat{C} := \left\{\bar{x} \in \mathbb{R}^n : \ \bar{x} \in \text{argmin}_{x\in C} \sum_{i\in I} \lambda_i d^2(x, D_i)\right\}, \tag{1.2}$$

where $D_i := \{x \in \mathbb{R}^n : \ t_i(x) \geq \delta_i\}$, $\tau_i \in (0, 1)$ such that $\sum_{i\in I} \tau_i = 1$ and $d(x, D_i) := \min\{\|x - y\| : \ y \in D_i\}$. Set

$$T(x) := Pr_C\left(\sum_{i\in I} \tau_i d(x, D_i)\right). \tag{1.3}$$

Then T is nonexpansive, and $\bar{x} \in \hat{C}$ if and only if $\bar{x} \in Fix(T)$.

In this paper, we investigate a new and efficient global algorithm for solving variational inequalities over the fixed point set of a firmly nonexpansive mapping. To solve the problem, most of current algorithms are based on the metric projection onto a nonempty closed convex constraint set, in general, which is not easy to compute. The fundamental difference here is that, at each main iteration in the proposed algorithm, we only require computing the simple projection. Moreover, by choosing suitable regularization parameters, we show that the iterative sequence globally converges to a solution of Problem $VI(F, Fix(T))$.

The paper is organized as follows. Section 2 recalls some concepts related to variational inequalities over the fixed point set of nonexpansive mapping, that will be used in the sequel and a new iteration scheme. Section 3 investigates the convergence theorem of the iteration sequences presented in Section 2 as the main results of our paper. Application to the problem is a power control for a direct-sequence code-division multiple-access data network is presented in Section 4.

2. Preliminaries

We list some well known definitions and the projection under the Euclidean norm, which will be required in our following analysis.

Definition 2.1. Let C be a nonempty closed convex subset of $\mathbb{R}^n$, we denote the metric projection on C by $Pr_C(\cdot)$, *i.e.*,

$$Pr_C(x) = \text{argmin}\{\|y - x\| : \ y \in C\}, \ \ \forall x \in \mathbb{R}^n.$$

The mapping $F : C \to \mathbb{R}^n$ is said to be

(i) monotone on C if for each $x, y \in C$,

$$\langle F(x) - F(y), x - y\rangle \geq 0;$$

(ii) pseudomonotone on C if for each $x, y \in C$,

$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(y), x - y\rangle \geq 0.$$

It is well-known that the gradient method in [20] solves the convex optimization problem:

$$\min\{f(x) : \ x \in C\}, \tag{2.1}$$

where C_i is a closed convex subset of $\mathbb{R}^n$ for all $i = 1, \cdots, m$, $C := \cap_{i=1}^m C_i$, and f is a differentiable convex function on C. The iteration sequence $\{x^k\}$ of the method is defined by $x^{k+1} := P_C(x^k - \lambda \nabla f(x^k))$. When C is arbitrary closed convex, in general, computation of the metric projection P_C is not necessarily easy and hence it is not effective for solving the convex optimization problem. To overcome this drawback, Yamada in [21] proposed a fixed point iteration method

$$x^{k+1} := T(x^k - \lambda_k \nabla f(x^k)),$$

where T is a nonexpansive mapping defined by $T(x) := \sum_{i=1}^m \beta_i P_{C_i}(x)$ for all $x \in C, \beta_i \in (0, 1)$ such that $\sum_{i=1}^m \beta_i = 1$. Under certain parameters $\beta_i (i = 1, \cdots, m)$, the sequence $\{x^k\}$ converges a solution to Problem (2.1). Also this method has applied for signal processing problems (see [15, 19]). Motivated the fixed point iteration method, Iiduka and Yamada in [12] proposed a subgradient-type method for the variational inequalities over the fixed point set of a nonexpansive mapping $VI(F, Fix(T))$ and applied for the Nash equilibrium model in noncooperative games. Under an asymtotic optimization condition is satisfied. The authors showed that the iterative sequence converges a point in the problem without the metric projection onto a closed convex set. Very recently, Iiduka in [11] proposed the fixed point optimization algorithm for solving the following variational inequalities:

$$\text{Finding} \quad x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall\, x \in C,$$

where C is a nonempty closed convex subset of $\mathbb{R}^n, F : C \to \mathbb{R}^n$, over the fixed point set $Fix(T)$ of a firmly nonexpansive mapping $T : C \to \mathbb{R}^n$. In each iteration of the algorithm, in order to get the next iterate x^{k+1}, one orthogonal projection onto C included $Fix(T)$ is calculated, according to the following iterative step. Given the current iterate x^k, calculate

$$\begin{cases} y^k := T(x^k - \lambda_k F(x^k)), \\ x^{k+1} := Pr_C(\alpha_k x^k + (1-\alpha_k) y^k). \end{cases}$$

Under certain conditions over parameters $\lambda_k, \alpha_k (k \geq 1)$, and asymtotic optimization conditions $\bigcap_{k=1}^{\infty} \left\{ u \in Fix(T) : \ \langle F(x^k), x - x^k \rangle \leq 0 \right\} \neq \emptyset$ is satisfied. Then, the iterative sequence $\{x^k\}$ converges a solution to the variational inequalities over the fixed point set of the firmly nonexpansive mapping. In fact, the asymtotic optimization condition, in some cases, is is very difficult to define. In order to avoid this requirement, we propose a new iteration method without both the asymtotic optimization condition and computing the metric projection on a closed convex set. Our algorithm is described more detailed as follows.

Algorithm 2.2. (Initialization) Take a point $x^0 \in \mathbb{R}^n$ such that $M \leq \|x^0\|$, $\eta_0 := \|x^0\|$, a positive number $\rho > 0$, and the positive sequences $\{\beta_k\}, \{\rho_k\}, \{\epsilon_k\}$ verifying the following conditions:

$$\rho < \rho_k, \quad \lim_{k \to \infty} \epsilon_k = 0, \quad \sum_{k=0}^{\infty} \frac{\beta_k}{\rho_k} = +\infty, \quad \sum_{k=0}^{\infty} \beta_k^2 < \infty. \tag{2.2}$$

Step 1. Let $x^k \in \mathbb{R}^n$. Choose arbitrary $\lambda_k \in (0,1)$ such that

$$(1-\lambda_k)(\|x^k\| + M) \leq \beta_k$$

for all $k \geq 0$. Define $\gamma_k := \max\{\rho_k, \|F(x^k)\|\}$, $\alpha_k := \frac{\beta_k}{\gamma_k}$ and $C_k := \{x \in \mathbb{R}^n : \ \|x\| \leq \eta_k + 1\}$. Evaluate $t^k := P_{C_k}(x^k - \alpha_k F(x^k))$.

Step 2. Compute

$$x^{k+1} := T(\lambda_k x^k + (1-\lambda_k) t^k), \quad \eta_{k+1} := \max\{\eta_k, \|x^{k+1}\|\}, \quad k := k+1.$$

Note that $C_k = \{x \in \mathbb{R}^n : \ \|x\| \leq \eta_k + 1\}$ is a closed ball. Therefore, the metric projection $P_{C_k}(x^k - \alpha_k F(x^k))$ is computed by

$$t^k = \frac{\eta_k + 1}{\|x^k - \alpha_k F(x^k)\|}(x^k - \alpha_k F(x^k)).$$

3. Convergent results

To investigate the convergence of Algorithm 2.2, we recall the following technical lemmas, which will be used in the sequel.

Lemma 3.1. ([16]) *Let $\{a_k\}, \{b_k\}$ and $\{c_k\}$ be the three nonnegative sequences satisfying the following condition:*

$$a_{k+1} \leq (1+b_k)a_k + c_k.$$

If $\sum_{i=1}^{\infty} b_k < \infty$ and $\sum_{i=1}^{\infty} c_k < \infty$, then $\lim_{k\to\infty} a_k$ exists.

We are now in a position to prove some convergence theorems.

Theorem 3.2. *Let C be a nonempty closed convex subset of $\mathbb{R}^n$, $T : C \to \mathbb{R}^n$ is a firmly nonexpansive mapping such that $Fix(T)$ is bounded by $M > 0$. Then, the sequence $\{x^k\}$ generalized by Algorithm 2.2 converges to a solution of Problem $VI(F, Fix(T))$.*

Proof. We divide the proof into four steps.

Step 1. For each $x^* \in Sol(F, Fix(T))$, we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2(2-\lambda_k)\beta_k^2. \tag{3.1}$$

Indeed, from $t^k := P_{C_k}(x^k - \alpha_k F(x^k))$, it follows that

$$\langle \alpha_k F(x^k) + t^k - x^k, x - t^k\rangle \geq 0, \quad \forall\, x \in C_k. \tag{3.2}$$

Using the assumption $\|x\| \leq \|x^0\|$ for all $x \in Fix(T)$ and $C_k \subseteq C_{k+1}$ for all $k \geq 0$, we have $Fix(T) \subseteq C_k$. Then, substituting $x = x^*$ into (3.2), we get

$$\langle \alpha_k F(x^k) + t^k - x^k, x^* - t^k\rangle \geq 0.$$

Combinating this and the inequality

$$\|t^k - x^*\|^2 = \|x^k - x^*\|^2 - \|t^k - x^k\|^2 + 2\langle x^k - t^k, x^* - t^k\rangle,$$

we have

$$\|t^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|t^k - x^k\|^2 + 2\alpha_k\langle F(x^k), x^* - t^k\rangle. \tag{3.3}$$

Since (3.3), $x^{k+1} := T(\lambda_k x^k + (1-\lambda_k)t^k)$ and the equaltity

$$\begin{aligned} &\|\lambda x + (1-\lambda)y\|^2 \\ &= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \ \ \forall\,\lambda \in [0,1],\ x, y \in \mathbb{R}^n, \end{aligned} \tag{3.4}$$

we get

$$
\begin{aligned}
&\|x^{k+1} - x^*\|^2 \\
&= \|T(\lambda_k x^k + (1-\lambda_k)t^k) - T(x^*)\|^2 \\
&\le \|\lambda_k x^k + (1-\lambda_k)t^k - x^*\|^2 \\
&= \lambda_k\|x^k - x^*\|^2 + (1-\lambda_k)\|t^k - x^*\|^2 - \lambda_k(1-\lambda_k)\|t^k - x^k\|^2 \\
&\le \lambda_k\|x^k - x^*\|^2 + (1-\lambda_k)\left[\|x^k - x^*\|^2 - \|t^k - x^k\|^2 + 2\alpha_k\langle F(x^k), x^* - t^k\rangle\right] \\
&\quad - \lambda_k(1-\lambda_k)\|t^k - x^k\|^2 \\
&= \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\alpha_k(1-\lambda_k)\langle F(x^k), x^* - t^k\rangle.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 \\
+ 2\alpha_k(1-\lambda_k)\langle F(x^k), x^* - t^k\rangle.
\end{aligned} \tag{3.5}
$$

From $\gamma_k := \max\{\rho_k, \|F(x^k)\|\}$ and $\alpha_k := \frac{\beta_k}{\gamma_k}$, it follows that

$$
\alpha_k\|F(x^k)\| = \frac{\beta_k}{\gamma_k}\|F(x^k)\| = \frac{\beta_k\|F(x^k)\|}{\max\{\rho_k, \|F(x^k)\|\}} \le \beta_k. \tag{3.6}
$$

By the definition of the metric projection Pr_{C_k} and (3.6), we have

$$
\|t^k - x^k\|^2 \le \langle \alpha_k F(x^k), x^k - t^k\rangle \le \alpha_k\|F(x^k)\|\|x^k - t^k\| \le \beta_k\|t^k - x^k\|. \tag{3.7}
$$

Combinating (3.5), (3.6) and (3.7), we get

$$
\begin{aligned}
&\|x^{k+1} - x^*\|^2 \\
&\le \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\alpha_k(1-\lambda_k)\langle F(x^k), x^* - x^k\rangle \\
&\quad + 2\alpha_k(1-\lambda_k)\|F(x^k)\|\|x^k - t^k\| \\
&\le \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\alpha_k(1-\lambda_k)\|F(x^k)\|\|x^* - x^k\| \\
&\quad + 2(1-\lambda_k)\beta_k^2 \\
&\le \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2\beta_k(1-\lambda_k)(\|x^k\| + M) \\
&\quad + 2(1-\lambda_k)\beta_k^2 \\
&\le \|x^k - x^*\|^2 - (1-\lambda_k^2)\|t^k - x^k\|^2 + 2(2-\lambda_k)\beta_k^2.
\end{aligned}
$$

This implies (3.1).

Step 2. Claim that $\lim_{k\to\infty}\|x^{k+1}-x^k\|=0$ and $\lim_{k\to\infty}\|x^k-T(x^k)\|=0$.
Indeed, using (3.3), (3.4) and the definition of the firmly nonexpansive mapping, we have

$$\begin{aligned}
&\|x^{k+1}-x^*\|^2\\
&=\|T(\lambda_k x^k+(1-\lambda_k)t^k)-T(x^*)\|^2\\
&\le\langle\lambda_k x^k+(1-\lambda_k)t^k-x^*,x^{k+1}-x^*\rangle\\
&=\frac{1}{2}[\|\lambda_k x^k+(1-\lambda_k)t^k-x^*\|^2+\|x^{k+1}-x^*\|^2-\|\lambda_k x^k+(1-\lambda_k)t^k-x^{k+1}\|^2]\\
&\le\frac{1}{2}[\lambda_k\|x^k-x^*\|^2+(1-\lambda_k)\|t^k-x^*\|^2-\lambda_k(1-\lambda_k)\|x^k-t^k\|^2\\
&\quad+\|x^{k+1}-x^*\|^2]-\frac{1}{2}[\lambda_k\|x^k-x^{k+1}\|^2+(1-\lambda_k)\|t^k-x^{k+1}\|^2\\
&\quad-\lambda_k(1-\lambda_k)\|x^k-t^k\|^2]\\
&=\frac{1}{2}[\lambda_k\|x^k-x^*\|^2+\|x^{k+1}-x^*\|^2-\lambda_k\|x^k-x^{k+1}\|^2-(1-\lambda_k)\|t^k-x^{k+1}\|^2\\
&\quad+(1-\lambda_k)\|t^k-x^*\|^2]\\
&\le\frac{1}{2}\big[\lambda_k\|x^k-x^*\|^2+\|x^{k+1}-x^*\|^2-\lambda_k\|x^k-x^{k+1}\|^2-(1-\lambda_k)\|t^k-x^{k+1}\|^2\\
&\quad+(1-\lambda_k)\big(\|x^k-x^*\|^2-\|t^k-x^k\|^2+2\alpha_k\langle F(x^k),x^*-t^k\rangle\big)\big].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x^{k+1}-x^*\|^2\le\|x^k-x^*\|^2-\lambda_k\|x^k-x^{k+1}\|^2-(1-\lambda_k)\|t^k-x^{k+1}\|^2\\
-(1-\lambda_k)\|t^k-x^k\|^2+2(1-\lambda_k)\langle F(x^k),x^*-t^k\rangle. \qquad (3.8)
\end{aligned}$$

Applying Lemma 3.1 for the sequences in the inequality (3.1), there exists

$$A:=\lim_{k\to\infty}\|x^k-x^*\|. \qquad (3.9)$$

From Initialization of Algorithm 2.2 that $\lambda_k\in(0,1),(1-\lambda_k)(\|x^k\|+\|x^0\|)\le\beta_k$ and $\sum_{k=0}^{\infty}\beta_k^2<\infty$, it follows that

$$\lim_{k\to\infty}\lambda_k=1.$$

Combinating this, (3.8) and (3.9), we get

$$\lim_{k\to\infty}\|x^{k+1}-x^k\|=0.$$

Using this, the nonexpansive property of T and $\lim_{k\to\infty} \lambda_k = 1$, we have

$$\begin{aligned}\|x^{k+1} - T(x^{k+1})\| =&\|T(\lambda_k x^k + (1-\lambda_k)t^k) - T(x^{k+1})\| \\ \leq&\|\lambda_k x^k + (1-\lambda_k)t^k - x^{k+1}\| \\ \leq&\lambda_k\|x^k - x^{k+1}\| + (1-\lambda_k)\|t^k - x^{k+1}\| \\ \to&0 \quad as \quad k\to\infty.\end{aligned}$$

Since $\{x^k\}$ is bounded, there exists $\eta := \sup\{\eta_k : \ k \geq 0\} < \infty$ and a subsequence $\{x^{k_i}\}$ which converges to $\bar{x}$ as $i \to \infty$.

Step 3. Claim that $\bar{x} \in Fix(T) \cap B(0, \eta + 1 - \delta)$, where $\delta \in (0,1)$ and the open ball is defined by

$$B(0,\eta+1-\delta) := \{x \in \mathbb{R}^n : \ \|x\| < \eta + 1 - \delta\}.$$

Indeed, from $\eta := \sup\{\eta_k : \ k \geq 0\} < \infty$ and $\delta \in (0,1)$, it follows that the existence of k_0 such that $\eta_k \geq \eta - \delta$ for all $k \geq k_0$. It means that $B(0, \eta+1-\delta) \subseteq C_k$ for all $k \geq k_0$. Then, we have

$$\|\bar{x}\| = \lim_{i\to\infty} \|x^{k_i}\| \leq \eta < \eta + 1 - \delta.$$

Thus, $\bar{x} \in B(0, \eta+1-\delta)$.

Now we suppose that $\bar{x} \neq T(\bar{x})$. By Step 2 and Opial's condition, we get

$$\begin{aligned}\lim_{i\to\infty} \|x^{k_i} - \bar{x}\| &< \lim_{i\to\infty} \|x^{k_i} - T(\bar{x})\| \\ &\leq \lim_{i\to\infty} \Big(\|x^{k_i} - T(x^{k_i})\| + \|T(x^{k_i}) - T(\bar{x})\|\Big) \\ &= \lim_{i\to\infty} \|T(x^{k_i}) - T(\bar{x})\| \\ &\leq \lim_{i\to\infty} \|x^{k_i} - \bar{x}\|.\end{aligned}$$

This is a contradiction. So, $\bar{x} = T(\bar{x})$.

Step 4. Claim that $\bar{x} \in Sol(F, Fix(T))$ and the sequence $\{x^k\}$ converges to $\bar{x}$. Indeed, from (3.6), it follows that $0 \leq \rho\|F(x^k)\| \leq \|F(x^k)\| \max\{\rho_k, \|F(x^k)\|\} = \alpha_k\|F(x^k)\| \leq \beta_k$. Using $\sum_{k=0}^{\infty} \beta_k < \infty$ and $\rho > 0$, we have $\lim_{k\to\infty} \|F(x^k)\| = 0$. Combinating this and Step 3, we have

$$\langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \forall\, y \in Fix(T) \cap B(0, \eta+1-\delta).$$

Denote $g(z) := \langle F(\bar{x}), z - \bar{x}\rangle$. Then, g is convex and

$$g(z) \geq g(\bar{x}) = 0, \quad \forall\, z \in Fix(T) \cap B(0,\eta+1-\delta).$$

Thus, $\bar{x}$ is a local minimizer of g. Since $Fix(T)$ is nonempty convex, $\bar{x}$ is also a global minimizer of g, i.e., $g(z) \geq g(\bar{x})$ for all $z \in Fix(T)$. This means that

$$\langle F(\bar{x}), z - \bar{x} \rangle \geq 0, \quad \forall\, z \in Fix(T).$$

So, $\bar{x} \in Sol(F, Fix(T))$.

To prove $\{x^k\}$ converges to $\bar{x}$, we suppose that the subsequence $\{x^{k_j}\}$ also converges to $\hat{x}$ as $j \to \infty$. By a same way, we also have $\hat{x} \in VI(F, Fix(T))$. Suppose that $\bar{x} \neq \hat{x}$. Then, using Opial's condition, we have

$$\begin{aligned}
\lim_{k\to\infty} \|x^k - \bar{x}\| &= \lim_{i\to\infty} \|x^{k_i} - \bar{x}\| \\
&< \lim_{i\to\infty} \|x^{k_i} - \hat{x}\| \\
&= \lim_{k\to\infty} \|x^k - \hat{x}\| \\
&= \lim_{j\to\infty} \|x^{k_j} - \hat{x}\| \\
&< \lim_{j\to\infty} \|x^{k_j} - \bar{x}\| \\
&= \lim_{k\to\infty} \|x^k - \bar{x}\|.
\end{aligned}$$

This is a contradiction. Thus, the sequence $\{x^k\}$ converges to $\bar{x} \in Sol(F, Fix(T))$. □

4. Application to power-control problem

Now we illustrate our algorithm to present a computational method for the control-power problem for cdoe-division multiple-access systems. Note that

(i) The common power set $C := C_1 \times \cdots \times C_s$, the required SINR for the ith user in the network $D_i := \{x \in \mathbb{R}^n : t_i(x) \geq \delta_i\}$, where t_i is defined by (1.1).

(ii) The generalized convex feasible set $\hat{C}$ is given by (1.2).

Power control problem is to find a power-control solution $x^* \in \hat{C}$ for efficient resource allocation and interference management in the uplink and downlink of CDMA systems. We define two mappings $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$, as follows: For each $x \in \mathbb{R}^n$, $T(x)$ is defined by (1.3) and

$$S(x) := \frac{1}{2}\left(x + T(x)\right).$$

Then, Yamada in [21] showed that T is nonexpansive on $\mathbb{R}^n$ and S is firmly nonexpansive on $\mathbb{R}^n$. Moreover, $\hat{C} \neq \emptyset$ and $\hat{C} = Fix(S)$ hold. Therefore, we can see that the power control problem can be formulated as Problem $VI(F, Fix(S))$, where the bifunction f defined by $f(x, y) := \langle F(x), y - x \rangle$, the operator $F : \mathbb{R}^n \to \mathbb{R}^n$, and the derivatives $\nabla_{x_i} f_i(x)$ of the function f_i forward to x_i at x defined by

$$F(x) := \big(-\nabla_{x_1} f_1(x), -\nabla_{x_2} f_2(x), \cdots, -\nabla_{x_s} f_s(x)\big)^T$$

and

$$\nabla_{x_i} f_i(x) = \frac{Lc_i e^{-t_i(x)} h_i^2 \big[1 - e^{-t_i(x)}\big]^{K-1}}{\sigma^2 + \frac{1}{N} \sum\limits_{j \neq i} x_j h_j^2}.$$

By Algorithm 2.2 and Theorem 3.2, we get the following algorithm and convergent theorem for the power control problem.

Algorithm 4.1. (Initialization) Take a point $x^0 \in \mathbb{R}^n$ such that $M \leq \|x^0\|$, $\eta_0 := \|x^0\|$, a positive number $\rho > 0$ and the positive sequences $\{\beta_k\}, \{\rho_k\}$, $\{\epsilon_k\}$ verifying the following conditions:

$$\lim_{k\to\infty} \epsilon_k = 0, \ \sum_{k=0}^{\infty} \frac{\beta_k}{\rho_k} = +\infty, \ \sum_{k=0}^{\infty} \beta_k^2 < \infty.$$

- Let $x^k \in \mathbb{R}^n$. Choose $\lambda_k \in (0, 1)$ such that $(1 - \lambda_k)(\|x^k\| + M) \leq \beta_k$ for all $k \geq 0$. Define $\gamma_k := \max\{\rho_k, \|F(x^k)\|\}$, $\alpha_k := \frac{\beta_k}{\gamma_k}$, and $C_k := \{x \in \mathbb{R}^n : \|x\| \leq \eta_k + 1\}$. Evaluate $t^k := P_{C_k}(x^k - \alpha_k F(x^k))$.
- Compute

$$x^{k+1} := S(\lambda_k x^k + (1 - \lambda_k)t^k), \quad \eta_k := \max\{\eta_k, \|x^{k+1}\|\}.$$

Note that $P_{C_k}(x^k - \alpha_k F(x^k))$ is computed by

$$t^k = \frac{\eta_k + 1}{\|x^k - \alpha_k F(x^k)\|}(x^k - \alpha_k F(x^k)).$$

Theorem 4.2. *Let $\{x^n\}$ be the sequence generated by Algorithm 4.1. Assume that $\hat{C}$ is bounded by $M > 0$. Then, the sequence $\{x^k\}$ converges to a power-control solution $x^* \in \hat{C}$ for efficient resource allocation and interference management in the uplink and downlink of CDMA systems of the power control problem.*

To illustrate the inexact subgradient algorithm, we consider the power control problem for CDMA systems based on the model proposed by Meshkati *et. al.*, in [15]. In this model, each packet contains $100b$ of informations for 9 users and

- The processing gain: $N = 100$.
- The total number of bits in a packet: $K = 100$.
- The number of information bits: $L = 100$.
- The transmission rate for the ith user: $c_i = 10^4 bits/s$.
- The noise power: $\sigma^2 = 10 \times 10^{-14}$ watts.

- The strategy set of the ith user: $a_i = 0.1$ watts, $b_i = 1$ watts.
- The channel gain for the ith user: $h_i = \frac{0.3}{d_i^2}$, where d_i is the distance from the ith user to the base station. Now we assume that $d_1 = 310m, d_2 = 460m, d_3 = 570m, d_4 = 660m, d_5 = 740m, d_6 = 810m, d_7 = 880m, d_8 = 940m, d_9 = 1000m$.
- The coefficient of the nonexpansive mapping T defined by (1.3): $\tau_i = \frac{i}{45}$.
- The coefficient δ_i of the required SINR for the ith user in the network D_i: $\delta_i = 1$.

The parameters in Algorithm 4.1 are chosen as follows:

$$\beta_k := \frac{1}{(k+1)^2}, \quad \rho_k := 1 + \frac{1}{k+1}, \quad \lambda_k = 1 - \frac{1}{2k^2(\|x^k\| + K)}.$$

From $a_i = 0.1$ watts and $b_i = 1$ watts, it follows that $\|x\| \leq 3$ for all $x \in Fix(S)$. Therefore, we can choose $M = 3$. It follows from Algorithm 4.1 and Theorem 4.2 that x^k is an ϵ- power control point in uplink and downlink of CDMA systems if

$$\|x^{k+1} - x^k\| \leq \epsilon.$$

The following computations are performed by Matlab R2008a running on a PC Desktop Intel(R) Core(TM)i5 650@3.2GHz 3.33GHz 4Gb RAM.

It	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k	x_8^k	x_9^k
0	1	1	1	1	1	1	1	1	1
1	0.8616	0.9827	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.7324	0.959	0.9935	0.9971	0.9995	1.0000	1.0000	1.0000	1.0000
3	0.6182	0.9369	0.9855	0.9932	0.9976	0.9995	1.0000	1.0000	1.0000
4	0.5169	0.9168	0.9778	0.9893	0.9955	0.9984	1.0000	1.0000	1.0000
5	0.4271	0.8989	0.9707	0.9856	0.9934	0.9971	1.0000	1.0000	1.0000
6	0.3478	0.8830	0.9642	0.9822	0.9915	0.9959	1.0000	1.0000	1.0000
7	0.2776	0.8689	0.9585	0.9792	0.9897	0.9948	1.0000	1.0000	1.0000
8	0.2156	0.8564	0.9534	0.9765	0.9881	0.9938	1.0000	1.0000	1.0000
9	0.1607	0.8453	0.9489	0.9740	0.9867	0.9928	1.0000	1.0000	1.0000
10	0.1304	0.8354	0.9448	0.9719	0.9854	0.9920	1.0000	1.0000	1.0000
11	0.1152	0.8263	0.9410	0.9698	0.9842	0.9912	0.9995	1.0000	1.0000
12	0.1076	0.8175	0.9374	0.9679	0.9830	0.9904	0.9990	1.0000	1.0000
13	0.1038	0.8088	0.9338	0.9659	0.9818	0.9896	0.9984	1.0000	1.0000
14	0.1019	0.8002	0.9302	0.9640	0.9806	0.9888	0.9979	1.0000	1.0000
15	0.1009	0.7917	0.9267	0.9621	0.9794	0.9881	0.9973	1.0000	1.0000

Table 1: Numerical results of Algorithm 4.1 with $\epsilon = 10^{-3}$.

The approximate solution obtained after 15 iterations is

$$x^{15} = (0.1009, 0.7917, 0.9267, 0.9621, 0.9794, 0.9881, 0.9973, 1.0000, 1.0000)^T.$$

The following tables list the numerical results of Algorithm 4.1 with some changes such as the tolerance $\epsilon = 10^{-6}$, the coefficient $\tau_i = \frac{1}{9}(i = 1, \cdots, 9)$ of the nonexpansive mapping T, and the starting point $x^0 = (0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)^T$.

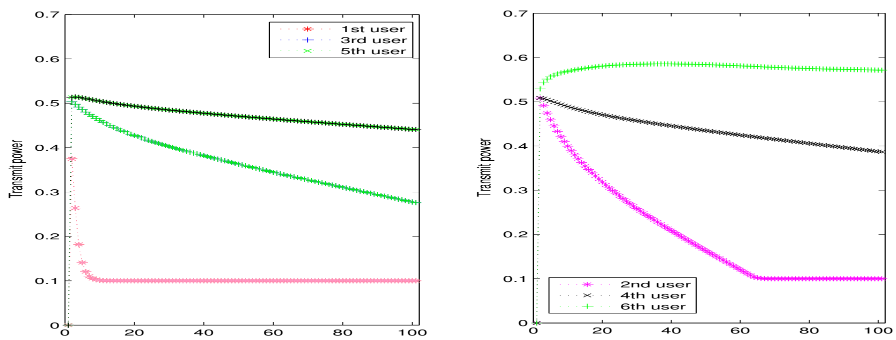

FIGURE 1. The iterative graph of the users.

Base on the preliminary numerical results reported in the tables, we can see that the same as some other well-known algorithms such as the proximal point algorithm for solving variational inequalities in [14], the fixed point optimization algorithm for solving equilibrium problems on the fixed point set of a nonexpansive mapping in [2, 11, 13], and other methods in [1, 6, 7, 9], the rapicdity of the algorithm depends very much on the starting point. Moreover, the efficiency of the algorithm depends very much on the choice of the parameters ρ, β_k, ρ_k and the coefficients of the firmly nonexpansive mapping S.

5. Conclusion

This paper presented an iterative algorithm for solving variational inequalities over the fixed point set of a nonexpansive mapping T. By choosing the suitable regular parameters, we show that the sequences generated by the algorithm globally converge to a solution of Problem $VI(F, fix(T))$. Comparing with the current methods, the fundamental difference here is that, the algorithm only requires the continuity of the mapping F and convergence of the proposed algorithms does not require F to satisfy any type of monotonicity. Moreover, in general, computing the exact subgradient of a subdifferentiable function is too expensive, our algorithm only requires to compute approximate. From the preliminary numerical results reported in the tables, the algorithm seems to be efficient to solve a power control for a direct-sequence code-division multiple-access data network.

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