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SOLUTION SENSITIVITY FOR A SYSTEM OF GENERALIZED NONLINEAR EQUATIONS IN BANACH SPACES

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Abstract. In this work, we study the behaviour and sensitivity analysis of solution set for a system of generalized nonlinear equations with parametricaly (A, η, m)-accretive mapping in q-uniformly smooth Banach spaces.

1. Introduction

Nonlinear variational inequalities and variational inclusions are providing mathematical models to some problems arising in optimization and controls, economics and engineering sciences [3, 4, 15]. Sensitivity analysis for solutions of variational inequalities with single-valued mappings have been studied by many authors (see [7, 16]).

By using the resolvent operator techniques, Agarwal et al. [1], Jeong [13] studied a new system of parametric generalized nonlinear mixed quasi variational inclusion in Hilbert spaces and in $L_p(p \geq 2)$ spaces, respectively. In 2008, using the concepts and techniques of resolvent operator, Lan [18] studied the behaviour and sensitivity analysis of solution set for a new system of

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generalized parametric variational inclusions with (A,η)-accretive mappings in Banach spaces.

Recently Kim et al. [17] considered the methods of parametric (A,η,m)-proximal operator to studied the behaviour and sensitivity analysis of the solution set for a system of equations in Hilbert spaces.

In this work, we study the behaviour and sensitivity analysis of solution set for a system of generalized nonlinear equations in q-uniformly smooth Banach spaces. The present results improve and extend many results in the literatures.

2. Basic Foundation

Let X be a Banach space with dual space X^* and $\langle\cdot,\cdot\rangle$ be the dual pairing between X and $X^*, CB(X)$ denotes the family of all nonempty closed bounded subsets of X and 2^X denote the family of all nonempty subset of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\right\} \forall x \in X,$$

where $q > 1$ is a constant. In particular $J = J_2$ is called the normalized duality mapping. It is well known that J_q is single-valued if X^* is strictly convex and that

$$J_q(x) = \|x\|^{q-2}J_2(x),\ \forall x \neq 0.$$

If $X = H$ is a Hilbert space, then J_2 becomes the identity mapping of H. The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. It is know that

$$L_p(l_p) \text{ or } W_m^p = \begin{cases} p - \text{uniformly smooth} & \text{if } 1 < p < 0, \\ 2 - \text{uniformly smooth} & \text{if } p \geq 2. \end{cases}$$

A Banach space X is said to be uniformly convex if given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$,

$$\|\frac{1}{2}(x+y)\| \leq 1-\delta.$$

It is well known that L_p, l_p and Sobolev spaces $W_m^p(1 < p < \infty)$ are uniformly convex.

Lemma 2.1. *Let r and s be two non negative real numbers. Then*

$$(r+s)^q \leq 2^q(r^q+s^q).$$

Proof.

$$(r+s)^q \leq (2\max\{r,s\})^q = 2^q(\max\{r,s\})^q \leq 2^q(r^q+s^q).$$

□

Lemma 2.2. ([26]) *A space X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

In this paper, we consider the following system of (A,η,m)-proximal operator equation systems. For each fixed $(\omega,\lambda) \in \Omega \times \wedge$ finding $(z,t),(x,y) \in X_1 \times X_2$ such that $u \in S(x,\omega), v \in T(y,\lambda)$ and

$$\begin{cases} p(x,\omega) + \rho^{-1}R^{M(\cdot,x,\omega)}_{\rho,A_1}(z) = E(x,v,\omega), \\ h(y,\lambda) + \varrho^{-1}R^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t) = F(u,y,\lambda), \end{cases} \tag{2.1}$$

where Ω and $\wedge$ are two nonempty open subsets of q-uniformly smooth Banach spaces in which the parameter ω and λ takes values, respectively.

$S : X_1 \times \Omega \to 2^{X_1}$ and $T : X_2 \times \wedge \to 2^{X_2}$ are set-valued mappings, $E : X_1 \times X_2 \times \Omega \to X_1, F : X_2 \times X_1 \times \wedge \to X_2, f : X_1 \times \Omega \to X_1, g : X_2 \times \wedge \to X_2, \eta_1 : X_1 \times X_1 \times \Omega \to X_1, \eta_2 : X_2 \times X_2 \times \wedge \to X_2, p : X_1 \times \Omega \to X_1$ and $h : X_2 \times \wedge \to X_2$ are nonlinear single-valued mappings, $A_1 : X_1 \to X_1, A_2 : X_2 \to X_2$ are mappings, $M : X_1 \times X_1 \times \Omega \to 2^{X_1}$ is an (A_1,η_1,m_1)-accretive mapping with $f(X_1,\omega) \cap domM(\cdot,z,\omega) \neq \emptyset$ and for all $(t,\lambda) \in X_2 \times \wedge, N : X_2 \times X_2 \times \wedge \to 2^{X_2}$ is an (A_2,η_2,m_2)-accretive mapping with $g(X_2,\lambda) \cap domN(\cdot,t,\lambda) \neq \emptyset$, respectively.

$R^{M(\cdot,x,\omega)}_{\rho,A_1} = I - A_1\left(J^{M(\cdot,x,\omega)}_{\rho,A_1}\right)$ and $R^{N(\cdot,y,\lambda)}_{\varrho,A_2} = I - A_2\left(J^{N(\cdot,y,\lambda)}_{\varrho,A_2}\right)$,
where I is an identity mapping.

$A_1\left(J^{M(\cdot,x,\omega)}_{\rho,A_1}(z)\right) = A_1\left(J^{M(\cdot,x,\omega)}_{\rho,A_1}\right)(z), A_2\left(J^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t)\right) = A_2\left(J^{N(\cdot,y,\lambda)}_{\varrho,A_2}\right)(t)$ and $R^{M(\cdot,x,\omega)}_{\rho,A_1} = (A_1 + \rho M(\cdot,x,\omega))^{-1}$, $R^{N(\cdot,y,\lambda)}_{\varrho,A_2} = (A_2 + \varrho N(\cdot,y,\lambda))^{-1}$ for all $x,z \in X_1, y,t \in X_2, u \in S(x,\omega), v \in T(y,\lambda)$ and $(\omega,\lambda) \in \Omega \times \wedge$.

For appropriate and suitable choice of $E,F,M,N,S,T,f,g,p,h,A_i,\eta_i$ and X_i for $i = 1,2$, one see that problem (2.1) is a generalized version of some problems which includes a number (systems) of (parametric) quasi variational inclusions, (parametric) generalized quasi variational inclusions studied by many authors as special cases (see [2, 5, 8, 9, 10, 12, 19, 23, 24, 25]).

3. Preliminaries

In the sequel, let $\wedge$ be a nonempty open subset of q-uniformly smooth Banach space X in which the parameter λ takes values.

Definition 3.1. Let $A : X \times \wedge \to X, \eta : X \times X \times \wedge \to X$ be single-valued mappings. The mapping A is said to be

(i) parametrically accretive if

$$\langle A(x,\lambda) - A(y,\lambda), j_q(x-y)\rangle \geq 0,\ \forall x, y \in X, \lambda \in \wedge;$$

(ii) parametrically strictly accretive if

$$\langle A(x,\lambda) - A(y,\lambda), j_q(x-y)\rangle = 0,\ x \neq y, \forall x, y \in X, \lambda \in \wedge;$$

(iii) parametrically γ-strongly accretive if

$$\langle A(x,\lambda) - A(y,\lambda), j_q(x-y)\rangle \geq \gamma\|x-y\|^q,\ \forall x, y \in X, \lambda \in \wedge;$$

(iv) parametrically r-strongly η-accretive if

$$\langle A(x,\lambda) - A(y,\lambda), j_q(\eta(x,y))\rangle \geq r\|x-y\|^q,\ \forall x, y \in X, \lambda \in \wedge.$$

Definition 3.2. A single-valued mapping $\eta : X \times X \times \wedge \to X$ is said to be parametrically τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y,\lambda)\| \leq \|x-y\|,\ \forall x, y \in X, \lambda \in \wedge.$$

Definition 3.3. Let $A : X \times \wedge \to X, \eta : X \times X \times \wedge \to X$ be two single-valued mappings. The set-valued mapping $M : X \times X \times \wedge \to 2^X$ is said to be

(i) parametrically m-relaxed η-accretive if there exists a constant $m > 0$ such that

$$\langle u - v, j_q(\eta(x,y,\lambda))\rangle \geq -m\|x-y\|^q,$$

for all $x, y \in X, u \in M(\cdot, x, \lambda), v \in M(\cdot, y, \lambda)$;

(ii) parametrically (A, η, m)-accretive if
 (1) M is parametrically m-relaxed η-accretive mapping;
 (2) $(A + \rho M)(X) = X$ for every $\rho > 0$.

Definition 3.4. A mapping $T : X \times X \times \wedge \to X$ is said to be

(i) parametrically m-relaxed accretive in the first argument if there exists a constant $m > 0$ such that

$$\langle T(x,u,\lambda) - T(y,u,\lambda), j_q(x-y)\rangle \geq -m\|x-y\|^q,$$

for all $(x, y, u, \lambda) \in X \times X \times X \times \wedge$;

(ii) parametrically s-cocoercive in the first argument if there exists a constant $s > 0$ such that

$$\langle T(x,u,\lambda) - T(y,u,\lambda), j_q(x-y)\rangle \geq s\|T(x,u,\lambda) - T(y,u,\lambda)\|^q,$$

for all $(x,y,u,\lambda) \in X \times X \times X \times \wedge$;

(iii) parametrically γ-relaxed cocoercive with respect to $A : X \times \wedge \to X$ in the first argument of T if there exists a constant $\gamma > 0$ such that

$$\langle T(x,u,\lambda) - T(y,u,\lambda), j_q(A(x) - A(y))\rangle \geq -\gamma\|T(x,u,\lambda) - T(y,u,\lambda)\|^q,$$

for all $(x,y,u,\lambda) \in X \times X \times X \times \wedge$;

(iv) parametrically (γ,α)-relaxed cocoercive with respect to $A : X \times \wedge \to X$ in the first argument of T if there exists constants $\epsilon > 0$ and $\alpha > 0$ such that

$$\begin{aligned}\langle T(x,u,\lambda) - T(y,u,\lambda), j_q(A(x) - A(y))\rangle \geq -\gamma\|T(x,u,\lambda) - T(y,u,\lambda)\|^q \\ + \alpha\|A(x) - A(y)\|^q,\end{aligned}$$

for all $(x,y,u,\lambda) \in X \times X \times X \times \wedge$;

(v) parametrically μ-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\|T(x,u,\lambda) - T(y,u,\lambda)\| \leq \mu\|x-y\|,$$

for all $(x,y,u,\lambda) \in X \times X \times X \times \wedge$.

Remark 3.5. When $X = H$ is a real Hilbert space, then the Definition 3.1 reduces to the definition of parametrically monotonicity, parametrically strict monotonicity and parametrically strong monotonicity with respect to A, respectively (see [6, 14]).

Example 3.6. Let $T : X \times \wedge \to X$ be a parametrically nonexpansive mapping. If we set $F = I - T$ where I is an identity mapping, then F is parametrically $\frac{1}{2}$-cocoercive.

Proof. For any two elements $x, y \in X, \lambda \in \wedge$, we have

$$\begin{aligned}&\|F(x,\lambda) - F(y,\lambda)\|^2 \\ &= \|(I-T)(x,\lambda) - (I-T)(y,\lambda)\|^2 \\ &= \langle (I-T)(x,\lambda) - (I-T)(y,\lambda), (I-T)(x,\lambda) - (I-T)(y,\lambda)\rangle \\ &\leq 2[\|x-y\|^2 - \langle x-y, T(x,\lambda) - T(y,\lambda)\rangle] \\ &= 2\langle x-y, F(x,\lambda) - F(y,\lambda)\rangle.\end{aligned}$$

Hence F is parametrically $\frac{1}{2}$-cocoercive. □

Example 3.7. Let C be a nonempty closed convex subset of X and a projection mapping $P : X \times \wedge \to C$ be a parametrically nonexpansive. Then P is parametrically 1-cocoercive.

Proof. For any $x, y \in X, \lambda \in \wedge$, we have

$$\begin{aligned} \|P(x,\lambda) - P(y,\lambda)\|^2 &= \langle P(x,\lambda) - P(y,\lambda), P(x,\lambda) - P(y,\lambda)\rangle \\ &\leq \langle x - y, P(x,\lambda) - P(y,\lambda)\rangle. \end{aligned}$$

Thus P is parametrically 1-cocoercive. □

Example 3.8. A parametricaly r-strongly monotone (and hence parametrically r-expanding) mapping $T : X \times \wedge \to X$ is parametrically $(r+r^2, 1)$-relaxed cocoercive with respect to I.

Proof. For any two elements $x, y \in X, \lambda \in X$, we have

$$\|T(x,\lambda) - T(y,\lambda)\|^2 \geq r\|x - y\|,$$

$$\langle T(x,\lambda) - T(y,\lambda), x - y\rangle \geq r\|x - y\|^2$$

and so

$$\|T(x,\lambda) - T(y,\lambda)\|^2 + \langle T(x,\lambda) - T(y,\lambda), x - y\rangle \geq (r + r^2)\|x - y\|^2,$$

for all $x, y \in X, \lambda \in \wedge$. Hence, we have

$$\langle T(x,\lambda) - T(y,\lambda), x - y\rangle \geq (-1)\|T(x,\lambda) - T(y,\lambda)\|^2 + (r + r^2)\|x - y\|^2,$$

for all $x, y \in X, \lambda \in \wedge$. Therefore T is parametrically $(r + r^2, 1)$-relaxed cocoercive with respect to I. □

Remark 3.9. Clearly every parametrically m-cocoercive mapping is parametrically m-relaxed cocoercive while each parametrically r-strongly monotone mapping is parametrically $(r + r^2, 1)$-relaxed cocoercive with respect to I.

Definition 3.10. A mapping $p : X \times \wedge \to X$ is said to be

(i) parametrically δ-strongly accretive with respect to the first argument if there exists a constant $\delta \in (0, 1)$ such that

$$\langle p(x,\lambda) - p(y,\lambda), j_q(x - y)\rangle \geq \delta\|x - y\|^q, \forall x, y \in X, \lambda \in \wedge;$$

(ii) parametrically σ-Lipschitz continuous with respect to the first argument if there exists a constant $\sigma > 0$ such that

$$\|p(x,\lambda) - p(y,\lambda)\| \geq \sigma\|x - y\|^q,\ \forall x, y \in X, \lambda \in \wedge.$$

Definition 3.11. Let $F : X \times \wedge \to 2^X$ be a multi-valued mapping. Then F is said to be parametricaly τ-$\widetilde{\mathcal{H}}$-Lipschitz continuous in the first argument if there exists a constant $\tau > 0$ such that

$$\widetilde{\mathcal{H}}(F(x,\lambda), F(y,\lambda)) \leq \tau\|x - y\|,\ \forall x, y \in X, \lambda \in \wedge,$$

where $\widetilde{\mathcal{H}} : 2^X \times 2^X \to (-\infty, +\infty) \cup \{+\infty\}$ is the Hausdorff metric *i.e.,*

$$\widetilde{\mathcal{H}}(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{x\in B}\inf_{y\in A}\|x-y\|\right\}, \forall A, B \in 2^X.$$

Lemma 3.12. ([21]) *Let (X, d) be a complete metric space and $T_1, T_2 : X \to CB(X)$ be two set-valued contractive mappings with same contractive constant $t \in (0, 1)$ i.e.,*

$$\widetilde{\mathcal{H}}(T_i(x), T_i(y)) \le td(x, y),\ \forall x, y \in X, i = 1, 2.$$

Then

$$\widetilde{\mathcal{H}}(F(T_i), F(T_i)) \le \frac{1}{1-t}\sup_{x\in X}\widetilde{\mathcal{H}}(T_1(x), T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are fixed point sets of T_1 and T_2, respectively.

Lemma 3.13. *Let $\eta : X \times X \times \wedge \to X$ be a single-valued mapping. Let $A : X \times \wedge \to X$ be a parametrically r-strictly η-accretive mapping and $M : X \times \wedge \to 2^X$ be a parametrically (A, η)-accretive mapping. Then for a constant $\rho > 0$, the parametric resolvent operator associated with A and M is defined by*

$$R^{M,\eta}_{\rho,A}(x) = (A + \rho M)^{-1}(x),\ \forall x \in X.$$

Note that $R^{M,\eta}_{\rho,A} = (A+\rho M)^{-1}$ is a single-valued mapping. We remark that M is a parametrically (A, η, m)-accretive mapping with respect to the first argument for any fixed $(z, \lambda) \in X \times \wedge$, we define

$$R^{M(\cdot,z,\lambda),\eta}_{\rho,A}(x) = (A + \rho M(\cdot, z, \lambda))^{-1}(x),\ \forall x \in D(M),$$

which is called a parametric resolvent operator associated with A and $M(\cdot, z, \lambda)$.

Lemma 3.14. *Let X be a q-uniformly smooth Banach space, $\eta : X \times X \times \wedge \to X$ be a single-valued parametrically τ-Lipschitz continuous mapping, $A : X \times \wedge \to X$ be a parametrically r-strongly η-accretive mapping and $M : X \times X \times \wedge \to 2^X$ be a parametrically (A, η, m)-accretive mapping. Then the parametric resolvent operator $R^{M,\eta}_{\rho,A} : X \to X$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, i.e.,*

$$\|R^{M(\cdot,z,\lambda),\eta}_{\rho,A}(x) - R^{M(\cdot,z,\lambda),\eta}_{\rho,A}(y)\| \le \frac{\tau^{q-1}}{r-\rho m}\|x-y\|, \forall x, y \in X, \lambda \in \wedge.$$

In connection with the parametric (A, η, m)-proximal operator equation systems (2.1), we consider the following generalized parametric variational inclusion systems: for each fixed $(\omega, \lambda) \in \Omega \times \wedge$ finding $(x, u) \in X_1 \times X_2, u \in S(x, \omega), v \in T(y, \lambda)$ and

$$\begin{cases} 0 \in p(x, \omega) - E(x, v, \omega) + M(x, x, \omega), \\ 0 \in h(y, \lambda) - F(u, y, \lambda) + N(y, y, \lambda). \end{cases} \tag{3.1}$$

Now, for each fixed $(\omega,\lambda)\in\Omega\times\wedge$, the solution set $Q(\omega,\lambda)$ of problem (2.1) is denoted by

$$Q(\omega,\lambda)=\Big\{(z,t,x,y)\in X_1\times X_2\times X_1\times X_2:\exists u\in S(x,\omega),v\in T(y,\lambda)\text{ such that } p(x,\omega)+\rho^{-1}R^{M(\cdot,x,\omega)}_{\rho,A_1}(z)=E(x,v,\omega)\text{ and } h(y,\lambda)+\varrho^{-1}R^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t)=F(u,y,\lambda)\Big\}.$$

In this works, our aim is to study the behaviour of the solution set $Q(\omega,\lambda)$ and the conditions on these operators $T,S,F,E,M,N,p,h,\eta_1,\eta_2,A_1,A_2$ under which the function $Q(\omega,\lambda)$ is continuous or Lipschitz continuous with respect to the parameter $(\omega,\lambda)\in\Omega\times\wedge$.

4. Sensitivity Analysis for Solution sets

In the sequel, we first transfer the problem (3.1) into a problem of finding parametric fixed point of the associated parametric (A,η,m)-resolvent operator.

Lemma 4.1. *For each fixed $(\omega,\lambda)\in\Omega\times\wedge$, an elements $(x,y)\in Q(\omega,\lambda)$ is a solution of problem (3.1) if and only if there are $(x,y)\in X_1\times X_2, u\in S(x,\omega), v\in T(y,\lambda)$ such that*

$$\begin{cases} x=R^{M(\cdot,x,\omega)}_{\rho,A_1}[A_1(x)-\rho(p(x,\omega)-E(x,v,\omega))],\\ y=R^{N(\cdot,y,\lambda)}_{\varrho,A_2}[A_2(y)-\varrho(h(y,\lambda)-F(u,y,\lambda))],\end{cases}\tag{4.1}$$

where $R^{M(\cdot,x,\omega)}_{\rho,A_1}=(A_1+\rho M(\cdot,x,\omega))^{-1}$ and $R^{N(\cdot,y,\lambda)}_{\varrho,A_2}=(A_2+\varrho N(\cdot,y,\lambda))^{-1}$ are the corresponding parametric resolvent operator in the first argument of parametrically (A_1,η_1)-accretive operator of $M(\cdot,\cdot,\cdot)$, parametric (A_2,η_2)-accretive operator of $N(\cdot,\cdot,\cdot)$, respectively, A_i is a parametrically r_i-strongly accretive mapping for $i=1,2$ and $\rho,\varrho>0$.

Proof. For each fixed $(\omega,\lambda)\in\Omega\times\wedge$, from the definition of the parametric resolvent operator $R^{M(\cdot,x,\omega)}_{\rho,A_1}=(A_1+\rho M(\cdot,x,\omega))^{-1}$ of $M(\cdot,x,\omega)$ and $R^{N(\cdot,y,\lambda)}_{\varrho,A_2}=(A_2+\varrho N(\cdot,y,\lambda))^{-1}$ of $N(\cdot,y,\lambda)$, respectively, we know that there exists $x\in X_1, y\in X_2, u\in S(x,\omega), v\in T(y,\lambda)$ such that (3.1) holds if and only if

$$\begin{cases} A_1(x)-\rho(p(x,\omega)-E(x,v,\omega))\in A_1(x)+\rho M(x,x,\omega),\\ A_2(y)-\varrho(h(y,\lambda)-F(u,y,\lambda))\in A_2(y)+\varrho N(y,y,\lambda).\end{cases}$$

It follows from the definition of $Q(\omega,\lambda)$ that $(x,y)\in Q(\omega,\lambda)$ is a solution of problem (3.1) if and only if there exists $(x,y)\in X_1\times X_2, u\in S(x,\omega), v\in T(y,\lambda)$ such that equations (4.1) holds. □

Now we prove that problem (2.1) is equivalent to problem (3.1).

Lemma 4.2. *Problem* (2.1) *has a solution* (z,t,x,y,u,v) *with* $u \in S(x,\omega)$, $v \in T(y,\lambda)$ *if and only if problem* (3.1) *has a solution* (x,y,u,v) *with* $u \in S(x,\omega), v \in T(y,\lambda)$*, where*

$$x = R^{M(\cdot,x,\omega)}_{\rho,A_1}(z),\ y = R^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t) \tag{4.2}$$

and

$$\begin{aligned} z &= A_1(x) - \rho(p(x,\omega) - E(x,v,\omega)),\\ t &= A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda)). \end{aligned}$$

Proof. Let (x,y,u,v) with $u \in S(x,\omega), v \in T(y,\lambda)$ be a solution of problem (3.1). Then from Lemma 4.1. it is a solution of the following system of equations:

$$\begin{aligned} x &= R^{M(\cdot,x,\omega)}_{\rho,A_1}[A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))],\\ y &= R^{N(\cdot,y,\lambda)}_{\varrho,A_2}[A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda))]. \end{aligned}$$

By using the fact $R^{M(\cdot,x,\omega)}_{\rho,A_1} = I - A_1(J^{M(\cdot,x,\omega)}_{\rho,A_1}), R^{N(\cdot,y,\lambda)}_{\varrho,A_2} = I - A_2(J^{N(\cdot,y,\lambda)}_{\varrho,A_2})$ and (4.1), we have

$$\begin{aligned} &R^{M(\cdot,x,\omega)}_{\rho,A_1}[A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))]\\ &= [A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))]\\ &\quad - A_1(J^{M(\cdot,x,\omega)}_{\rho,A_1}[A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))])\\ &= A_1(x) - \rho(p(x,\omega) - E(x,v,\omega)) - A_1(x)\\ &= -\rho(p(x,\omega) - E(x,v,\omega)) \end{aligned}$$

and

$$\begin{aligned} &R^{N(\cdot,y,\lambda)}_{\varrho,A_2}[A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda))]\\ &= A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda))\\ &\quad - A_2(J^{N(\cdot,y,\lambda)}_{\varrho,A_2}[A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda))])\\ &= A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda)) - A_2(y)\\ &= -\varrho(h(y,\lambda) - F(u,y,\lambda)) \end{aligned}$$

which imply that

$$\begin{aligned} p(x,\omega) + \rho^{-1}R^{M(\cdot,x,\omega)}_{\rho,A_1}(z) &= E(x,v,\omega),\\ h(y,\lambda) + \varrho^{-1}R^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t) &= F(u,y,\lambda) \end{aligned}$$

with

$$z = A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))$$

and

$$t = A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda)).$$

That is, (z,t,x,y,u,v) with $u\in S(x,\omega), v\in T(y,\lambda)$ is a solution of problem (2.1).

Conversely, letting (z,t,x,y,u,v) with $u\in S(x,\omega), v\in T(y,\lambda)$ is a solution of problem (2.1), then

$$\begin{aligned}
p(x,\omega)-E(x,v,\omega) &= -\rho^{-1}R^{M(\cdot,x,\omega)}_{\rho,A_1}(z),\\
h(y,\lambda)-F(u,y,\lambda) &= -\varrho^{-1}R^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t),\\
\rho[p(x,\omega)-E(x,v,\omega)] &= -R^{M(\cdot,x,\omega)}_{\rho,A_1}(z)=A_1(J^{M(\cdot,x,\omega)}_{\rho,A_1}(z))-z,\\
\varrho[h(y,\lambda)-F(u,y,\lambda)] &= -R^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t)=A_2(J^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t))-t. \qquad (4.3)
\end{aligned}$$

It follows that (4.2) and (4.3) that

$$\begin{aligned}
\rho[p(x,\omega)-E(x,v,\omega)] &= A_1(J^{M(\cdot,x,\omega)}_{\rho,A_1}(A_1(x)-\rho(p(x,\omega)-E(x,v,\omega))))\\
&\quad -A_1(x)+\rho(p(x,\omega)-E(x,v,\omega)),
\end{aligned}$$

$$\begin{aligned}
\varrho[h(y,\lambda)-F(u,y,\lambda)] &= A_2(J^{N(\cdot,y,\lambda)}_{\varrho,A_2}(A_2(y)-\varrho(h(y,\lambda)-F(u,y,\lambda))))\\
&\quad -A_2(y)+\varrho(h(y,\lambda)-F(u,y,\lambda)),
\end{aligned}$$

which imply that

$$\begin{aligned}
A_1(x) &= A_1(J^{M(\cdot,x,\omega)}_{\rho,A_1}(A_1(x)-\rho(p(x,\omega)-E(x,v,\omega)))),\\
A_2(y) &= A_2(J^{N(\cdot,y,\lambda)}_{\varrho,A_2}(A_2(y)-\varrho(h(y,\lambda)-F(u,y,\lambda)))).
\end{aligned}$$

Hence

$$\begin{aligned}
x &= J^{M(\cdot,x,\omega)}_{\rho,A_1}(A_1(x)-\rho(p(x,\omega)-E(x,v,\omega))),\\
y &= J^{N(\cdot,y,\lambda)}_{\varrho,A_2}(A_2(y)-\varrho(h(y,\lambda)-F(u,y,\lambda))),
\end{aligned}$$

that is, (x,y,u,v) with $u\in S(x,\omega), v\in T(y,\lambda)$ is a solution of problem (3.1).

Alternative Proof. Let

$$\begin{aligned}
z &= A_1(x)-\rho(p(x,\omega)-E(x,v,\omega)),\\
t &= A_2(y)-\varrho(h(y,\lambda)-F(u,y,\lambda)).
\end{aligned}$$

Then by (4.2) we have

$$x=J^{M(\cdot,x,\omega)}_{\rho,A_1}(z),\ \ y=J^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t)$$

and

$$\begin{aligned}
z &= A_1(J^{M(\cdot,x,\omega)}_{\rho,A_1}(z))-\rho(p(x,\omega)-E(x,v,\omega)),\\
t &= A_2(J^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t))-\varrho(h(y,\lambda)-F(u,y,\lambda)).
\end{aligned}$$

Since

$$A_1(J^{M(\cdot,x,\omega)}_{\rho,A_1}(z))=A_1(J^{M(\cdot,x,\omega)}_{\rho,A_1})(z)$$

and

$$A_2(J_{\varrho,A_2}^{N(\cdot,y,\lambda)}(t)) = A_2(J_{\varrho,A_2}^{N(\cdot,y,\lambda)})(t),$$

we have

$$p(x,\omega) + \rho^{-1}R_{\rho,A_1}^{M(\cdot,x,\omega)}(z) = E(x,v,\omega),$$

$$h(y,\lambda) + \varrho^{-1}R_{\varrho,A_2}^{N(\cdot,y,\lambda)}(t) = F(u,y,\lambda),$$

which is required problem (2.1). □

From Lemma 4.1 and 4.2, we suggest the following sensitivity analysis results for the system of parametric (A,η,m)-proximal operator equations (2.1).

Theorem 4.3. *Let $A_i : X_i \to X_i$ be a parametrically r_i-strongly accretive and parametrically s_i-Lipschitz continuous mapping for each $i = 1,2$, $S : X_1 \times \Omega \to CB(X_1)$ be a parametrically $\kappa_S - \widetilde{\mathcal{H}}$-Lipschitz continuous mapping and $T : X_2 \times \wedge \to CB(X_2)$ be a parametrically $\kappa_T - \widetilde{\mathcal{H}}$-Lipschitz continuous mapping. Let $M : X_1 \times X_1 \times \Omega \to 2^{X_1}$ be parametrically (A_1,η_1)-accretive with constant m_1 in the first argument and $N : X_2 \times X_2 \times \wedge \to 2^{X_2}$ be parametrically (A_2,η_2)-accretive with constant m_2 in the first argument. Let $\eta_1 : X_1 \times X_1 \times \Omega \to X_1$ be a parametrically τ_2-Lipschitz continuous mapping, $\eta_2 : X_2 \times X_2 \times \wedge \to X_2$ be a parametrically τ_2-Lipschitz continuous mapping, $E : X_1 \times X_2 \times \Omega \to X_1$ be a parametrically Lipschitz continuous mapping with respect to first argument with constant $\beta_E > 0$, second argument with respect to the constant $\xi_E > 0$ and parametrically (γ_E,α_E)-relaxed cocoercive with respect to A_1 and first argument of E with constants $\gamma_E > 0, \alpha_E > 0$. Let $p : X_1 \times \Omega \to X_1$ be a parametrically Lipschitz continuous mapping with constant $\delta_p > 0$ and parametrically (γ_p,α_p)-relaxed cocoercive with respect to A_1 with constants $\gamma_p, \alpha_p \geq 0$. Let $F : X_1 \times X_2 \times \wedge \to X_2$ be parametrically Lipschitz continuous with respect to first and second argument with constants β_F, ξ_F, respectively. Let $h : X_2 \times \wedge \to X_2$ be parametrically Lipschitz continuous with constant $\delta_h > 0$ and parametrically (γ_h,α_h)-relaxed cocoercive with respect to A_2 with constants $\gamma_h > 0, \alpha_h > 0$, respectively. Let F be a parametrically (γ_F,α_F)-relaxed cocoercive mapping with respect to A_2 and second argument of F with constants $\gamma_F, \alpha_F > 0$, respectively. If*

$$\|J_{\rho,A_1}^{M(\cdot,x,\omega)}(z) - J_{\rho,A_1}^{M(\cdot,y,\omega)}(z)\| \leq \upsilon_1\|x-y\| \forall (x,y,z,\omega) \in X_1 \times X_1 \times X_1 \times \Omega; \quad (4.4)$$

$$\|J_{\varrho,A_2}^{N(\cdot,x,\lambda)}(z) - J_{\varrho,A_2}^{N(\cdot,y,\lambda)}(z)\| \leq \upsilon_2\|x-y\| \forall (x,y,z,\lambda) \in X_2 \times X_2 \times X_2 \times \wedge; \quad (4.5)$$

with $0 < \upsilon_i < 1$ for $i = 1,2$ and there exist constants $\rho \in \left(0, \frac{r_1}{m_1}\right)$, $\varrho \in \left(0, \frac{r_2}{m_2}\right)$ such that

$$\sqrt[q]{s_1^2 - q\rho(s_1^q(\alpha_p - \alpha_E) - \gamma_p\delta_p^q + \gamma_E\beta_E^q) + 2^q c_q\rho^q(\delta_p^q + \beta_E^q)}$$
$$< \tau_1^{1-q}(r_1 - \rho m_1)\left(1 - \upsilon_1 - \frac{\tau_2^{q-1}\varrho\beta_F\kappa_S}{r_2 - \varrho m_2}\right), \tag{4.6}$$
$$\sqrt[q]{s_2^2 - q\varrho(s_2^q(\alpha_h - \alpha_F) - \gamma_h\delta_h^q + \gamma_F\xi_F^q) + 2^q c_q\varrho^q(\delta_h^q + \xi_F^q)}$$
$$< \tau_2^{1-q}(r_2 - \varrho m_2)\left(1 - \upsilon_2 - \frac{\tau_1^{q-1}\rho\xi_E\kappa_T}{r_1 - \rho m_1}\right)$$

Then for each $(\omega, \lambda) \in \Omega \times \wedge$, the solution set $Q(\omega, \lambda)$ of problem (2.1) is a nonempty and closed subset in $X_1 \times X_2$.

Proof. In the sequel from (4.1), we first define the operator $\Phi_\rho : X_1 \times X_2 \times \Omega \times \wedge \to X_1$ and $\Psi_\varrho : X_1 \times X_2 \times \Omega \times \wedge \to X_2$ as follows:

$$\Phi_\rho(x, y, \omega, \lambda) = J_{\rho,A_1}^{M(\cdot,x,\omega)}[A_1(x) - \rho(p(x, \omega) - E(x, v, \omega))],$$
$$\Psi_\varrho(x, y, \omega, \lambda) = J_{\varrho,A_2}^{N(\cdot,y,\lambda)}[A_2(y) - \varrho(h(y, \lambda) - F(u, y, \lambda))] \tag{4.7}$$

for all $(x, y, \omega, \lambda) \in X_1 \times X_2 \times \Omega \times \wedge$. Now we define a norm $\|\cdot\|_1$ on $X_1 \times X_2$ by

$$\|(x, y)\|_1 = \|x\| + \|y\| \ \forall (x, y) \in X_1 \times X_2.$$

It is well known that $(X_1 \times X_2, \|\cdot\|_1)$ is a Banach spaces [11]. For any given $\rho > 0$ and $\varrho > 0$, define $G : X_1 \times X_2 \times \Omega \times \wedge \to 2^{X_1 \times X_2}$ by

$$G_{\rho,\varrho}(x, y, \omega, \lambda) = \{(\Phi_\rho(x, y, \omega, \lambda), \Psi_\varrho(x, y, \omega, \lambda)) : u \in S(x, \omega), v \in T(y, \lambda)\},$$

for all $(x, y, \omega, \lambda) \in X_1 \times X_2 \times \Omega \times \wedge$. Since $S(x, \omega) \in CB(X_1), T(y, \lambda) \in CB(X_2), A_1, A_2, \eta_1, \eta_2, E, F, p, h, J_{\rho,A_1}^{M(\cdot,x,\omega)}, J_{\varrho,A_2}^{N(\cdot,x,\lambda)}$ are continuous, we have

$$G_{\rho,\varrho}(x, y, \omega, \lambda) \in CB(X_1 \times X_2).$$

Now for each fixed $(\omega, \lambda) \in \Omega \times \wedge$, we show that $G_{\rho,\varrho}(x, y, \omega, \lambda)$ is a multi-valued contractive mapping. In fact, for any $(x, y, \omega, \lambda), (\overline{x}, \overline{y}, \omega, \lambda) \in X_1 \times X_2 \times \Omega \times \wedge$ and $(a_1, a_2) \in G_{\rho,\varrho}(x, y, \omega, \lambda)$ there exists $u \in S(x, \omega), v \in T(y, \lambda)$ such that

$$a_1 = J_{\rho,A_1}^{M(\cdot,x,\omega)}[A_1(x) - \rho(p(x, \omega) - E(x, v, \omega))],$$
$$a_2 = J_{\varrho,A_2}^{N(\cdot,y,\lambda)}[A_2(y) - \varrho(h(y, \lambda) - F(u, y, \lambda))].$$

Note that $S(\overline{x}, \omega) \in CB(X_1), T(\overline{y}, \lambda) \in CB(X_2)$. It follows from Nadler's Theorem [22] that there exists $\overline{u} \in S(\overline{x}, \omega) \in CB(X_1), \overline{v} \in T(\overline{y}, \lambda) \in CB(X_2)$ such that

$$\|u - \overline{u}\| \leq \widetilde{\mathcal{H}}(S(x, \omega), S(\overline{x}, \omega)),$$
$$\|v - \overline{v}\| \leq \widetilde{\mathcal{H}}(T(y, \lambda), T(\overline{y}, \lambda)). \tag{4.8}$$

Setting

$$b_1 = J_{\rho,A_1}^{M(\cdot,\overline{x},\omega)}[A_1(\overline{x}) - \rho(p(\overline{x},\omega) - E(\overline{x},\overline{v},\omega))],$$
$$b_2 = J_{\varrho,A_2}^{N(\cdot,\overline{y},\lambda)}[A_2(\overline{y}) - \varrho(h(\overline{y},\lambda) - F(\overline{u},\overline{y},\lambda))],$$

then we have $(b_1, b_2) \in G_{\rho,\varrho}(\overline{x},\overline{y},\omega,\lambda)$. It follows from (4.4) and Lemma 3.14 that

$$\begin{aligned}
\|a_1 - b_1\| &= \|J_{\rho,A_1}^{M(\cdot,x,\omega)}[A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))] \\
&\quad - J_{\rho,A_1}^{M(\cdot,\overline{x},\omega)}[A_1(\overline{x}) - \rho(p(\overline{x},\omega) - E(\overline{x},\overline{v},\omega))]\| \\
&\le \|J_{\rho,A_1}^{M(\cdot,x,\omega)}[A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))] \\
&\quad - J_{\rho,A_1}^{M(\cdot,\overline{x},\omega)}[A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))]\| \\
&\quad + \|J_{\rho,A_1}^{M(\cdot,\overline{x},\omega)}[A_1(x) - \rho(p(x,\omega) - E(x,v,\omega))] \\
&\quad - J_{\rho,A_1}^{M(\cdot,\overline{x},\omega)}[A_1(\overline{x}) - \rho(p(\overline{x},\omega) - E(\overline{x},\overline{v},\omega))]\| \qquad (4.9)\\
&\le \upsilon_1\|x - \overline{x}\| + \frac{\tau_1^{q-1}}{r_1 - \rho m_1}\|A_1(x) - A_1(\overline{x}) - \rho(p(x,\omega) - p(\overline{x},\omega) \\
&\quad - E(x,v,\omega) + E(\overline{x},\overline{v},\omega))\| \\
&\le \upsilon_1\|x - \overline{x}\| + \frac{\tau_1^{q-1}}{r_1 - \rho m_1}\|A_1(x) - A_1(\overline{x}) - \rho(p(x,\omega) - p(\overline{x},\omega) \\
&\quad - E(x,v,\omega) + E(\overline{x},v,\omega))\| + \frac{\tau_1^{q-1}}{r_1 - \rho m_1}\rho\|E(\overline{x},v,\omega) - E(\overline{x},\overline{v},\omega)\|.
\end{aligned}$$

Since E is parametrically Lipschitz continuous with respect to first and second argument with constants β_E, ξ_E, respectively and T is parametrically $\kappa_T - \widetilde{\mathcal{H}}$-Lipschitz continuous and p is a parametrically Lipschitz continuous mapping with constant $\delta_p > 0$ we have

$$\|p(x,\omega) - p(\overline{x},\omega)\| \le \delta_p\|x - \overline{x}\|, \qquad (4.10)$$
$$\|E(x,v,\omega) - E(\overline{x},v,\omega)\| \le \beta_E\|x - \overline{x}\| \qquad (4.11)$$

and

$$\begin{aligned}
\|E(\overline{x},v,\omega) - E(\overline{x},\overline{v},\omega)\| &\le \xi_E\|v - \overline{v}\| \\
&\le \xi_E\widetilde{\mathcal{H}}(T(y,\lambda), T(\overline{y},\lambda))\| \\
&\le \xi_E\kappa_T\|y - \overline{y}\|. \qquad (4.12)
\end{aligned}$$

Again from Lemma 2.1, Lemma 2.2 [26], A_1 is parametrically s_1-Lipschitz continuous and p is a parametricaly (γ_p, α_p)-relaxed cocoercive mapping with respect to A_1 and E is parametricaly (γ_E, α_E)-relaxed cocoercive mapping with respect to A_1 and first argument of E, and from (4.10)-(4.12) we have

$$
\begin{aligned}
&\|A_1(x) - A_1(\overline{x}) - \rho((p(x,\omega) - p(\overline{x},\omega)) - (E(x,v,\omega) - E(\overline{x},v,\omega)))\|^q \\
&\le \|A_1(x) - A_1(\overline{x})\|^q \\
&\quad - q\rho\langle (p(x,\omega) - p(\overline{x},\omega)) - (E(x,v,\omega) - E(\overline{x},v,\omega)), j_q(A_1(x) - A_1(\overline{x}))\rangle \\
&\quad + c_q\rho^q \|(p(x,\omega) - p(\overline{x},\omega)) - (E(x,v,\omega) - E(\overline{x},v,\omega))\|^q \\
&\le \|A_1(x) - A_1(\overline{x})\|^q - q\rho\langle p(x,\omega) - p(\overline{x},\omega), j_q(A_1(x) - A_1(\overline{x}))\rangle \\
&\quad + q\rho\langle E(x,v,\omega) - E(\overline{x},v,\omega), j_q(A_1(x) - A_1(\overline{x}))\rangle \\
&\quad + 2^q c_q\rho^q \left[\|p(x,\omega) - p(\overline{x},\omega)\|^q + \|E(x,v,\omega) - E(\overline{x},v,\omega)\|^q\right] \\
&\le s_1^q\|x - \overline{x}\|^q - q\rho(-\gamma_p\|p(x,\omega) - p(\overline{x},\omega)\|^q + \alpha_p\|A_1(x) - A_1(\overline{x})\|^q) \\
&\quad + q\rho(-\gamma_E\|E(x,v,\omega) - E(\overline{x},v,\omega)\|^q + \alpha_E\|A_1(x) - A_1(\overline{x})\|^q) \\
&\quad + 2^q c_q\rho^q \left[\delta_p^q\|x - \overline{x}\|^q + \beta_E^q\|x - \overline{x}\|^q\right] \\
&\le s_1^q\|x - \overline{x}\|^q - q\rho(-\gamma_p\delta_p^q\|x - \overline{x}\|^q + \alpha_p s_1^q\|x - \overline{x}\|^q) \\
&\quad + q\rho(-\gamma_E\beta_E^q\|x - \overline{x}\|^q + \alpha_E s_1^q\|x - \overline{x}\|^q) + 2^q c_q\rho^q\left[\delta_p^q + \beta_E^q\right]\|x - \overline{x}\|^q \\
&\le \left[s_1^q - q\rho(s_1^q(\alpha_p - \alpha_E) - \gamma_p\delta_p^q + \gamma_E\beta_E^q) + 2^q c_q\rho^q(\delta_p^q + \beta_E^q)\right]\|x - \overline{x}\|^q.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\|A_1(x) - A_1(\overline{x}) - \rho((p(x,\omega) - p(\overline{x},\omega)) - (E(x,v,\omega) - E(\overline{x},v,\omega)))\| \\
&\le \sqrt[q]{s_1^q - q\rho(s_1^q(\alpha_p - \alpha_E) - \gamma_p\delta_p^q + \gamma_E\beta_E^q) + 2^q c_q\rho^q(\delta_p^q + \beta_E^q)}\,\|x - \overline{x}\|. \quad (4.13)
\end{aligned}
$$

Combining (4.9)-(4.13), we have

$$
\begin{aligned}
&\|a_1 - b_1\| \\
&\le \upsilon_1\|x - \overline{x}\| \\
&+ \frac{\tau_1^{q-1}}{r_1 - \rho m_1}\sqrt[q]{s_1^q - q\rho(s_1^q(\alpha_p - \alpha_E) - \gamma_p\delta_p^q + \gamma_E\beta_E^q) + 2^q c_q\rho^q(\delta_p^q + \beta_E^q)}\,\|x - \overline{x}\| \\
&+ \frac{\tau_1^{q-1}}{r_1 - \rho m_1}\rho\xi_E\kappa_T\|y - \overline{y}\| \\
&\le \theta_1\|x - \overline{x}\| + \vartheta_1\|y - \overline{y}\|, \qquad (4.14)
\end{aligned}
$$

where

$$
\theta_1 = \upsilon_1 + \frac{\tau_1^{q-1}}{r_1 - \rho m_1}\sqrt[q]{s_1^q - q\rho(s_1^q(\alpha_p - \alpha_E) - \gamma_p\delta_p^q + \gamma_E\beta_E^q) + 2^q c_q\rho^q(\delta_p^q + \beta_E^q)}
$$

and

$$
\vartheta_1 = \frac{\tau_1^{q-1}}{r_1 - \rho m_1}\rho\xi_E\kappa_T.
$$

Similarly from the assumptions of S, A_2, F, h, Lemma 3.1 and Lemma 3.2 [26], we have

$$\begin{aligned}\|a_2 - b_2\| &= \|J_{\varrho,A_2}^{N(\cdot,y,\lambda)}[A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda))] \\ &\quad - J_{\varrho,A_2}^{N(\cdot,\overline{y},\lambda)}[A_2(\overline{y}) - \varrho(h(\overline{y},\lambda) - F(\overline{u},\overline{y},\lambda))]\| \\ &\le \|J_{\varrho,A_2}^{N(\cdot,y,\lambda)}[A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda))] \\ &\quad - J_{\varrho,A_2}^{N(\cdot,\overline{y},\lambda)}[A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda))]\| \\ &\quad + \|J_{\varrho,A_2}^{N(\cdot,\overline{y},\lambda)}[A_2(y) - \varrho(h(y,\lambda) - F(u,y,\lambda))] \\ &\quad - J_{\varrho,A_2}^{N(\cdot,\overline{y},\lambda)}[A_2(\overline{y}) - \varrho(h(\overline{y},\lambda) - F(\overline{u},\overline{y},\lambda))]\| \\ &\le \upsilon_2\|y-\overline{y}\| + \frac{\tau_2^{q-1}}{r_2 - \varrho m_2}\|A_2(y) - A_2(\overline{y}) - \varrho(h(y,\lambda) - h(\overline{y},\lambda) \\ &\quad - F(u,y,\lambda) + F(\overline{u},\overline{y},\lambda))\| \\ &\le \upsilon_2\|y-\overline{y}\| + \frac{\tau_2^{q-1}}{r_2 - \varrho m_2}\|A_2(y) - A_2(\overline{y}) - \varrho(h(y,\lambda) - h(\overline{y},\lambda) \\ &\quad - F(u,y,\lambda) + F(u,\overline{y},\lambda))\| \\ &\quad + \frac{\tau_2^{q-1}}{r_2 - \varrho m_2}\varrho\|F(u,\overline{y},\lambda) - F(\overline{u},\overline{y},\lambda)\|. \end{aligned} \tag{4.15}$$

Similarly F is parametrically Lipschitz continuous with respect to first and second argument with constants β_F, ξ_F and S is parametrically $\kappa_S - \widetilde{\mathcal{H}}$-Lipschitz continuous and h is a parametrically Lipschitz continuous mapping with constant $\delta_h > 0$, we have

$$\|h(y,\lambda) - h(\overline{y},\lambda)\| \le \delta_h\|y-\overline{y}\|, \tag{4.16}$$

$$\|F(\overline{u},y,\lambda) - F(\overline{u},\overline{y},\lambda)\| \le \xi_F\|y-\overline{y}\| \tag{4.17}$$

and

$$\begin{aligned}\|F(u,y,\lambda) - F(\overline{u},y,\lambda)\| &\le \beta_F\|u-\overline{u}\| \\ &\le \beta_F\widetilde{\mathcal{H}}(S(x,\omega), S(\overline{x},\omega))\| \\ &\le \beta_F\kappa_S\|x-\overline{x}\|. \end{aligned} \tag{4.18}$$

Again from Lemma 2.1, Lemma 2.2 [26], A_2 is parametricaly s_2-Lipschitz continuous and h is a parametrically (γ_h, α_h)-relaxed cocoercive mapping with respect to A_2 and F is a parametrically (γ_F, α_F)-relaxed cocoercive mapping with respect to A_2 and second argument, and from (4.16)-(4.17), we have

$$
\begin{aligned}
&\|A_2(y) - A_2(\overline{y}) - \varrho((h(y,\lambda) - h(\overline{y},\lambda)) - (F(u,y,\lambda) - F(u,\overline{y},\lambda)))\|^q \\
&\le \|A_2(y) - A_2(\overline{y})\|^q \\
&\quad - q\varrho\langle (h(y,\lambda) - h(\overline{y},\lambda)) - (F(u,y,\lambda) - F(u,\overline{y},\lambda)), j_q(A_2(y) - A_2(\overline{y}))\rangle \\
&\quad + c_q\varrho^q \|(h(y,\lambda) - h(\overline{y},\lambda)) - (F(u,y,\lambda) - F(u,\overline{y},\lambda))\|^q \\
&\le \|A_2(y) - A_2(\overline{y})\|^q - q\varrho\langle (h(y,\lambda) - h(\overline{y},\lambda)), j_q(A_2(y) - A_2(\overline{y}))\rangle \\
&\quad + q\varrho\langle F(u,y,\lambda) - F(u,\overline{y},\lambda), j_q(A_2(y) - A_2(\overline{y}))\rangle \\
&\quad + 2^q c_q \varrho^q \left[\|h(y,\lambda) - h(\overline{y},\lambda)\|^q + \|F(u,y,\lambda) - F(u,\overline{y},\lambda)\|^q\right] \\
&\le s_2^q\|y - \overline{y}\|^q - q\varrho(-\gamma_h\|h(y,\lambda) - h(\overline{y},\lambda)\|^q + \alpha_h\|A_2(y) - A_2(\overline{y})\|^q) \\
&\quad + q\varrho(-\gamma_F\|F(u,y,\lambda) - F(u,\overline{y},\lambda)\|^q + \alpha_F\|A_2(y) - A_2(\overline{y})\|^q) \\
&\quad + 2^q c_q\varrho^q \left[\delta_h^q\|y - \overline{y}\|^q + \xi_F^q\|y - \overline{y}\|^q\right] \\
&\le s_2^q\|y - \overline{y}\|^q - q\varrho(-\gamma_h\delta_h^q\|y - \overline{y}\|^q + \alpha_h s_2^q\|y - \overline{y}\|^q) \\
&\quad + q\varrho(-\gamma_F\xi_F^q\|y - \overline{y}\|^q + \alpha_F s_2^q\|y - \overline{y}\|^q) + 2^q c_q\varrho^q\left[\delta_h^q + \xi_F^q\right]\|y - \overline{y}\|^q \\
&\le \left[s_2^q - q\varrho(s_2^q(\alpha_h - \alpha_F) - \gamma_h\delta_h^q + \gamma_F\xi_F^q) + 2^q c_q\varrho^q(\delta_h^q + \xi_F^q)\right]\|y - \overline{y}\|^q.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\|A_2(y) - A_1(\overline{y}) - \varrho((h(y,\lambda) - h(\overline{y},\lambda)) - (F(u,y,\lambda) - F(u,\overline{y},\lambda)))\| \\
&\le \sqrt[q]{s_2^q - q\varrho(s_2^q(\alpha_h - \alpha_F) - \gamma_h\delta_h^q + \gamma_F\xi_F^q) + 2^q c_q\varrho^q(\delta_h^q + \xi_F^q)}\,\|y - \overline{y}\|. \qquad (4.19)
\end{aligned}
$$

Combining (4.15)-(4.19), we have

$$
\begin{aligned}
&\|a_2 - b_2\| \\
&\le \upsilon_2\|y - \overline{y}\| \\
&\quad + \frac{\tau_2^{q-1}}{r_2 - \varrho m_2}\sqrt[q]{s_2^q - q\varrho(s_2^q(\alpha_h - \alpha_F) - \gamma_h\delta_h^q + \gamma_F\xi_F^q) + 2^q c_q\varrho^q(\delta_h^q + \xi_F^q)}\,\|y - \overline{y}\| \\
&\quad + \frac{\tau_2^{q-1}}{r_2 - \varrho m_2}\varrho\beta_F\kappa_S\|x - \overline{x}\| \\
&\le \theta_2\|x - \overline{x}\| + \vartheta_2\|y - \overline{y}\|, \qquad (4.20)
\end{aligned}
$$

where

$$
\vartheta_2 = \upsilon_2 + \frac{\tau_2^{q-1}}{r_2 - \varrho m_2}\sqrt[q]{s_2^q - q\varrho(s_2^q(\alpha_h - \alpha_F) - \gamma_h\delta_h^q + \gamma_F\xi_F^q) + 2^q c_q\varrho^q(\delta_h^q + \xi_F^q)}
$$

and

$$
\theta_2 = \frac{\tau_2^{q-1}}{r_2 - \varrho m_2}\varrho\beta_F\kappa_S.
$$

It follows from (4.14) and (4.20) that

$$\begin{aligned}\|a_1-b_1\|+\|a_2-b_2\| &\le (\theta_1+\theta_2)\|x-\overline{x}\|+(\vartheta_1+\vartheta_2)\|y-\overline{y}\| \\ &\le \sigma(\|x-\overline{x}\|+\|y-\overline{y}\|), \end{aligned} \tag{4.21}$$

where $\sigma=\max\{\theta_1+\theta_2,\vartheta_1+\vartheta_2\}$. From conditions (4.6), we know that $\sigma<1$. Hence from (4.21), we get

$$\begin{aligned} d((a_1,a_2),G_{\rho,\varrho}(\overline{x},\overline{y},\omega,\lambda)) &= \inf_{(b_1,b_2)\in G_{\rho,\varrho}(\overline{x},\overline{y},\omega,\lambda)}(\|a_1-b_1\|+\|a_2-b_2\|) \\ &\le -\sigma\|(x,y)-(\overline{x},\overline{y})\|. \end{aligned}$$

Since $(a_1,a_2)\in G_{\rho,\varrho}(x,y,\omega,\lambda)$ is arbitrary, we obtain

$$\sup_{(a_1,a_2)\in G_{\rho,\varrho}(x,y,\omega,\lambda)} d((a_1,a_2),G_{\rho,\varrho}(\overline{x},\overline{y},\omega,\lambda)) \le -\sigma\|(x,y)-(\overline{x},\overline{y})\|.$$

By the same argument we can prove

$$\sup_{(b_1,b_2)\in G_{\rho,\varrho}(\overline{x},\overline{y},\omega,\lambda)} d((b_1,b_2),G_{\rho,\varrho}(x,y,\omega,\lambda)) \le -\sigma\|(x,y)-(\overline{x},\overline{y})\|.$$

It follows from the definition of Hausdorff metric $\widetilde{\mathcal{H}}$ on $CB(X_1\times X_2)$ that

$$\widetilde{\mathcal{H}}(G_{\rho,\varrho}(x,y,\omega,\lambda),G_{\rho,\varrho}(\overline{x},\overline{y},\omega,\lambda)) \le -\sigma\|(x,y)-(\overline{x},\overline{y})\|.$$

for all $(x,\overline{x},\omega)\in X_1\times X_1\times\Omega, (y,\overline{y},\lambda)\in X_2\times X_2\times\wedge$, that is, $G_{\rho,\varrho}(x,y,\omega,\lambda)$ is a multi-valued contractive mapping which is uniform with respect to $(\omega,\lambda)\in\Omega\times\wedge$. By the fixed point theorem of Nadler [22], for each $(\omega,\lambda)\in\Omega\times\wedge, G_{\rho,\varrho}(x,y,\omega,\lambda)$ has a fixed point $(x(\omega),y(\lambda))\in X_1\times X_2$, that is, $(x,y)\in G_{\rho,\varrho}(x,y,\omega,\lambda)$. By the definition of G, there exists $u\in S(x,\omega), v\in T(y,\lambda)$ such that (4.1) holds. Thus it follows from Lemma 4.1 that (x,y,u,v) with $u\in S(x,\omega), v\in T(y,\lambda)$ is a solution of problem (3.1). Hence from Lemma 4.2 that (z,t,x,y,u,v) with $u\in S(x,\omega), v\in T(y,\lambda)$ is a solution of (3.1). Therefore $Q(\omega,\lambda)\neq\emptyset$ for all $(\omega,\lambda)\in\Omega\times\wedge$. Next, we prove the closedness of the solution set $Q(\omega,\lambda)$. For each $(\omega,\lambda)\in\Omega\times\wedge$, let $\{(z_n,t_n,x_n,y_n)\}\subset Q(\omega,\lambda)$ and $z_n\to z_0, t_n\to t_0, x_n\to x_0, y_n\to y_0$ as $n\to\infty$. Then we know that there exist $u_n\in S(x_n,\omega), v_n\in T(y_n,\lambda)$ and $(x_n,y_n)\in G_{\rho,\varrho}(x_n,y_n,\omega,\lambda), z_n = A_1(x_n)-\rho(p(x_n,\omega)-E(x_n,v_n,\omega)), t_n=A_2(y_n)-\varrho(h(y_n,\lambda)-F(u_n,y_n,\lambda))$, and $z_0=A_1(x_0)-\rho(p(x_0,\omega)-E(x_0,v_0,\omega))$,
$t_0=A_2(y_0)-\varrho(h(y_0,\lambda)-F(u_0,y_0,\lambda))$. Note that for all $(\omega,\lambda)\in\Omega\times\wedge$,

$$\widetilde{\mathcal{H}}(G_{\rho,\varrho}(x_n,y_n,\omega,\lambda),G_{\rho,\varrho}(x_0,y_0,\omega,\lambda)) \le -\sigma\|(x_n,y_n)-(x_0,y_0)\|,$$

It follows that

$$\begin{aligned} d((x_0,y_0),G_{\rho,\varrho}(x_0,y_0,\omega,\lambda)) &\le \|(x_0,y_0)-(x_n,y_n)\| \\ &\quad + d((x_n,y_n),G_{\rho,\varrho}(x_n,y_n,\omega,\lambda)) \\ &\quad + \widetilde{\mathcal{H}}(G_{\rho,\varrho}(x_n,y_n,\omega,\lambda),G_{\rho,\varrho}(x_0,y_0,\omega,\lambda)) \\ &\le (1+\sigma)\|(x_n,y_n)-(x_0,y_0)\|. \end{aligned}$$

Hence, we have $(x_0,y_0)\in G_{\rho,\varrho}(x_0,y_0,\omega,\lambda)$ and $(x_0,y_0)\in Q(\omega,\lambda)$. Therefore $Q(\omega,\lambda)$ is a closed subset of $X_1\times X_2$. □

Theorem 4.4. *Under the assumptions of Theorem 4.3, suppose that*

(i) *for $x\in X_1,\omega\to S(x,\omega)$ is parametrically $\ell_S-\widetilde{\mathcal{H}}$-Lipschitz continuous (or continuous);*

(ii) *for $y\in X_2,\lambda\to T(y,\lambda)$ is parametrically $\ell_T-\widetilde{\mathcal{H}}$-Lipschitz continuous (or continuous);*

(iii) *for $x,z\in X_1,y,t\in X_2,\omega\to p(x,\omega),\omega\to E(x,y,\omega),\omega\to J^{M(\cdot,x,\omega)}_{\rho,A_1}(z)$, $\lambda\to h(y,\lambda),\lambda\to F(x,y,\lambda)$ and $\lambda\to J^{N(\cdot,y,\lambda)}_{\varrho,A_2}(t)$ are parametrically Lipschitz continuous (or continuous) with parametrically Lipschitz constants $\ell_p,\ell_E,\ell_{J_1},\ell_h,\ell_F$ and ℓ_{J_2}, respectively.*

Then the solution set $Q(\omega,\lambda)$ of problem (2.1) *is parametrically Lipschitz continuous (or continuous) from $\Omega\times\wedge$ to $X_1\times X_2$.*

Proof. From the assumptions of Theorem 4.3, for any $(\omega,\lambda),(\overline{\omega},\overline{\lambda})\in\Omega\times\wedge$, we know that $Q(\omega,\lambda)$ and $Q(\overline{\omega},\overline{\lambda})$ are nonempty closed subsets of $X_1\times X_2$. From the proof of Theorem 4.3, $G_{\rho,\varrho}(x,y,\omega,\lambda)$ and $G_{\rho,\varrho}(x,y,\overline{\omega},\overline{\lambda})$ are contractive mappings with same contractive constant $\sigma\in(0,1)$ and have fixed points $(x(\omega,\lambda),y(\omega,\lambda))$ and $(x(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}))$, respectively. It follows from Lemma 3.12 and 4.2 that

$$\begin{aligned} &\widetilde{\mathcal{H}}(Q(\omega,\lambda),Q(\overline{\omega},\overline{\lambda})) \\ &\le \frac{1}{1-\sigma}\sup_{(x,y)\in X_1\times X_2}\widetilde{\mathcal{H}}(G_{\rho,\varrho}(x(\omega,\lambda),y(\omega,\lambda),\omega,\lambda),G_{\rho,\varrho}(x(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\overline{\omega},\overline{\lambda})). \end{aligned} \tag{4.22}$$

Setting $(a_1,a_2)\in G_{\rho,\varrho}(x(\omega,\lambda),y(\omega,\lambda),\omega,\lambda)$, then there exist $u(\omega,\lambda)\in S(x(\omega,\lambda),\omega)$, and $v(\omega,\lambda)\in T(y(\omega,\lambda),\lambda)$ such that

$$a_1 = J^{M(\cdot,x(\omega,\lambda),\omega)}_{\rho,A_1}[A_1(x(\omega,\lambda))-\rho(p(x(\omega,\lambda),\omega)-E(x(\omega,\lambda),v(\omega,\lambda),\omega))],$$

$$a_2 = J^{N(\cdot,y(\omega,\lambda),\lambda)}_{\varrho,A_2}[A_2(y(\omega,\lambda))-\varrho(h(y(\omega,\lambda),\lambda)-F(u(\omega,\lambda),y(\omega,\lambda),\lambda))].$$

Since $S(x(\omega,\lambda),\omega),S(x(\overline{\omega},\overline{\lambda}),\overline{\omega})\in CB(X_1)$ and $T(y(\omega,\lambda),\lambda),T(y(\overline{\omega},\overline{\lambda}),\overline{\lambda})\in CB(X_2)$, It follows from Nadler's Theorem [22] that there exist $u(\overline{\omega},\overline{\lambda})\in$

$S(x(\overline{\omega},\overline{\lambda}),\overline{\omega}) \in CB(X_1), v(\overline{\omega},\overline{\lambda}) \in T(y(\overline{\omega},\overline{\lambda}),\overline{\lambda}) \in CB(X_2)$ such that

$$\|u(\omega,\lambda) - u(\overline{\omega},\overline{\lambda})\| \leq \widetilde{\mathcal{H}}(S(x(\omega,\lambda),\omega), S(x(\overline{\omega},\overline{\lambda}),\overline{\omega})),$$
$$\|v(\omega,\lambda) - v(\overline{\omega},\overline{\lambda})\| \leq \widetilde{\mathcal{H}}(T(y(\omega,\lambda),\lambda), T(y(\overline{\omega},\overline{\lambda}),\overline{\lambda})). \tag{4.23}$$

Let

$$b_1 = J_{\rho,A_1}^{M(\cdot,x(\overline{\omega},\overline{\lambda}),\overline{\omega})}[A_1(x(\overline{\omega},\overline{\lambda})) - \rho(p(x(\overline{\omega},\overline{\lambda}),\overline{\omega}) - E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\overline{\omega}))],$$

$$b_2 = J_{\varrho,A_2}^{N(\cdot,y(\overline{\omega},\overline{\lambda}),\overline{\lambda})}[A_2(y(\overline{\omega},\overline{\lambda})) - \varrho(h(y(\overline{\omega},\overline{\lambda}),\overline{\lambda}) - F(u(\overline{\omega},\overline{\lambda}), y(\overline{\omega},\overline{\lambda}),\overline{\lambda}))].$$

Then we have $(b_1, b_2) \in G_{\rho,\varrho}(x(\overline{\omega},\overline{\lambda}), y(\overline{\omega},\overline{\lambda}),\overline{\omega},\overline{\lambda})$. It follows from the assumptions on $J_{\rho,A_1}^{M(\cdot,\cdot,\cdot)}$, E, A_1, p and T that

$$\begin{aligned}
&\|a_1 - b_1\| \\
&= \|J_{\rho,A_1}^{M(\cdot,x(\omega,\lambda),\omega)}[A_1(x(\omega,\lambda)) - \rho(p(x(\omega,\lambda),\omega) - E(x(\omega,\lambda), v(\omega,\lambda),\omega))] \\
&\quad - J_{\rho,A_1}^{M(\cdot,x(\overline{\omega},\overline{\lambda}),\overline{\omega})}[A_1(x(\overline{\omega},\overline{\lambda})) - \rho(p(x(\overline{\omega},\overline{\lambda}),\overline{\omega}) - E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\overline{\omega}))]\| \\
&\leq \|J_{\rho,A_1}^{M(\cdot,x(\omega,\lambda),\omega)}[A_1(x(\omega,\lambda)) - \rho(p(x(\omega,\lambda),\omega) - E(x(\omega,\lambda), v(\omega,\lambda),\omega))] \\
&\quad - J_{\rho,A_1}^{M(\cdot,x(\overline{\omega},\overline{\lambda}),\omega)}[A_1(x(\overline{\omega},\overline{\lambda})) - \rho(p(x(\overline{\omega},\overline{\lambda}),\omega) - E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\omega))]\| \\
&\quad + \|J_{\rho,A_1}^{M(\cdot,x(\overline{\omega},\overline{\lambda}),\omega)}[A_1(x(\overline{\omega},\overline{\lambda})) - \rho(p(x(\overline{\omega},\overline{\lambda}),\omega) - E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\omega))] \\
&\quad - J_{\rho,A_1}^{M(\cdot,x(\overline{\omega},\overline{\lambda}),\overline{\omega})}[A_1(x(\overline{\omega},\overline{\lambda})) - \rho(p(x(\overline{\omega},\overline{\lambda}),\omega) - E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\omega))]\| \\
&\quad + \|J_{\rho,A_1}^{M(\cdot,x(\overline{\omega},\overline{\lambda}),\overline{\omega})}[A_1(x(\overline{\omega},\overline{\lambda})) - \rho(p(x(\overline{\omega},\overline{\lambda}),\omega) - E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\omega))] \\
&\quad - J_{\rho,A_1}^{M(\cdot,x(\overline{\omega},\overline{\lambda}),\overline{\omega})}[A_1(x(\overline{\omega},\overline{\lambda})) - \rho(p(x(\overline{\omega},\overline{\lambda}),\overline{\omega}) - E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\overline{\omega}))]\| \\
&\leq \theta_1\|x(\omega,\lambda) - x(\overline{\omega},\overline{\lambda})\| + \vartheta_1\|y(\omega,\lambda) - y(\overline{\omega},\overline{\lambda})\| + \ell_{J_1}\|\omega - \overline{\omega}\| \\
&\quad + \frac{\tau_1^{q-1}\rho}{r_1 - \rho m_1}[\|p(x(\overline{\omega},\overline{\lambda}),\omega) - p(x(\overline{\omega},\overline{\lambda}),\overline{\omega})\| \\
&\quad + \|E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\omega) - E(x(\overline{\omega},\overline{\lambda}), v(\overline{\omega},\overline{\lambda}),\overline{\omega})\|] \\
&\leq \theta_1\|x(\omega,\lambda) - x(\overline{\omega},\overline{\lambda})\| + \vartheta_1\|y(\omega,\lambda) - y(\overline{\omega},\overline{\lambda})\| + \ell_{J_1}\|\omega - \overline{\omega}\| \\
&\quad + \frac{\tau_1^{q-1}\rho}{r_1 - \rho m_1}[\ell_p\|\omega - \overline{\omega}\| + \ell_E\|\omega - \overline{\omega}\|] \\
&\leq \theta_1\|x(\omega,\lambda) - x(\overline{\omega},\overline{\lambda})\| + \vartheta_1\|y(\omega,\lambda) - y(\overline{\omega},\overline{\lambda})\| + \chi_1\|\omega - \overline{\omega}\|,
\end{aligned} \tag{4.24}$$

where θ_1 and ϑ_1 are the constants of (4.14) and

$$\chi_1 = \ell_{J_1} + \frac{\rho\tau_1^{q-1}(\ell_p + \ell_E)}{r_1 - \rho m_1}.$$

Similarly, from the assumptions on $h, F, A_2, S, J_{\varrho,A_2}^{N(\cdot,\cdot,\cdot)}$, we have

$$
\begin{aligned}
&\|a_2 - b_2\| \\
&= \|J_{\varrho,A_2}^{N(\cdot,y(\omega,\lambda),\lambda)}[A_2(y(\omega,\lambda)) - \varrho(h(y(\omega,\lambda),\lambda) - F(u(\omega,\lambda),y(\omega,\lambda),\lambda))] \\
&\quad - J_{\varrho,A_2}^{N(\cdot,y(\overline{\omega},\overline{\lambda}),\overline{\lambda})}[A_2(y(\overline{\omega},\overline{\lambda})) - \varrho(h(y(\overline{\omega},\overline{\lambda}),\overline{\lambda}) - F(u(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\overline{\lambda}))]\| \\
&\le \|J_{\varrho,A_2}^{N(\cdot,y(\omega,\lambda),\lambda)}[A_2(y(\omega,\lambda)) - \varrho(h(y(\omega,\lambda),\lambda) - F(u(\omega,\lambda),y(\omega,\lambda),\lambda))] \\
&\quad - J_{\varrho,A_2}^{N(\cdot,y(\overline{\omega},\overline{\lambda}),\lambda)}[A_2(y(\overline{\omega},\overline{\lambda})) - \varrho(h(y(\overline{\omega},\overline{\lambda}),\lambda) - F(u(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\lambda))]\| \\
&\quad + \|J_{\varrho,A_2}^{N(\cdot,y(\overline{\omega},\overline{\lambda}),\lambda)}[A_2(y(\overline{\omega},\overline{\lambda})) - \varrho(h(y(\overline{\omega},\overline{\lambda}),\lambda) - F(u(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\lambda))] \\
&\quad - J_{\varrho,A_2}^{N(\cdot,y(\overline{\omega},\overline{\lambda}),\overline{\lambda})}[A_2(y(\overline{\omega},\overline{\lambda})) - \varrho(h(y(\overline{\omega},\overline{\lambda}),\lambda) - F(u(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\lambda))]\| \\
&\quad + \|J_{\varrho,A_2}^{N(\cdot,y(\overline{\omega},\overline{\lambda}),\overline{\lambda})}[A_2(y(\overline{\omega},\overline{\lambda})) - \varrho(h(y(\overline{\omega},\overline{\lambda}),\lambda) - F(u(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\lambda))] \\
&\quad - J_{\varrho,A_2}^{N(\cdot,y(\overline{\omega},\overline{\lambda}),\overline{\lambda})}[A_2(y(\overline{\omega},\overline{\lambda})) - \varrho(h(y(\overline{\omega},\overline{\lambda}),\overline{\lambda}) - F(u(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\overline{\lambda}))]\| \\
&\le \theta_2\|x(\omega,\lambda) - x(\overline{\omega},\overline{\lambda})\| + \vartheta_2\|y(\omega,\lambda) - y(\overline{\omega},\overline{\lambda})\| + \ell_{J_2}\|\lambda - \overline{\lambda}\| \\
&\quad + \frac{\tau_2^{q-1}\varrho}{r_2 - \varrho m_2}[\|h(y(\overline{\omega},\overline{\lambda}),\lambda) - h(y(\overline{\omega},\overline{\lambda}),\overline{\lambda})\| \\
&\quad + \|F(u(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\lambda) - F(u(\overline{\omega},\overline{\lambda}),y(\overline{\omega},\overline{\lambda}),\overline{\lambda})\|] \\
&\le \theta_2\|x(\omega,\lambda) - x(\overline{\omega},\overline{\lambda})\| + \vartheta_2\|y(\omega,\lambda) - y(\overline{\omega},\overline{\lambda})\| + \ell_{J_2}\|\lambda - \overline{\lambda}\| \\
&\quad + \frac{\tau_2^{q-1}\varrho}{r_2 - \varrho m_2}\left[\ell_h\|\lambda - \overline{\lambda}\| + \ell_F\|\lambda - \overline{\lambda}\|\right] \\
&\le \theta_2\|x(\omega,\lambda) - x(\overline{\omega},\overline{\lambda})\| + \vartheta_2\|y(\omega,\lambda) - y(\overline{\omega},\overline{\lambda})\| + \chi_2\|\lambda - \overline{\lambda}\|, \qquad (4.25)
\end{aligned}
$$

where θ_2 and ϑ_2 are the constants of (4.20) and

$$\chi_2 = \ell_{J_2} + \frac{\varrho\tau_2^{q-1}(\ell_h + \ell_F)}{r_2 - \rho m_2}.$$

It follows from (4.24), (4.25) and (4.1) that

$$
\begin{aligned}
\|a_1 - b_1\| + \|a_2 - b_2\| &\le (\theta_1 + \theta_2)\|x(\omega,\lambda) - x(\overline{\omega},\overline{\lambda})\| \\
&\quad + (\vartheta_1 + \vartheta_2)\|y(\omega,\lambda) - y(\overline{\omega},\overline{\lambda})\| \\
&\quad + \chi_1\|\omega - \overline{\omega}\| + \chi_2\|\lambda - \overline{\lambda}\| \\
&\le \sigma(\|a_1 - b_1\| + \|a_2 - b_2\|) + \chi_1\|\omega - \overline{\omega}\| + \chi_2\|\lambda - \overline{\lambda}\|,
\end{aligned}
$$

where σ is the constant of (4.21) which implies that

$$\|a_1 - b_1\| + \|a_2 - b_2\| \le \Theta(\|\omega - \overline{\omega}\| + \|\lambda - \overline{\lambda}\|), \qquad (4.26)$$

where

$$\Theta = \frac{1}{1-\sigma}\max\{\chi_1, \chi_2\}.$$

Hence from (4.26) we obtain

$$\sup_{(a_1,a_2)\in G_{\rho,\varrho}(x,y,\omega,\lambda)} d((a_1,a_2), G_{\rho,\varrho}(x,y,\overline{\omega},\overline{\lambda})) \le \Theta\|(\omega,\lambda)-(\overline{\omega},\overline{\lambda})\|.$$

By using a similar argument as above, we get

$$\sup_{(b_1,b_2)\in G_{\rho,\varrho}(x,y,\overline{\omega},\overline{\lambda})} d(G_{\rho,\varrho}(x,y,\omega,\lambda),(b_1,b_2)) \le \Theta\|(\omega,\lambda)-(\overline{\omega},\overline{\lambda})\|.$$

It implies that

$$\widetilde{\mathcal{H}}(G_{\rho,\varrho}(x,y,\omega,\lambda), G_{\rho,\varrho}(x,y,\overline{\omega},\overline{\lambda})) \le \Theta\|(\omega,\lambda)-(\overline{\omega},\overline{\lambda})\|,$$

for all $(x,y,\omega,\overline{\omega},\lambda,\overline{\lambda}) \in X_1 \times X_2 \times \Omega \times \Omega \times \wedge \times \wedge$. Thus, it follows from (4.22) that

$$\widetilde{\mathcal{H}}(Q(\omega,\lambda), Q(\overline{\omega},\overline{\lambda})) \le \frac{\Theta}{1-\sigma}\|(\omega,\lambda)-(\overline{\omega},\overline{\lambda})\|.$$

This proves that $Q(\omega,\lambda)$ is parametrically Lipschitz continuous in $(\omega,\lambda) \in \Omega\times\wedge$. If each operator with conditions (i) and (ii) is assumed to be continuous in $(\omega,\lambda)\in\Omega\times\wedge$, then by similar argument as above, we show that $S(\omega)$ and $T(\lambda)$ are parametrically continuous in $(\omega,\lambda)\in\Omega\times\wedge$. □

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References

[1] R.P. Agarwal, N.J. Huang and M.Y. Tan, *Sensitivity analysis for a new system of generalized nonlinear mixed quasi variational inclusions*, Appl. Math. Lett., **17** (2004), 345–352.

[2] G.A. Anastassiou, Salahuddin and M.K. Ahmad, *Sensitivity analysis for generalized set valued variational inclusions*, J. Concrete Appl. Math., **11**(3-4) (2013), 292–302.

[3] C. Baiocchia and A. Capelo, *Variational and Quasi Variational Inequalities*, Wiley, Neq York, 1984.

[4] F.E. Browder, *Fixed point theory and nonlinear problems*, Proc. Symp. Pure Math., **39** (1980), 49–87.

[5] H.W. Cao, *Sensitivity analysis for a system of generalized nonlinear mixed quasi variational inclusions with H-monoton operators*, J. Appl. Math., 2011 Article ID 92 1835 (2011).

[6] S.S. Chang, H.W.J. Lee and C.K. Chan, *Generalized system for relaxed cocoercive variational inequalities in Hilbert space*, Applied Math. Lett., **20**(3) (2007), 329–334.

[7] S. Dafermos, *Sensitivity analysis in variational inequalities*, Math. Oper. Res., **13**(3) (1988), 421–434.

[8] X. P. Ding, *Sensitivity analysis for a system of generalized mixed implicit equilibrium problems in uniformly smooth Banach spaces*, Nonlinear Analysis, TMA, **73**(5) (2010), 1264–1276.

[9] K. Ding, Z.Q. He and J. Li, *Sensitivity analysis for a new class of generalized strongly monlinear projection equations*, Nonlinear Funct. Anal. Appl., **8**(3) (2003), 423–431.

[10] X.P. Ding and Z.B. Wang, *Sensitivity analysis for a system of parametric generalized mixed quasi variational inclusions involving* (K, η)*-monotone mappings*, Appl. Math. Comput., **214**(1) (2009), 318–327.

[11] Y.P. Fang and N.J. Huang, *Iterative algorithm for a system of variational inclusions involving* H*-accretive operators in Banach spaces*, Acta Math. Hung, **108**(3) (2005), 183–195.

[12] A. Farajzadeh and Salahuddin, *Sensitivity analysis for nonlinear set valued variational equations in Banach frame work*, J. Funct. Spaces and Appl. 2013 Article ID 258543 (2013).

[13] J.U. Jeong, *Sensitivity analysis for a system of generalized nonlinear mixed quasi variational inclusions in* q*-uniformly Banach spaces*, Appl. Math. Comput., **202**(1) (2008), 178–187.

[14] M.M. Jin, *Iterative algorithms for a new system of nonlinear variational inclusions with* (A, η)*-accretive mappings in Banach spaces*, Comput. Math. Appl., **54** (2007), 579–588.

[15] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalies and Their Applications*, Academic Press, New York, 1980.

[16] J.K. Kim, *Sensitivity analysis for general nonlinear nonconvex set-valued variational inequalities in Banach spaces*, Jour. of Comput. Anal. and Appl., **22**(2) (2017), 327–335,

[17] J.K. Kim, H.Y. Lan and Y.J. Cho, *Solution sensitivity of generalized nonlinear parametric* (A, η, m)*-proximal operator system of equations in Hilbert spaces*, J. Inequal. Appl., 2014, 2014:362.

[18] H.Y. Lan, *Nonlinear parametric multivalued variational inclusions systems involving* (A, η)*-accretive mappings in Banach spaces*, Nonlinear Anal., **69**(5-6) (2008), 1757–1767.

[19] H.Y. Lan, Y.J. Cho and R.U. Verma, *On solution sensitivity of generalized relaxed cocoercive implicit quasi variational inclusions with* A*-monotone mappings*, J. Comput. Anal Appl., **8**(1) (2006), 75–89.

[20] H.Y. Lan, Y.J. Cho and R.U. Verma, *Nonlinear relaxed cocoercive variational inclusions involving* $(A, \cdot)$*-accretive mappings in Bamach spaces*, Comput. Math. Appl., **51**(9-10) (2006), 1529–1538.

[21] T.C. Lim, *On fixed point stability for set valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl., **110**(2) (1985), 436–441.

[22] Jr.S.B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488.

[23] Salahuddin, *Parametric generalized set valued variational inclusions and resolvent equations*, J. Math. Anal. Appl., **298** (2004), 146–156.

[24] R.U. Verma, *Sensitivity analysis for relaxed cocoercive nonlinear quasi variational inclusions*, J. Appl. Math. Stoch. Anal., 2006, Article ID 52041 (2006).

[25] R.U. Verma, *Role of* A*-monotonicity in sensitivity analysis for relaxed cocoercive nonlinear quasi variational inclusions and resolvent operator techniques*, Nonlinear Funct. Anal. Appl., **11**(4) (2006), 577–588.

[26] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16**(12) (1991), 1127–1138.