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A COMMON FIXED POINT THEOREM OF INTEGRAL TYPE USING IMPLICIT RELATION

H. K. Pathak[1], Rosana Rodríguez-López[2] and R. K. Verma[3]

[1]School of Studies in Mathematics, Pt. Ravishankar Shukla University,
Raipur (C.G) 492010, India
e-mail: hkpathak05@gmail.com

[2]Departamento de Análisis Matemático, Facultad de Matemáticas,
Universidad de Santiago de Compostela, 15782-Santiago de Compostela, Spain
e-mail: rosana.rodriguez.lopez@usc.es

[3]Department of Mathematics, Govt. C.L.C. College Patan,
Dist.-Durg (C.G.) 491111, India
e-mail: rohitverma1967@sify.com

Abstract. In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings in a metric space satisfying a contractive condition of integral type by using an implicit relation of Popa [7] and the property (E.A) introduced recently by Aamri and Mautawakil [1] as a generalization of noncompatible mappings. Our theorem generalizes Theorem 2 of Aamri and Mautawakil in the sense that we can obtain its contractive condition as an special case of our contractive condition. Further, our theorem is a slight variation of Theorem 5 of Popa in the sense that we have replaced the Meir-Keeler type contractive condition to impose the property (E.A). Thus we have unified and generalized both results by using implicit relation and property (E.A) under the integral type mappings.

1. Introduction

The notion of weak commutativity of Sessa [8] is generalized by Jungck [3] for compatible mappings and further generalized by Jungck and Rhoades [4] for weakly compatible mappings. In the sequel, the noncompatibility and various types of compatibility were used to study the existence of a common fixed point. The noncompatibility as a tool for finding fixed points is introduced

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by Pant [5, 6]. The noncompatibility is further generalized by introducing property (E.A) in a metric space by Aamri and Mautawakil [1]. They established some common fixed point theorems under strict contractive condition for weakly compatible mappings satisfying property (E.A).

On the other hand, Popa [7] used the implicit relation for two pairs of weakly compatible self-maps of Meir-Keeler type contractive condition to relax the continuity of mappings in the metric space.

2. Preliminaries and Definitions

In 1982, Sessa introduced the notion of weak commutativity as follows:

Definition 2.1. [8] *Two self-maps A and S of a metric space (X, d) are said to be weakly commuting if* $d(ASx, SAx) \leq d(Ax, Sx),\ \forall x \in X$.

It is clear that two commuting mappings are weakly commuting but the converse is not true as shown in [8]. Jungck [3] extended this concept in the following way:

Definition 2.2. [3] *Let A and S be two self-maps of a metric space (X, d). A and S are said to be compatible if*

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0, \tag{2.1}$$

whenever there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in X$.

Obviously, two weakly commuting mappings are compatible, but the converse is not true as shown in [3]. Note that, if the limit on the left hand side of (2.1) is either nonzero or nonexistent, then the pair is called *noncompatible.*

In 1998, Jungck introduced weakly compatible maps as follows:

Definition 2.3. [4] *Two self-maps A and S of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points; i.e.,*

$$ASu = SAu, \text{ for } u \in X \text{ whenever } Au = Su. \tag{2.2}$$

It is easy to see that two compatible maps are weakly compatible but the converse is not true as shown in [4]. A noncompatible pair may also satisfy weakly compatible property (see Examples 2.5 and 2.6 below).

Recently, Aamri and Mautawakil [1] generalized the notion of noncompatibility by introducing the property (E.A) in the following way:

Definition 2.4. [1] *Let A and S be two self-maps of a metric space (X, d) then they are said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X. \tag{2.3}$$

Notice that weakly compatibility and property (E.A) are independent to each other.

Example 2.5. *Let $X = [0, 1]$ and d be the usual metric on X. Define $f, g : X \to X$ by $fx = (\sqrt{5 - 4(2x - 1)^2} - 1)/4$ and $gx = (\frac{1}{3})$ fractional part of $(1 - x)$, $\forall x \in X$. Then we observe that the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$ satisfies (2.3) for $t = 0$ and (f, g) satisfies property (E.A), but (f, g) is noncompatible; as $\lim_{n\to\infty} fx_n = 0 = \lim_{n\to\infty} gx_n$ but $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$. Further, f and g are weakly compatible since they commute at their coincidence points $x = 0, \frac{1}{4}$ and 1.*

Example 2.6. *Let $X = [0, 2]$ and d be the usual metric on X. Define $f, g : X \to X$ by:*

$$fx = 0, \text{ if } 0 < x \leq 1 \text{ and } fx = 1, \text{ if } x = 0 \text{ or } 1 < x \leq 2; \text{ and}$$

$$gx = [x], \text{ the greatest integer less than or equal to } x, \ \forall x \in X.$$

Consider the sequence $\{x_n = 1 - \frac{1}{n}\}_{n\geq 2}$ in $(0, 1)$ (or $\{x_n = 1 + \frac{1}{n}\}_{n\geq 2}$ in $(1, 2)$) then we have $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in [0, 2]$. Thus the pair (f, g) satisfies property (E.A). But f and g are not weakly compatible; as each $u_1 \in (0, 1)$ and $u_2 \in (1, 2)$ are coincidence points of f and g, where they do not commute. Moreover, they commute at $x = 0, 1$ and 2 but none of these points are coincidence points of f and g. Further, (f, g) is noncompatible for all the sequences in $[0, 2]$. Hence, (E.A) does not imply weak compatibility.

Example 2.7. *To check that weakly compatible property does not imply (E.A), it is enough to consider $X = [0, 1]$, d the usual metric on X, and $f(x) = 0$, $g(x) = 1$, $\forall x \in X$. Hence, for all sequence $\{x_n\}$ in X, $\lim_{n\to\infty} fx_n = 0 \neq 1 = \lim_{n\to\infty} gx_n$.*

3. Implicit Relation

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real and non-negative real numbers, respectively, throughout our further discussion. We now state an implicit relation [7] as follows:

Let $\mathcal{F}$ be the set of all continuous functions

$$F : (t_1, \cdots, t_6) \in \mathbb{R}_+^6 \longrightarrow F(t_1, \cdots, t_6) \in \mathbb{R}$$

satisfying the following conditions:

$$(F_1): \ F(u,\ 0,\ u,\ 0,\ 0,\ u) \leq 0 \Longrightarrow u = 0, \tag{3.1}$$

$(F_2): \ F(u,\ 0,\ 0,\ u,\ u,\ 0) \leq 0 \Longrightarrow u = 0.$ (3.2)

The function $F : \mathbb{R}^6_+ \to \mathbb{R}$ is said to satisfy condition (F_u) if:

$(F_u): \ F(u,\ u,\ 0,\ 0,\ u,\ u) \geq 0,\ \forall u > 0.$ (3.3)

The following are some examples of implicit relation satisfying $(F_1), (F_2), (F_u)$.

Example 3.1. *Let* $F(t_1, ..., t_6) = pt_1 - qt_2 + r(t_3 - t_4) + s(-t_5 + t_6)$, *where* $r + s < p$, $-r - s < p$ *and* $q \leq p$. *Then:*
$(F_1): \ F(u, 0, u, 0, 0, u) = u(p + r + s) \leq 0$ *implies* $u = 0$;
$(F_2): \ F(u, 0, 0, u, u, 0) = u(p - r - s) \leq 0$ *implies* $u = 0$ *and*
$(F_u): \ F(u, u, 0, 0, u, u) = u(p - q) \geq 0,\ \forall u > 0.$

Example 3.2. *Let* $F(t_1, ..., t_6) = pt_1 + max\{-qt_2,\ (t_3 - t_4)/2,\ -s(t_5 - t_6)/2\}$, *where* $0 \leq s, q$, *and* $0 < p$. *Then:*
$(F_1): \ F(u, 0, u, 0, 0, u) = pu + max\{0, u/2, su/2\} = u(p + max\{1/2, s/2\}) \leq 0 \Rightarrow u = 0$;
$(F_2): \ F(u, 0, 0, u, u, 0) = pu + max\{0, -u/2, -su/2\} = up \leq 0 \Rightarrow u = 0$;
$(F_u): \ F(u, u, 0, 0, u, u) = pu + max\{-qu, 0, 0\} = up \geq 0,\ \forall u > 0.$

Example 3.3. *Let* $F(t_1, ..., t_6) = t_1 - max\{qt_2,\ -r(t_3 - t_4)/2,\ (t_5 - t_6)/2\}$, *where* $0 \leq q \leq 1$ *and* $0 \leq r < 2$. *Then:*
$(F_1): \ F(u, 0, u, 0, 0, u) = u - max\{0, -ru/2, -u/2\} = u \leq 0 \Rightarrow u = 0$;
$(F_2): \ F(u, 0, 0, u, u, 0) = u - max\{0, ru/2, u/2\} = u(1 - max\{r/2, 1/2\}) \leq 0 \Rightarrow u = 0$;
$(F_u): \ F(u, u, 0, 0, u, u) = u - max\{qu, 0, 0\} = u - qu = u(1 - q) \geq 0,\ \forall u > 0.$

Example 3.4. *Let* $F(t_1, ..., t_6) = t_1 - h\, max\{t_2,\ t_4 - t_3,\ t_5 - t_6\}$, *where* $0 \leq h < 1$. *Then:*
$(F_1): \ F(u, 0, u, 0, 0, u) = u - h\, max\{0, -u, -u\} = u \leq 0 \Rightarrow u = 0$;
$(F_2): \ F(u, 0, 0, u, u, 0) = u - h\, max\{0, u, u\} = u(1 - h) \leq 0 \Rightarrow u = 0$;
$(F_u): \ F(u, u, 0, 0, u, u) = u - h\, max\{u, 0, 0\} = u(1 - h) \geq 0,\ \forall u > 0.$

Example 3.5. *Let* $F(t_1, ..., t_6) = t_1^2 - at_2^2 + t_3t_4 - bt_5^2 + ct_6^2$, *where* $a, b, c \geq 0$, $1 > b$ *and* $a + b - c \leq 1$. *Then:*
$(F_1): \ F(u, 0, u, 0, 0, u) = u^2(1 + c) \leq 0 \Rightarrow u = 0$;
$(F_2): \ F(u, 0, 0, u, u, 0) = u^2(1 - b) \leq 0 \Rightarrow u = 0$ *and*
$(F_u): \ F(u, u, 0, 0, u, u) = u^2(1 - a - b + c) \geq 0,\ \forall u > 0.$

Example 3.6. *Let* $F(t_1, ..., t_6) = t_1^2 - at_2^2 + t_3^2 - t_4^2 + bt_5^2 + ct_6^2$, *where* $a, b, c \geq 0$, $b > 0$ *and* $a - b - c \leq 1$. *Then:*
$(F_1): \; F(u, 0, u, 0, 0, u) = u^2 + u^2 + cu^2 = (2 + c)u^2 \leq 0 \Rightarrow u = 0;$
$(F_2): \; F(u, 0, 0, u, u, 0) = bu^2 \leq 0 \Rightarrow u = 0$ *and*
$(F_u): \; F(u, u, 0, 0, u, u) = u^2(1 - a + b + c) \geq 0, \; \forall u > 0.$

Example 3.7. *Let* $F(t_1, ..., t_6) = t_1^3 - k(t_2^3 - t_3^3 + t_4^3 + t_5^3 - t_6^3)$, *where* $0 \leq k < 1/2$. *Then:*
$(F_1): \; F(u, 0, u, 0, 0, u) = u^3(1 + 2k) \leq 0 \Rightarrow u = 0,$
$(F_2): \; F(u, 0, 0, u, u, 0) = u^3(1 - 2k) \leq 0 \Rightarrow u = 0$ *and*
$(F_u): \; F(u, u, 0, 0, u, u) = u^3(1 - k) \geq 0, \; \forall u > 0.$

We will use the implicit relation of Popa [7] to relax the continuity of two pairs of weakly compatible mappings satisfying property (E.A) and a contractive condition of integral type mapping. The main purpose of our paper is to prove a common fixed point theorem for generalized noncompatible weakly compatible non continuous pairs of self-mappings satisfying a Lebesgue-integral type contractive condition. We will use the method of Aliouche [2] to prove the existence of coincidence and fixed point.

4. Main Result

Throughout this section, let ψ be a non-negative real-valued function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, which is a Lebesgue-integrable mapping such that
(a) ψ is summable and non-negative,
(b) $\int_0^{\epsilon} \psi(t)dt > 0$, for all $\epsilon > 0$,
(c) $\int \psi(t)dt$ is a non-decreasing function in $\mathbb{R}_+$.

Let $\mathbb{N}$ denote the set of positive integer numbers. Let $\mathcal{F}$ be the set of all continuous functions $F : (t_1, ..., t_6) \in \mathbb{R}_+^6 \longrightarrow F(t_1, ..., t_6) \in \mathbb{R}$ which also satisfy $(F_1), (F_2)$ and (F_u).

Now we state and prove our main theorem.

Theorem 4.1. *Let* A, B, S *and* T *be four self-mappings of a metric space* (X, d) *such that*
(i) $A(X) \subseteq T(X), \; B(X) \subseteq S(X)$,
(ii) suppose there exists a continuous function $F \in \mathcal{F}$ *such that*

$$F\Big(\int_0^{d(Ax,By)} \psi(t)dt, \; \int_0^{d(Sx,Ty)} \psi(t)dt, \; \int_0^{d(Ax,Sx)} \psi(t)dt,$$
$$\int_0^{d(By,Ty)} \psi(t)dt, \; \int_0^{d(By,Sx)} \psi(t)dt, \; \int_0^{d(Ax,Ty)} \psi(t)dt\Big) < 0,$$

for all $x, y \in X$ where $F \in \mathcal{F}$ satisfies conditions $(F_1), (F_2)$ and (F_u), and ψ satisfies the conditions (a), (b) and (c),
(iii) (A, S) and (B, T) are weakly compatible,
(iv) (A, S) or (B, T) satisfies property (E.A).

If the range of one of the mappings is a complete subspace of X, then A, B, S and T have a unique common fixed point.

Proof. Suppose (B, T) satisfies property (E.A), then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$ for all $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty} d(Sy_n, Tx_n) = 0$. Now we show that $\lim_{n\to\infty} d(Ay_n, z) = 0$. Indeed, in view of implicit relation (ii), we have

$$F\Big(\int_0^{d(Ay_n,Bx_n)} \psi(t)dt,\ \int_0^{d(Sy_n,Tx_n)} \psi(t)dt,\ \int_0^{d(Ay_n,Sy_n)} \psi(t)dt,$$
$$\int_0^{d(Bx_n,Tx_n)} \psi(t)dt,\ \int_0^{d(Bx_n,Sy_n)} \psi(t)dt,\ \int_0^{d(Ay_n,Tx_n)} \psi(t)dt\Big) < 0,$$

i.e., $F\Big(\int_0^{d(Ay_n,Bx_n)} \psi(t)dt,\ \int_0^{d(Bx_n,Tx_n)} \psi(t)dt,\ \int_0^{d(Ay_n,Bx_n)} \psi(t)dt,$

$$\int_0^{d(Bx_n,Tx_n)} \psi(t)dt,\ 0,\ \int_0^{d(Ay_n,Tx_n)} \psi(t)dt\Big) < 0.$$

Note that $\limsup_{n\to\infty} \int_0^{d(Ay_n,Tx_n)} \psi(t)dt = \limsup_{n\to\infty} \int_0^{d(Ay_n,Bx_n)} \psi(t)dt$. Indeed,

$$\left| \int_0^{d(Ay_n,Tx_n)} \psi(t)dt - \int_0^{d(Ay_n,Bx_n)} \psi(t)dt \right| = \left| \int_{d(Ay_n,Tx_n)}^{d(Ay_n,Bx_n)} \psi(t)dt \right|,$$

and the measure of the interval tends to zero as $n \to \infty$:

$$|d(Ay_n, Bx_n) - d(Ay_n, Tx_n)| \le d(By_n, Tx_n),\ \forall n.$$

Besides, if $\int_0^{d(Ay_{n_k},Bx_{n_k})} \psi(t)dt$ tends to $\limsup_{n\to\infty} \int_0^{d(Ay_n,Bx_n)} \psi(t)dt$, as $k \to \infty$, then $\int_0^{d(Ay_{n_k},Tx_{n_k})} \psi(t)dt$ also tends to $\limsup_{n\to\infty} \int_0^{d(Ay_n,Bx_n)} \psi(t)dt$, as $k \to \infty$. Thus, taking into account that F is continuous, and using that

$$\lim_{n\to\infty} \int_0^{d(Bx_n,Tx_n)} \psi(t)dt = 0,$$

it yields, taking *lim sup* in the inequality deduced from the implicit relation,

$$F\Big(\limsup_{n\to\infty} \int_0^{d(Ay_n,Bx_n)} \psi(t)dt,\ 0, \limsup_{n\to\infty} \int_0^{d(Ay_n,Bx_n)} \psi(t)dt,$$

$$0,\ 0,\ \limsup_{n\to\infty} \int_0^{d(Ay_n,Bx_n)} \psi(t)dt\Big) \leq 0.$$

Using (F_1), we obtain $\limsup_{n\to\infty} \int_0^{d(Ay_n,Bx_n)} \psi(t)dt = 0$. Whence by (b),

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = z. \tag{4.1}$$

Next, suppose that $S(X)$ is a complete subspace of X, then for this $z \in X$, there exists some $u \in X$ such that $z = Su$. As a consequence, we obtain

$$\lim_{n\to\infty} d(Ay_n, Su) = \lim_{n\to\infty} d(Bx_n, Su) = \lim_{n\to\infty} d(Tx_n, Su) = \lim_{n\to\infty} d(Sy_n, Su) = 0.$$

Now we claim that $Au = z$. If not, then using the implicit relation (ii) we have

$$F\Big(\int_0^{d(Au,Bx_n)} \psi(t)dt,\ \int_0^{d(Su,Tx_n)} \psi(t)dt,\ \int_0^{d(Au,Su)} \psi(t)dt,$$
$$\int_0^{d(Bx_n,Tx_n)} \psi(t)dt,\ \int_0^{d(Bx_n,Su)} \psi(t)dt,\ \int_0^{d(Au,Tx_n)} \psi(t)dt\Big) < 0.$$

Letting $n \to \infty$, it yields

$$F\Big(\lim_{n\to\infty} \int_0^{d(Au,Bx_n)} \psi(t)dt,\ \lim_{n\to\infty} \int_0^{d(Su,Tx_n)} \psi(t)dt,$$
$$\lim_{n\to\infty} \int_0^{d(Au,Su)} \psi(t)dt,\ \lim_{n\to\infty} \int_0^{d(Bx_n,Tx_n)} \psi(t)dt,$$
$$\lim_{n\to\infty} \int_0^{d(Bx_n,Su)} \psi(t)dt,\ \lim_{n\to\infty} \int_0^{d(Au,Tx_n)} \psi(t)dt\Big) \leq 0.$$

Now, the continuity of the integral operator with (b) implies that,

$$F\Big(\int_0^{d(Au,z)} \psi(t)dt,\ 0,\ \int_0^{d(Au,z)} \psi(t)dt,\ 0,\ 0,\ \int_0^{d(Au,z)} \psi(t)dt\Big) \leq 0,$$

which, on using (F_1) yields $\int_0^{d(Au,z)} \psi(t)dt = 0$. So that, by (b), $Au = z$. Therefore u is a coincidence point of A and S.

Further, since $A(X) \subseteq T(X)$, then $z = Au$ implies $z \in T(X)$. Let $v \in X$ such that $Tv = z$. We claim that $Bv = z$. For, setting $x = y_n$ and $y = v$ in the implicit relation (ii), we have

$$F\Big(\int_0^{d(Ay_n,Bv)} \psi(t)dt,\ \int_0^{d(Sy_n,Tv)} \psi(t)dt,\ \int_0^{d(Ay_n,Sy_n)} \psi(t)dt,$$
$$\int_0^{d(Bv,Tv)} \psi(t)dt,\ \int_0^{d(Bv,Sy_n)} \psi(t)dt,\ \int_0^{d(Ay_n,Tv)} \psi(t)dt\Big) < 0,$$

letting $n \to \infty$ and then using condition (b) it yields

$$F\Big(\int_0^{d(z,Bv)}\psi(t)dt,\ 0,\ 0,\ \int_0^{d(Bv,z)}\psi(t)dt,\ \int_0^{d(Bv,z)}\psi(t)dt,\ 0\Big) \le 0,$$

using (F_2) it implies $\int_0^{d(z,Bv)}\psi(t)dt = 0$, yielding $Bv = z$. Therefore v is a coincidence point of B and T.

The weak compatibility of A with S and B with T implies that $Sz = SAu = ASu = Az$ and $Tz = TBv = BTv = Bz$.

In order to show that z is a coincidence point of A, B, S and T, let us show that $Az = Bz$. Contrary, let $Az \neq Bz$. Then, setting $x = z$ and $y = z$ in (ii), we have successively

$$F\Big(\int_0^{d(Az,Bz)}\psi(t)dt,\ \int_0^{d(Sz,Tz)}\psi(t)dt,\ \int_0^{d(Az,Sz)}\psi(t)dt,\ \int_0^{d(Bz,Tz)}\psi(t)dt,\ \int_0^{d(Bz,Sz)}\psi(t)dt,\ \int_0^{d(Az,Tz)}\psi(t)dt\Big) < 0,$$

$$F\Big(\int_0^{d(Az,Bz)}\psi(t)dt, \int_0^{d(Az,Bz)}\psi(t)dt,\ 0,\ 0,\ \int_0^{d(Bz,Az)}\psi(t)dt, \int_0^{d(Az,Bz)}\psi(t)dt\Big) < 0,$$

which contradicts (F_u). So that $Az = Bz$. Therefore z is a coincidence point of A, B, S and T.

Now, we claim that z is a common fixed point of A, B, S and T. If $Az \neq z$, then by putting z for x and v for y in (ii), we have successively

$$F\Big(\int_0^{d(Az,Bv)}\psi(t)dt,\ \int_0^{d(Sz,Tv)}\psi(t)dt,\ \int_0^{d(Az,Sz)}\psi(t)dt,\ \int_0^{d(Bv,Tv)}\psi(t)dt,\ \int_0^{d(Bv,Sz)}\psi(t)dt,\ \int_0^{d(Az,Tv)}\psi(t)dt\Big) < 0,$$

$$F\Big(\int_0^{d(Az,z)}\psi(t)dt, \int_0^{d(Az,z)}\psi(t)dt, 0, 0, \int_0^{d(z,Az)}\psi(t)dt, \int_0^{d(Az,z)}\psi(t)dt\Big) < 0,$$

which contradicts (F_u). Thus z is a common fixed point of A, B, S and T.

Similar arguments arise if we assume that the range of either of the mappings A, B or T is a complete subspace of X. The uniqueness of z follows easily by using (ii) and then (b). This completes the proof. □

Remark 4.2. *Note that, in the implicit relation, the strict '$<$' sign can be replaced by '$\leq$' just by considering the strict inequality in condition (F_u), that is, $F(u,\ u,\ 0,\ 0,\ u,\ u) > 0,\ \forall u > 0$.*

Remark 4.3. *In Theorem 4.1, if we replace condition a) by the following assumption:*

- *ψ summable on each compact interval, but not summable on $\mathbb{R}_+$, and non-negative,*

then, in order to guarantee (see the first part of the proof of Theorem 4.1) that $\limsup_{n\to\infty} \int_0^{d(Ay_n,Bx_n)} \psi(t)dt$ is finite, we must admit that the sequence Ay_n is bounded. Hence, in this more general case, we must add the following hypothesis:

(v):
- *$\{By_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that $\{Ty_n\}$ is convergent (in case (A,S) satisfies property (E.A)), and*
- *$\{Ay_n\}$ is a bounded sequence for every $\{y_n\} \subseteq X$ such that $\{Sy_n\}$ is convergent (in case (B,T) satisfies property (E.A)).*

Alternatively, we can consider the following condition:

(vi):
- **Case** (A,S) **satisfies (E.A)**: *If $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ are non-negative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and*
$$F(z_n, r_n, r_n, z_n, w_n, 0) \leq 0,\ n \in \mathbb{N},$$
then $\{r_n\} \not\to 0$, as $n \to \infty$.
- **Case** (B,T) **satisfies (E.A):** *If $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ are non-negative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and*
$$F(z_n, r_n, z_n, r_n, 0, w_n) \leq 0,\ n \in \mathbb{N},$$
then $\{r_n\} \not\to 0$, as $n \to \infty$.

For instance, in the proof of Theorem 4.1, assuming that (B,T) satisfies (E.A), we get

$$F\Big(\int_0^{d(Ay_n,Bx_n)} \psi(t)dt,\ \int_0^{d(Bx_n,Tx_n)} \psi(t)dt,\ \int_0^{d(Ay_n,Bx_n)} \psi(t)dt,$$
$$\int_0^{d(Bx_n,Tx_n)} \psi(t)dt,\ 0,\ \int_0^{d(Ay_n,Tx_n)} \psi(t)dt\Big) < 0.$$

If $\{Ay_n\}$ is not bounded, then $\{d(Ay_n, Bx_n)\}$ is not bounded and, thus, there exists a subsequence such that $\{d(Ay_{n_k}, Bx_{n_k})\} \to \infty$. Since ψ is not summable on $\mathbb{R}_+$, then $\int_0^{d(Ay_{n_k},Bx_{n_k})} \psi(t)dt \to \infty$ and $\int_0^{d(Ay_{n_k},Tx_{n_k})} \psi(t)dt \to \infty$, as $k \to \infty$. This joint to the previous inequality and condition (vi) implies that $\int_0^{d(Bx_{n_k},Tx_{n_k})} \psi(t)dt \not\to 0$, which is a contradiction. We proceed similarly in the case where (A,S) satisfies (E.A).

Example 4.4. *For function F in Example 3.1, $F(t_1,...,t_6) = pt_1 - qt_2 + r(t_3 - t_4) + s(-t_5 + t_6)$, where $r + s < p$, $-r - s < p$ and $q \leq p$, condition (vi) is satisfied, adding additional conditions on the constants. Consider either $p > r$ and $s \leq 0$, or $p \geq r$ and $s < 0$. Under these conditions, if $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ are nonnegative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and*

$$F(z_n, r_n, r_n, z_n, w_n, 0) \leq 0,\ n \in \mathbb{N},$$

then

$$pz_n - qr_n + r(r_n - z_n) + s(-w_n) \leq 0,\ n \in \mathbb{N},$$

which yields

$$(p - r)z_n - sw_n \leq (q - r)r_n,\ n \in \mathbb{N}.$$

This inequality is not possible if $q - r \leq 0$ and, for $q - r > 0$, we obtain $\{r_n\} \to \infty$, as $n \to \infty$. On the other hand, consider that either $p + r > 0$ and $s \geq 0$, or $p + r \geq 0$ and $s > 0$. If $F(z_n, r_n, z_n, r_n, 0, w_n) \leq 0$, $n \in \mathbb{N}$, then

$$pz_n - qr_n + r(z_n - r_n) + sw_n \leq 0,\ n \in \mathbb{N},$$

which implies

$$(p + r)z_n + sw_n \leq (q + r)r_n,\ n \in \mathbb{N}.$$

This inequality is not possible if $q + r = 0$ and, for $q + r > 0$, we obtain $\{r_n\} \to \infty$, as $n \to \infty$. Note that we must impose different conditions to the constants, depending on the pair which satisfies property (E.A), to deduce the validity of condition (vi).

Example 4.5. *For function F in Example 3.2,*

$$F(t_1,...,t_6) = pt_1 + max\{-qt_2,\ (t_3 - t_4)/2,\ -s(t_5 - t_6)/2\},$$

where $0 \leq s, q$, and $0 < p$, (vi) is valid. Consider $\{z_n\}$, $\{r_n\}$ and $\{w_n\}$ nonnegative sequences such that $\{z_n\} \to \infty$, $\{w_n\} \to \infty$, as $n \to \infty$ and

$$F(z_n, r_n, r_n, z_n, w_n, 0) = pz_n + max\{-qr_n,\ (r_n - z_n)/2,\ -sw_n/2\} \leq 0,\ n \in \mathbb{N},$$

then $pz_n \leq min\{qr_n,\ (z_n - r_n)/2,\ sw_n/2\}$ and $pz_n \leq qr_n$, $n \in \mathbb{N}$. If $q = 0$, this inequality is not valid and, if $q > 0$, $\{r_n\} \to \infty$, as $n \to \infty$. On the other hand, if

$$F(z_n, r_n, z_n, r_n, 0, w_n) = pz_n + max\{-qr_n,\ (z_n - r_n)/2,\ sw_n/2\} \leq 0,\ n \in \mathbb{N},$$

then

$$pz_n \leq min\{qr_n,\ (r_n - z_n)/2,\ -sw_n/2\},\ n \in \mathbb{N},$$

and, similarly, $\{r_n\} \to \infty$, as $n \to \infty$. Hence (vi) holds.

Taking into account Remarks 4.2 and 4.3, if we put $\psi(t) = 1$ in condition (ii) we get the following Corollary.

Corollary 4.6. *Let A, B, S and T be four self-mappings of a metric space (X, d) such that (i), (iii), (iv) and one of the conditions (v) or (vi) hold. Further,*
(ii)^o there exists a continuous function $F \in \mathcal{F}$ satisfying $(F_1), (F_2)$ and (F_u) such that for all $x, y \in X$, the contractive condition:

$$F\big(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty)\big) \le 0,$$

holds. If the range of one of the mappings is a complete subspace of X, then A, B, S and T have a unique common fixed point.

Remark 4.7. *Since property (E.A) and weak compatibility are independent to each other, we can not remove condition (iii) or (iv) from Theorem 4.1.*

Remark 4.8. *If we take $S = T = id_X$ (the identity map on X) in Corollary 4.6, we get the implicit relation*

$$F\big(d(Ax, By), d(x, y), d(Ax, x), d(By, y), d(By, x), d(Ax, y)\big) \le 0,$$

for x, $y \in X$. Choosing

$$F(t_1, t_2, ..., t_6) = G(t_1) - \phi\left(G\left(max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}\right)\right),$$

where G and ϕ are continuous, then the implicit relation can be written as

$$\begin{aligned} &G(d(Ax, By)) \le \\ &\le \phi\left(G\left(max\left\{d(x, y), d(Ax, x), d(By, y), \frac{1}{2}\left(d(By, x) + d(Ax, y)\right)\right\}\right)\right), \end{aligned}$$

for x, $y \in X$, which is similar to the condition in Theorem 1 [9]*. Note that conditions (F_1), (F_2) and (F_u) hold for this choice of F if $G(t) > 0$, for $t > 0$ and $\phi(t) < t$, for $t > 0$.*

Remark 4.9. *Taking G the identity map in Remark 4.8, then $F(t_1, t_2, ..., t_6) = t_1 - \phi\left(max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}\right)$, with ϕ continuous, and we obtain the implicit relation*

$$\begin{aligned} &\int_0^{d(Ax,By)} \psi(t)dt \\ &\qquad \le \phi\left(\int_0^{max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),\frac{1}{2}(d(By,Sx)+d(Ax,Ty))\}} \psi(t)dt\right), \end{aligned}$$

for x, $y \in X$. Taking $S = T = id_X$ in this inequality, we get the implicit relation in Corollary 1 [9]*:*

$$\int_0^{d(Ax,By)} \psi(t)dt \le \phi\left(\int_0^{max\{d(x,y),d(Ax,x),d(By,y),\frac{1}{2}(d(By,x)+d(Ax,y))\}} \psi(t)dt\right).$$

Remark 4.10. *Taking* $F(t_1, t_2, ..., t_6) = t_1 - \phi\,(max\,\{t_2, t_4, t_5\})$ *, for* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *continuous, we obtain the implicit relation*

$$\int_0^{d(Ax,By)} \psi(t)dt \leq \phi\left(\int_0^{max\{d(Sx,Ty),d(By,Ty),d(By,Sx)\}} \psi(t)dt\right),$$

for $x, y \in X$*, which coincides with Condition (1) in Theorem 1* [2].

Taking $\psi(t) = 1$ *in this inequality, we get Condition (1) in Theorem 2* [1]*:*

$$d(Ax, By) \leq \phi\,(max\{d(Sx, Ty), d(By, Ty), d(By, Sx)\}),$$

for $x, y \in X$*. Note that* (F_1)*,* (F_2) *and* (F_u) *hold for* F *if* $\phi(0) = 0$*, and* $\phi(t) < t$*, for* $t > 0$*. Moreover, under these assumptions, condition (vi) of Theorem 4.1 holds (case* (B, T) *satisfies (E.A)).*

Taking into account that continuity of F can be weakened in Theorem 4.1, we can obtain results which extend the above mentioned Theorems.

References

[1] M. Aamri and D. El. Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270** (2002), 181–188.

[2] A. Aliouche, *A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type*, J. Math. Anal. Appl., **322** (2006), 796–802.

[3] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **9** (1986), 771–779.

[4] G. Jungck and B. E. Rhoades, *Fixed points for set-valued functions without continuity*, Indian J. Pure Appl. Math., **29** (3)(1998), 227–238.

[5] R. P. Pant, *Common fixed points of sequences of mappings*, Ganita **47** (1996), 43–49.

[6] R. P. Pant, *Common fixed points of contractive maps*, J. Math. Anal. Appl., **226** (1998), 251–258.

[7] V. Popa, *A general common fixed point theorem of Meir and Keeler type for noncontinuous weak compatible mappings*, Filomat(Niš) **18** (2004), 33–40.

[8] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math., (Beograd) **32** (1982), 149–153.

[9] X. Zhang, *Common fixed point theorems for some new generalized contractive type mappings*, J. Math. Anal. Appl., **333** (2007), 780–786.