Nonlinear Functional Analysis and Applications
Vol. 22, No. 1 (2017), pp. 41-58
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

IMPROVED CONVERGENCE ANALYSIS FOR THE KURCHATOV METHOD

Ioannis K. Argyros[1] **and Santhosh George**[2]

[1]Department of Mathematicsal Sciences
Cameron University, Lawton, OK 73505, USA
e-mail: ioannisa@cameron.edu

[2]Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, India
e-mail: sgeorge@nitk.ac.in

Abstract. We present a new convergence analysis for the Kurchatov method using our new idea of restricted convergence domains in order to solve nonlinear equations in a Banach space setting. The sufficient convergence conditions are weaker than in earlier studies. Hence, we extend the applicability of this method. Moreover, our radius of convergence is larger leading to a wider choice of initial guesses and fewer iterations to achieve a desired error tolerance. Numerical examples are also provided showing the advantages of our approach over earlier work.

1. Introduction

In [8], Argyros and Ren studied the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is Fréchet-differentiable operator defined on a convex subset of a Banach space $\mathbb{B}_1$ with values in a Banach space $\mathbb{B}_2$. In the present paper, we study the local as well as semilocal convergence of the method considered in [8], using the idea of restricted convergence domains. That is we consider the

[0]Received March 14, 2016. Revised June 28, 2016.
[0]2010 Mathematics Subject Classification: 65D10, 65D99, 47J25, 47J05.
[0]Keywords: Kurchatov method, Newton's method, Banach space, local-semilocal convergence, divided difference.

quadratically convergent Kurchatov method defined for each $n = 0, 1, \cdots$ by

$$x_{n+1} = x_n - A_n^{-1} F(x_n), \tag{1.2}$$

where, x_{-1}, x_0 are initial points and $A_n = [2x_n - x_{n-1}, x_{n-1}; F]$ and $[x, y; F]$ denotes the divided difference of operator F at the points $x, y \in \Omega$.

We need the definition of divided differences of order one and two.

Definition 1.1. Let x, y be two points in Ω. The linear operator $[x, y.; F] \in L(\mathbb{B}_1, \mathbb{B}_2)$ is called a divided difference of order one of operator F at the x, y if the following holds

$$[x, y; F](x - y) = F(x) - F(y) \quad \text{for any} \quad x, y \in \Omega \quad \text{with} \quad x \neq y.$$

If F is Fréchet-differentiable on Ω, then $F'(x) = [x, x; F]$. Similarly, $[x, y, z; F]$ is a divided difference of order two at points $x, y, z \in \Omega$, if

$$[x, y, z; F](y - z) = [x, y; F] - [x, z; F].$$

If F is twice-Fréchet-differentiable on Ω, then $\frac{1}{2} F''(x) = [x, x, x; F]$.

The convergence of the Kurchatov method has been studied under hypotheses up to the divided difference of order two in [1, 3] and up to the divided difference of one in [2]-[7]. In particular, the condition

$$\|A_0^{-1}([u, x, y; F] - [v, x, y; F])\| \leq \delta \|u - v\|, \tag{1.3}$$

for each $x, y, u, v \in \Omega$ has been used. However, there are examples where (1.3) is violated or $[\cdot, \cdot, \cdot; F]$ does not exist. For an example, define function $f : [-1, 1] \to (-\infty, \infty)$ by

$$f(x) = x^2 \ln x^2 + c_1 x^2 + c_2 x + c_3, \quad f(0) = c_3,$$

where c_1, c_2, c_3 are given real numbers. Then, we have that $\lim_{x \to 0} x^2 \ln x^2 = 0$, $\lim_{x \to 0} x \ln x^2 = 0$, $f'(x) = 2x \ln x^2 + 2(c_1 + 1)x + c_2$ and $f''(x) = 2(\ln x^2 + 3 + c_1)$. Then, function f does not satisfy (1.3).

Here, we use the idea of restricted convergence domains and the hypotheses on the divided difference of order one. When, we use divided difference of order two the convergence conditions are weaker than in [3]-[5], [8], [10]. Hence, the applicability of the Kurchatov method is extended.

The results in [8] are proved using the following Lemmas.

Lemma 1.2. ([8, Lemma 2.1]) *Let* $L > 0$, $L_i > 0, i = -1, 0, 1, 2, 3$, $t_0 \geq 0$, $t_1 > 0$ *be given parameters. Denote by* α *the smallest root of polynomial* p *in the interval* $(0, 1)$ *defined by*

$$p(t) = 2L_0 t^3 + (-L_0 + L_1 + L_2)t^2 + L_3 t - (L_2 + L_3).$$

Suppose that

$$0 < \frac{L_0(t_1 - t_{-1}) + L_1(t_0 - t_{-1})}{1 - (L_{-1}(2t_1 - t_0) + Lt_0)} \leq \alpha, \tag{1.4}$$

$$0 < \frac{L_2(t_2 - t_0) + L_3(t_1 - t_0)}{1 - (L_0(2t_2 - t_1) + L_1 t_1)} \leq \alpha \tag{1.5}$$

and

$$0 < \alpha \leq 1 - \frac{(L_0 + L_1)(t_1 - t_0)}{1 - (L_0 + L_1)t_0}, \tag{1.6}$$

where, $t_{-1} = 0$ *and*

$$t_2 = t_1 + \frac{L_0(t_1 - t_{-1}) + L_1(t_0 - t_{-1})}{1 - (L_{-1}(2t_1 - t_0) + Lt_0)}(t_1 - t_0).$$

Then, scalar sequence $\{t_n\}$ *defined by*

$$t_{n+2} = t_{n+1} + \tfrac{L_2(t_{n+1} - t_{n-1}) + L_3(t_n - t_{n-1})}{1 - (L_0(2t_{n+1} - t_n) + L_1 t_n)}(t_{n+1} - t_n) \text{ for each } n = 1, 2, \cdots \tag{1.7}$$

is well defined, increasing, bounded above by

$$t^{\star\star} = \frac{t_1 - t_0}{1 - \alpha} + t_0 \tag{1.8}$$

and converges to its unique least upper bound $t^\star$ *which satisfies*

$$t^\star \in [t_1, t^{\star\star}]. \tag{1.9}$$

Moreover, the following estimates hold for each $n = 0, 1, \cdots$

$$t_{n+2} - t_{n+1} \leq \alpha(t_{n+1} - t_n). \tag{1.10}$$

Next we present an alternative to Lemma 1.2.

Lemma 1.3. ([8, Lemma 2.2]) *Let* $L > 0$, $L_i > 0, i = -1, 0, 1, 2, 3$, $K \geq 0$, $s_0 \geq 0$, $s_1 > 0$ *be given parameters. Suppose* β *is the smallest positive root of polynomial* $\overline{p}$ *defined in interval* $[0, 1]$ *by*

$$\overline{p}(t) = 2L_0 t^3 + [\frac{L_2 + L_3}{2} + K(s_1 - s_0) - L_0 + L_1]t^2 - K(s_1 - s_0)t - \frac{L_2 + L_3}{2}. \tag{1.11}$$

Moreover, suppose

$$0 < \frac{\frac{L_0 + L_1}{2}(s_1 - s_{-1}) + K(s_0 - s_{-1})^2}{1 - [L_{-1}(2s_1 - s_0) + Ls_0]} \leq \beta, \tag{1.12}$$

$$0 < \frac{\frac{L_2 + L_3}{2}(s_2 - s_0) + K(s_1 - s_0)^2}{1 - [L_0(2s_2 - s_1) + L_1 s_1]} \leq \beta, \tag{1.13}$$

$$\beta \leq 1 - \frac{(L_0 + L_1)(s_1 - s_0)}{1 - (L_0 + L_1)s_0}, \tag{1.14}$$

where, $s_{-1}=0$ *and*

$$s_2 = s_1 + \frac{\frac{L_0+L_1}{2}(s_1-s_{-1})+K(s_0-s_{-1})^2}{1-[L_{-1}(2s_1-s_0)+Ls_0]}(s_1-s_0).$$

Then, scalar sequence $\{s_n\}$ *defined for each* $n=1,2,\cdots$ *by*

$$s_{n+2} = s_{n+1} + \frac{\frac{L_2+L_3}{2}(s_{n+1}-s_{n-1})+K(s_n-s_{n-1})^2}{1-[L_0(2s_{n+1}-s_n)+L_1 s_n]}(s_{n+1}-s_n) \tag{1.15}$$

is well defined, increasing, bounded above by

$$s^{\star\star} = \frac{s_1-s_0}{1-\beta}+s_0 \tag{1.16}$$

and converges to its unique least upper bound $s^\star$ *which satisfies*

$$s^\star \in [s_1, s^{\star\star}]. \tag{1.17}$$

Moreover, the following estimates hold for each $n=0,1,\cdots$

$$s_{n+2}-s_{n+1} \le \beta(s_{n+1}-s_n). \tag{1.18}$$

The paper is organized as follows: Section 2 contains results on the semilocal convergence analysis of the Kurchatov method while the local convergence analysis is presented in Section 3. Finally, the numerical examples are presented in the concluding Section 4.

2. Semilocal convergence

Let $U(x,\xi)$, $\overline{U}(x,\xi)$, stand respectively, for the open and closed balls in $\mathbb{B}_1$, with center $x \in \mathbb{B}_1$ and of radius $\xi > 0$.

Next, we present two semilocal convergence results for Kurchatov method (1.2). In the first one, we use Lemma 1.2 and hypotheses on the divided difference of order one for F. In the second result, we use Lemma 1.3 and divided differences up to order two.

Theorem 2.1. *Let* $F : \Omega \subset \mathbb{B}_1 \to \mathbb{B}_2$ *be a Fréchet-differentiable operator. Suppose that there exists a divided difference* $[\cdot,\cdot;F]$ *of order one for operator* F *on* $\Omega\times\Omega$*. Moreover, suppose that there exist* $x_{-1}, x_0 \in \Omega$, $L>0$, $L_i > 0, i=-1,0,1,2,3$, $t_0 \ge 0$, $t_1 > 0$ *such that for each* $x,y,z,v \in \Omega$

$$A_0^{-1} \in L(\mathbb{B}_2,\mathbb{B}_1), \tag{2.1}$$

$$\|x_0 - x_{-1}\| \le t_0, \tag{2.2}$$

$$\begin{gathered} \|A_0^{-1}F(x_0)\| \le t_1 - t_0, \\ \|A_0^{-1}(A_1-A_0)\| \le L_{-1}\|2(x_1-x_0)-(x_0-x_{-1})\| + L\|x_0-x_{-1}\|. \end{gathered} \tag{2.3}$$

Let $\bar{r}_0 = \frac{1}{2L_0+L_1}$. *Furthermore, suppose that for each* $x, y, z, v \in \Omega_1 := \Omega \cap U(x_0, \bar{r}_0)$

$$\|A_0^{-1}([x, y; F] - A_0)\| \leq L_0\|x - 2x_0 + x_{-1}\| + L_1\|y - x_{-1}\|, \tag{2.4}$$

$$\|A_0^{-1}([x, y; F] - [z, v; F])\| \leq L_2\|x - z\| + L_3\|y - v\|, \tag{2.5}$$

$$\overline{U}(x_0, r_0) \subseteq \Omega, \ r_0 = \max(2(t_1 - t_0), t^\star - t_0) \tag{2.6}$$

and hypotheses of Lemma 1.2 *hold, where* $x_1 = x_0 - A_0^{-1}F(x_0)$, $A_0 = [2x_0 - x_{-1}, x_{-1}; F]$ *and* $t^\star$ *is given in Lemma* 1.2. *Then, sequence* $\{x_n\}$ *generated by the Kurchatov method* (1.2) *is well defined, remains in* $\overline{U}(x_0, r_0)$ *and converges to a solution* $x^\star \in \overline{U}(x_0, r_0)$ *of equation* $F(x) = 0$. *Moreover, the following estimates hold for each* $n = 0, 1, \cdots$,

$$\|x_n - x^\star\| \leq t^\star - t_n. \tag{2.7}$$

Furthermore, if there exists $R > r_0$ *such that*

$$U(x_0, R) \subseteq \Omega \tag{2.8}$$

and

$$L_0 t^\star + L_1(R + t_0) < 1, \tag{2.9}$$

then the point $x^\star$ *is the only solution of* $F(x) = 0$ *in* $\overline{U}(x_0, R)$.

Proof. It follows from the corresponding proof in [8] by simply noticing that the iterates x_n lie in Ω_0 which is a more precise location than Ω used in [8], since $\Omega_0 \subseteq \Omega$. □

Theorem 2.2. *Let* $F : \Omega \subset \mathbb{B}_1 \to \mathbb{B}_2$ *be a Fréchet-differentiable operator. Suppose that there exist divided differences* $[\cdot, \cdot; F]$, $[\cdot, \cdot, \cdot; F]$ *of order one and order two for operator* F *on* $\Omega \times \Omega$ *and* $\Omega \times \Omega \times \Omega$, *respectively. Moreover, suppose that there exist* $x_{-1}, x_0 \in \Omega$, $L > 0$, $L_i > 0, i = -1, 0, 1, 2, 3$, $K \geq 0$, $s_0 \geq 0$, $s_1 > 0$ *such that for each* $x, y \in \Omega$

$$A_0^{-1} \in L(\mathbb{B}_2, \mathbb{B}_1),$$

$$\|x_0 - x_{-1}\| \leq s_0,$$

$$\|A_0^{-1}F(x_0)\| \leq s_1 - s_0,$$

$$\|A_0^{-1}(A_1 - A_0)\| \leq L_{-1}\|2(x_1 - x_0) - (x_0 - x_{-1})\| + L\|x_0 - x_{-1}\|,$$

$$\|A_0^{-1}([x, y; F] - A_0)\| \leq L_0\|x - 2x_0 + x_{-1}\| + L_1\|y - x_{-1}\|$$

and for each $x, y, z, v \in \Omega_1$,

$$\|A_0^{-1}([x, y; F] - [z, v; F])\| \leq L_2\|x - z\| + L_3\|y - v\|,$$

$$\|A_0^{-1}([x, v, y; F] - [x, z, y; F])\| \leq K\|v - z\|,$$

$$\overline{U}(x_0, R_0) \subseteq \Omega, \ R_0 = \max(2(s_1 - s_0), s^\star - s_0)$$

and hypotheses of Lemma 1.3 *hold, where* $x_1 = x_0 - A_0^{-1}F(x_0)$, $A_0 = [2x_0 - x_{-1}, x_{-1}; F]$ *and* $s^\star$ *is given in Lemma* 1.3*. Then, sequence* $\{x_n\}$ *generated by the Kurchatov method* (1.2) *is well defined, remains in* $\overline{U}(x_0, R_0)$ *and converges to a solution* $x^\star \in \overline{U}(x_0, R_0)$ *of equation* $F(x) = 0$*. Moreover, the following estimates hold for each* $n = 0, 1, \cdots$,

$$\|x_n - x^\star\| \leq s^\star - s_n. \tag{2.10}$$

Furthermore, if there exists $R > R_0$ *such that*

$$U(x_0, R) \subseteq \Omega$$

and

$$L_0 s^\star + L_1(R + s_0) < 1,$$

then the point $x^\star$ *is the only solution of* $F(x) = 0$ *in* $\overline{U}(x_0, R)$.

Remark 2.3. (a) The limit point $t^\star$ (or $s^\star$) can be replaced by $t^{\star\star}$ (or $s^{\star\star}$) given in the corresponding closed form by (2.5) (or (2.21)) in both theorems.

(b) We have that

$$L_{-1} \leq L_0, \tag{2.11}$$

$$L \leq L_1, \tag{2.12}$$

hold in general. If the divided difference $[x, y; F]$ is symmetric, then we have that $L_2 = L_3$ and $L_0 = L_1$. Notice that in the literature they use $L_{-1} = L = L_0 = L_1 = L_2 = L_3$ to study iterative methods with divided differences [8], [10]-[14]. However, if strict inequality holds in any of the inequality in (2.11) or (2.12), then, our approach leads to tighter majorizing sequences weaker sufficient convergence conditions and an at least as precise information on the location of the solutions [8], [10]-[14].

(c) Let us denote by $\bar{L}_2, \bar{L}_3$ the corresponding constants, when (1.18) holds on Ω. Then, we have that

$$L_2 \leq \bar{L}_2, \;\; L_3 \leq \bar{L}_3, \;\; L_0 \leq \bar{L}_2, \;\; L_1 \leq \bar{L}_3 \tag{2.13}$$

leading to even more improved results than in [8].

(d) Condition (2.6) can be replaced by for all $x, y \in \Omega \Rightarrow 2y - x \in \Omega$, which holds e.g., if $\mathbb{B}_1 = \mathbb{B}_2 = \Omega$ ([3, 4, 5]).

(f) The preceding results can be extended even further, if we work on the ball $\Omega_1^* = \Omega \cap U(x_1, \bar{r}_0 - t_1 + t_0)$ instead of the ball Ω_1. The new Lipschitz constants $L_0^1, L_1^1, L_2^1, L_3^1$ will be at least as small as L_0, L_1, L_2, L_3, respectively, since $\Omega_1^* \subset \Omega$ leading to even weaker sufficient semilocal convergence criteria, tighter error bounds on the distances involved and an at least as precise information on the location of the solution.

3. Local convergence

In this section, we present the local convergence of Kurchatov method (1.2).

Theorem 3.1. *Let F be a continuous nonlinear operator defined on an open subset Ω of a Banach space $\mathbb{B}_1$ with values in a Banach space $\mathbb{B}_2$. Suppose:*

(a) *equation $F(x) = 0$ has a solution $x^\star \in \Omega$ at which the Fréchet derivative exists and is invertible;*

(b) *there exist $M_i \geq 0, i = 0, 1, 2,$ $M \geq 0$ with $M_0 + M_1 + M_2 + M > 0$ such that the divided differences of order one and two of F on $\Omega_0 \subseteq \Omega$ satisfy the following Lipschitz conditions:*

$$\begin{aligned} &\|F'(x^\star)^{-1}([x, y; F] - F'(x^\star))\| \\ &\leq M_0\|x - x^\star\| + M_1\|y - x^\star\| \quad \text{for each } x, y \in \Omega_0. \end{aligned} \tag{3.1}$$

Let

$$\bar{r}^* = \frac{1}{3M_0 + M_1}.$$

Moreover, suppose

$$\begin{aligned} &\|F'(x^\star)^{-1}([x, x^\star; F] - [y, x^\star; F])\| \\ &\leq M_2\|x - y\| \quad \text{for each } x, y \in \Omega_2 := \Omega_0 \cap U(x^\star, \bar{r}^*), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\|F'(x^\star)^{-1}([x, v, y; F] - [x, z, y; F])\| \\ &\leq M\|v - z\| \quad \text{for each } x, y, z, v \in \Omega_2; \end{aligned} \tag{3.3}$$

(c) *the ball*

$$U^\star = U(x^\star, 3r^\star) \subseteq \Omega_0, \tag{3.4}$$

where, $r^\star$ is the unique positive root of polynomial q defined by

$$q(t) = 4M_2t^3 + 2Mt^2 + (3M_0 + M_1)t - 1. \tag{3.5}$$

Then, the sequence $\{x_n\}$ generated by method (1.2) is well defined, remains in $U(x^\star, r^\star)$ for each $n = 0, 1, 2, \cdots$ and converges to $x^\star$ provided that

$$x_{-1}, x_0 \in U(x^\star, r^\star). \tag{3.6}$$

Moreover, the following estimates hold for $n \geq 0$:

$$\begin{aligned} &\|x_{n+1} - x^\star\| \\ &\leq \frac{M\|x_{n-1} - x^\star\|\|x_n - x_{n-1}\| + M_2\|x_n - x_{n-1}\|^2\|x_n - x^\star\|}{1 - M_0(\|x_n - x^\star\| + \|x_n - x_{n-1}\|) - M_1\|x_{n-1} - x^\star\|}\|x_n - x^\star\| \\ &\leq \|x_n - x^\star\| < r^\star. \end{aligned} \tag{3.7}$$

Furthermore, if there exists $R^\star \in [r^\star, \frac{1}{M_0})$ $(M_0 \neq 0)$ such that $\overline{U}(x^\star, R^\star) \subseteq \Omega_0$, then the limit point $x^\star$ is the only solution of equation $F(x) = 0$ in $\overline{U}(x^\star, R^\star)$. If $M_0 = 0$, $x^\star$ is unique in $U(x^\star, r^\star)$.

Remark 3.2. If $M + M_2 > 0$, the estimates (3.7) means that sequence $\{x_n\}$ converges to $x^\star$ quadratically. In fact, by (3.7), we have that the following estimates holds for all $n \geq 0$:

$$\begin{aligned}
&\|x_{n+1} - x^\star\| \\
&\leq \frac{M\|x_{n-1}-x^\star\|\|x_n-x^\star\|+M\|x_{n-1}-x^\star\|^2+M_2(\|x_n-x^\star\|+\|x_{n-1}-x^\star\|)^2\|x_n-x^\star\|}{1 - M_0(2\|x_n - x^\star\| + \|x_{n-1} - x^\star\|) - M_1\|x_{n-1} - x^\star\|} \\
&\quad \times \|x_n - x^\star\| \\
&\leq \frac{Mr^\star\|x_n - x^\star\|^2 + M\|x_n - x^\star\|\|x_{n-1} - x^\star\|^2 + 4M_2{r^\star}^2\|x_n - x^\star\|^2}{1 - (3M_0 + M_1)r^\star},
\end{aligned} \tag{3.8}$$

which means

$$\begin{aligned}
&\frac{\|x_{n+1} - x^\star\|}{r^\star} \\
&\leq \frac{(M{r^\star}^2 + 4M_2{r^\star}^3)(\frac{\|x_n-x^\star\|}{r^\star})^2 + M{r^\star}^2\frac{\|x_n-x^\star\|}{r^\star}(\frac{\|x_{n-1}-x^\star\|}{r^\star})^2}{1 - (3M_0 + M_1)r^\star}, \quad n \geq 0.
\end{aligned} \tag{3.9}$$

If we drop the divided difference of order two from the hypotheses, we can show two more local convergence results along the same lines of Theorem 3.1. In the next result, we suppose the divided difference of order one $[x, y; F]$ can be expression as

$$[x, y; F] = \int_0^1 F(tx + (1 - t)y)dt, \quad \text{for } x, y \in \Omega_0 \subseteq \Omega,$$

which holds in many cases [2, 6].

Theorem 3.3. *Let $F : \Omega \subseteq \mathbb{B}_1 \to \mathbb{B}_2$ be a Fréchet-differentiable operator. Suppose:*

(a) *equation $F(x) = 0$ has a solution $x^\star \in \Omega$ at which the Fréchet-derivative exists and is invertible;*

(b) *there exist $N_i \geq 0, i = 0, 1$, $N \geq 0$ with $N_0 + N_1 + N > 0$ such that the divided difference of order one of F on $\Omega_0 \subseteq \Omega$ satisfies the following Lipschitz conditions*

$$\|F'(x^\star)^{-1}(F'(x) - F'(x^\star))\| \leq N_0\|x - x^\star\| \quad \textit{for each } x \in \Omega_0,$$

Let $0 < \bar{r}_0^* = \frac{1}{N_0} - \frac{1}{2}\|x_{-1} - x^\star\|$. For each $x, y \in \Omega_0 \cap U(x^\star, \bar{r}_0^*)$,

$$\|F'(x^\star)^{-1}([x, y; F] - [x, x^\star; F])\| \leq N\|y - x^\star\|,$$
$$\|F'(x^\star)^{-1}([x, x^\star; F] - [y, x^\star; F])\| \leq N_1\|x - y\|;$$

(c) *the ball*

$$U(x^\star, 3r_0^\star) \subseteq \Omega_0,$$

where,

$$r_0^\star = \frac{1}{\frac{3}{2}N_0 + N + 2N_1}.$$

Then, the sequence $\{x_n\}$ *generated by method* (1.2) *is well defined, remains in* $U(x^\star, r_0^\star)$ *for all* $n \geq 0$ *and converges to* $x^\star$ *provided that*

$$x_{-1}, x_0 \in U(x^\star, r_0^\star).$$

Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$*:*

$$\|x_{n+1} - x^\star\| \leq \frac{N\|x_{n-1} - x^\star\| + N_1\|x_n - x_{n-1}\|}{1 - N_0(\|x_n - x^\star\| + \frac{1}{2}\|x_{n-1} - x^\star\|)}\|x_n - x^\star\|. \tag{3.10}$$

Furthermore, if there exists $R_0^\star \in [r_0^\star, \frac{1}{N_1})$, $N_1 \neq 0$ *such that* $\overline{U}(x^\star, R_0^\star) \subseteq \Omega_0$, *then the limit point* $x^\star$ *is the only solution of equation* $F(x) = 0$ *in* $\overline{U}(x^\star, R_0^\star)$. *If* $N_1 = 0$, $x^\star$ *is unique in* $U(x^\star, r_0^\star)$.

Theorem 3.4. *Let* $F : \Omega \subseteq \mathbb{B}_1 \to \mathbb{B}_2$ *be a continuous operator. Suppose:*

(a) *equation* $F(x) = 0$ *has a solution* $x^\star \in \Omega$ *at which the Fréchet-derivative exists and is invertible;*

(b) *there exist* $H_i \geq 0, i = 0, 1, 2,$ $H \geq 0$ *with* $H_0 + H_1 + H_2 + H > 0$ *such that the divided difference of order one of* F *on* $\Omega_0 \subseteq \Omega$ *satisfies following Lipschitz conditions*

$$\|F'(x^\star)^{-1}([x, y; F] - F'(x^\star))\|$$
$$\leq H_0\|x - x^\star\| + H\|y - x^\star\| \quad for\ each\ \ x, y \in \Omega_0,$$

For $\bar{\bar{r}}^* = \frac{1}{3H_0 + H}$

$$\|F'(x^\star)^{-1}([y, x^\star; F] - [2y - x, x; F])\|$$
$$\leq H_1\|y - x\| + H_2\|x - x^\star\| \quad for\ each\ \ x, y \in \Omega_3 := \Omega_0 \cap U(x^\star, \bar{\bar{r}}^*);$$

(c) *the ball*

$$U(x^\star, 3r_1^\star) \subseteq \Omega_0,$$

where

$$r_1^\star = \frac{1}{H + 3H_0 + 2H_1 + H_2}.$$

Then the sequence $\{x_n\}$ *generated by method* (1.2) *is well defined, remains in* $U(x^\star, r_1^\star)$ *for all* $n \geq 0$ *and converges to* $x^\star$ *provided that*

$$x_{-1}, x_0 \in U(x^\star, r_1^\star).$$

Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$*:*

$$\|x_{n+1} - x^\star\| \leq \frac{H_1\|x_n - x_{n-1}\| + H_2\|x_{n-1} - x^\star\|}{1 - (H_0\|2x_n - x_{n-1} - x^\star\| + H\|x_{n-1} - x^\star\|)}\|x_n - x^\star\|.$$

Furthermore, if there exists $R_1^\star \in [r_1^\star, \frac{1}{H_0+H})$, $H_0+H \neq 0$ *such that* $\overline{U}(x^\star, R_1^\star) \subseteq \Omega_0$, *then the limit point* $x^\star$ *is the only solution of equation* $F(x) = 0$ *in* $\overline{U}(x^\star, R_1^\star)$. *If* $H_0 = H = 0$, $x^\star$ *is unique in* $U(x^\star, r_1^\star)$.

Remark 3.5. A comment similar to the one in Remark 3.2 (b) and (c) can now follow.

4. Numerical examples

In this section, we present some numerical examples.

Example 4.1. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}$, $\Omega_0 = \Omega = (-1, 1)$ and define F on Ω by

$$F(x) = e^x - 1. \tag{4.1}$$

Then, $x^\star = 0$ is a solution of (1.1) and $F'(x^\star) = 1$. Note that for any $x, y \in \Omega_0$, we have

$$\begin{aligned}
&|F'(x^\star)^{-1}([x, y; F] - [x^\star, x^\star; F])| = |\int_0^1 F'(tx + (1-t)y)dt - F'(x^\star)| \\
&= |\int_0^1 (e^{tx+(1-t)y} - 1)dt| \\
&= |\int_0^1 (tx + (1-t)y)(1 + \tfrac{tx+(1-t)y}{2!} + \tfrac{(tx+(1-t)y)^2}{3!} + \cdots)dt| \\
&\leq |\int_0^1 (tx + (1-t)y)(1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots)dt| \\
&\leq \tfrac{e-1}{2}(|x - x^\star| + |y - x^\star|),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&|F'(x^\star)^{-1}([x, x^\star; F] - [y, x^\star; F])| \\
&= |\int_0^1 F'(tx + (1-t)x^\star)dt - \int_0^1 F'(ty + (1-t)x^\star)dt| \\
&= |\int_0^1 (e^{tx+(1-t)x^\star} - e^{ty+(1-t)x^\star})dt| \\
&= |\int_0^1 (t(x-y) + \tfrac{t^2(x^2-y^2)}{2!} + \tfrac{t^3(x^3-y^3)}{3!} + \cdots)dt| \\
&= |\int_0^1 t(x-y)(1 + \tfrac{t(x+y)}{2!} + \tfrac{t^2(x^2+xy+y^2)}{3!} + \cdots)dt| \\
&\leq \int_0^1 t(1 + t + \tfrac{t^2}{2!} + \cdots)dt|x-y| = \int_0^1 te^t dt|x-y| = |x-y|
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&|F'(x^\star)^{-1}([x, v, y; F] - [x, z, y; F])| \\
&= |[x, v, z, y; F](v-z)| = |\tfrac{F'''(\xi)}{3!}(v-z)| \\
&= |\tfrac{e^\xi}{6}(v-z)| \leq \tfrac{e}{6}|v-z| \quad \textit{for some } \ \xi \in \Omega_2.
\end{aligned} \tag{4.4}$$

Then we can choose $M_0 = M_1 = \frac{e-1}{2}$, $M_2 = 1$ and $\bar{M} = \frac{e}{6}$ in the old Theorem 3.1. By (3.5), we get $r_{old}^\star \approx 0.254664790$ and $U(x^\star, 3r^\star) \subseteq \Omega_0$ holds. That is, all conditions in Theorem 3.1 are satisfied and Theorem 3.1 applies. Note that

$$
\begin{aligned}
&|F'(x^\star)^{-1}([x,y;F]-[z,u;F])| \\
&= |\int_0^1 (F'(tx+(1-t)y)-F'(tz+(1-t)u))dt| \\
&= |\int_0^1\int_0^1 (F''\big(\theta(tx+(1-t)y) \\
&\quad +(1-\theta)(tz+(1-t)u)\big)\big(tx+(1-t)y-(tz+(1-t)u)\big)d\theta dt| \qquad (4.5)\\
&= |\int_0^1\int_0^1 (e^{\theta(tx+(1-t)y)+(1-\theta)(tz+(1-t)u)}\big(tx+(1-t)y-(tz+(1-t)u)\big)d\theta dt| \\
&\le \int_0^1 e|t(x-z)+(1-t)(y-u)|dt \\
&\le \frac{e}{2}(|x-z|+|y-u|),
\end{aligned}
$$

then if we only use condition (4.5) instead of conditions (3.1) and (3.2) in old Theorem 3.1, we get $M_0 = M_1 = M_2 = \frac{e}{2}$, $\bar{M} = \frac{e}{6}$ and the small radius $\overline{r}^\star \approx 0.136065472$ of convergence ball for method (1.2) than $r^\star_{old}$.

However, if we use the new Theorem 3.1 with $M = \frac{e^{\frac{1}{2(e-1)}}}{6} < \bar{M}$, we obtain $r^*_{new} = 0.25633521748772$ and $\bar{r}^*_{new} > \bar{r}^*_{old}$. Let us choose $x_{-1} = 0.136, x_0 = 0.128$. Suppose sequences $\{x_n\}$ is generated by method (1.2). Tables give a comparison results of error estimates for Example 4.1, which shows that tighter error estimates can be obtained from (3.7) by using both condition (3.1) and condition (3.2) instead of by using only the condition (4.5). Moreover, the new results are more precise than the old ones. Notice that these advantages are obtained under the same computational cost, since in practice the computation of the old Lipschitz constants requires the Computation of the new constants as special cases.

TABLE 1. The comparison old results of error estimates for Example 4.1

n	the first estimates of (3.7) by using both conditions (3.1) and (3.2)	the first estimates of (3.7) by using only condition (4.5) instead of (3.1) and (3.2)
0	8.37016E-05	0.000102359
1	7.12968E-05	8.5783E-05
2	1.39545E-09	1.40671E-09
3	1.92636E-18	1.92646E-18

TABLE 2. The comparison new results of error estimates for Example 4.1

n	the first estimates of (3.7) by using both conditions (3.1) and (3.2)	the first estimates of (3.7) by using only condition (4.5) instead of (3.1) and (3.2)
0	7.2884e-05	8.9207e-05
1	6.2079e-05	7.4756e-05
2	1.2121e-09	1.2219e-09
3	1.6733e-18	1.6733e-18

Example 4.2. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}$, $\Omega_0 = \Omega = (0.72, 1.28)$ and define F on Ω by

$$F(x) = x^3 - 0.75. \tag{4.6}$$

Let $x_{-1} = 0.95$, $x_0 = 1$ be two initial points for the Kurchatov method (1.2). Then, we have $A_0 = 3.0025$, $x_1 \approx 0.916736053$, $t_0 = 0.05$ and $t_1 = t_0 + |A_0^{-1}F(x_0)| \approx 0.133263947$. Note that, for any $x, y, z, v \in \Omega$, we have

$$\begin{aligned}
&|A_0^{-1}([x, y; F] - [z, v; F])| \\
&= |A_0^{-1}(x^2 + xy + y^2 - (z^2 + zv + v^2))| \\
&= |A_0^{-1}((x + z)(x - z) + xy - xv + xv - zv + (y + v)(y - v))| \\
&= |A_0^{-1}((x + z + v)(x - z) + (x + y + v)(y - v))| \\
&\le |A_0^{-1}|(|x + z + v||x - z| + |x + y + v||y - v|).
\end{aligned} \tag{4.7}$$

Then condition (2.5) in Theorem 2.3 holds for constants $\bar{L}_2 = \bar{L}_3 = L_2 = L_3 = 3 \times 1.28 \times |A_0^{-1}| \approx 1.278934221$. Setting $z = 2x_0 - x_{-1}, v = x_{-1}$ in (4.7), we deduce that condition (2.4) holds for $L_0 = 3.28 \times |A_0^{-1}| \approx 1.092422981$ and $L_1 = 3.51 \times |A_0^{-1}| \approx 1.169025812$. Furthermore, setting $x = 2x_1 - x_0, y = x_0, z = 2x_0 - x_{-1}, v = x_{-1}$ in (4.7), we deduce that the second condition of (2.3) holds for $L_{-1} = |2x_1 + x_0| \times |A_0^{-1}| \approx 0.943704282$ and $L = |2x_1 + x_{-1}| \times |A_0^{-1}| \approx 0.927051493$.

Using method (1.2), we get that $t_2 \approx 0.155936165$, $t_3 \approx 0.164388084$, $t_4 \approx 0.165312739$, $t_5 \approx 0.165346407$, $t_6 \approx 0.165346537$ and $t_7 \approx 0.165346537$. That is to say, we have $t^\star \approx 0.165346537$. Then, we have $r_0 = \max(2(t_1 - t_0), t^\star - t_0) \approx 0.166527893$, and $\overline{U}(x_0, r_0) \approx (0.833472107, 1.166527893) \subset \Omega$.

Next, we verify that all conditions of Lemma 1.2 hold. In fact, by the definition of polynomial p, we get that $\alpha \approx 0.737640298$. We also have

$$0 < \frac{L_0(t_1 - t_{-1}) + L_1(t_0 - t_{-1})}{1 - (L_{-1}(2t_1 - t_0) + Lt_0)} \approx 0.272293345 \le \alpha,$$

$$0 < \frac{L_2(t_2 - t_0) + L_3(t_1 - t_0)}{1 - (L_0(2t_2 - t_1) + L_1 t_1)} \approx 0.372787434 \le \alpha$$

and

$$0 < \alpha \le 1 - \frac{(L_0 + L_1)(t_1 - t_0)}{1 - (L_0 + L_1)t_0} \approx 0.787697259.$$

By now, we see that all conditions of Theorem 2.3 are satisfied, so Theorem 2.3 applies. However, we will show some condition of [13, Theorem 3.1] is not satisfied and so it does not apply. In fact, we can obtain constants in [13, Theorem 3.1] as follows:

$$\begin{aligned}
&a = 0.05,\ c \approx 0.083263947,\ p_0 \approx 1.278934221,\ q_0 \approx 0.333055787, \\
&s \approx 1.636117697,\ r \approx 0.340814461,\ r_0 \approx 0.095245569.
\end{aligned}$$

Then the condition $V_0 = U(x_0, 3r_0) \approx (0.714263293, 1.285736707) \subset \Omega$ of [13, Theorem 3.1] is not satisfied.

Next we will give two examples with multivariable. For $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m (m > 1)$, we define $\|x\| = \|x\|_\infty = \max_{1\le i\le m} |x_i|$ and the corresponding norm on $A = (a_{ij}) \in \mathbb{R}^{m\times m}$ as

$$||A|| = \max_{1\le i\le m} \sum_{j=1}^{m} |a_{ij}|.$$

The divided difference $[u, v; F]$ of operator $F = (F_1, F_2, ..., F_m)^t$ at the points $u = (u_1, u_2, \cdots, u_m)^t, v = (v_1, v_2, \cdots, v_m)^t \in \mathbb{R}^m$ is an matrix in $\mathbb{R}^{m\times m}$ and its entries can be expressed by [8]

$$\begin{aligned}[u, v; F]_{ij} = &\frac{1}{u_j - v_j} (F_i(u_1, \cdots, u_j, v_{j+1}, \cdots, v_m) \\ &-F_i(u_1, \cdots, u_{j-1}, v_j, \cdots, v_m)) \ \text{ for } \ 1 \le i, j \le m.\end{aligned}$$

Example 4.3. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^3$, $\Omega_0 = \Omega = (-1, 1)^3$ and define $F = (F_1, F_2, F_3)^t$ on Ω by

$$F(x) = F(x_1, x_2, x_3) = (e^{x_1} - 1, {x_2}^2 + x_2, x_3)^t. \tag{4.8}$$

For the points $u = (u_1, u_2, u_3)^t, v = (v_1, v_2, v_3)^t \in \Omega$, we get

$$[u, v; F] = \begin{pmatrix} \frac{e^{u_1}-e^{v_1}}{u_1-v_1} & 0 & 0 \\ 0 & u_2 + v_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\overline{x}_{-1} = (0.1, 0.1, 0.1)^t$, $\overline{x}_0 = (0.11, 0.11, 0.11)^t$ be two initial points for the Kurchatov method (1.2). Here, we use $\overline{x}_n$ instead of x_n to distinct iterative points with its component for some integer $n \ge -1$. Then, we have

$$2\overline{x}_0 - \overline{x}_{-1} = (0.12, 0.12, 0.12), \quad t_0 = 0.01,$$

$$A_0 \approx \begin{pmatrix} 1.116296675 & 0 & 0 \\ 0 & 1.22 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ A_0^{-1} \approx \begin{pmatrix} 0.895819205 & 0 & 0 \\ 0 & 0.819672131 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$t_1 = t_0 + \|A_0^{-1} F(\overline{x}_0)\| = 0.12, \quad \overline{x}_1 \approx (0.005835871, 0.009918033, 0).$$

Note that, for any

$$x = (x_1, x_2, x_3)^t, \ \ y = (y_1, y_2, y_3)^t, \ \ z = (z_1, z_2, z_3)^t, \ \ v = (v_1, v_2, v_3)^t \in \Omega,$$

we have

$$[x, y; F] - [z, v; F] = \begin{pmatrix} \frac{e^{x_1}-e^{y_1}}{x_1-y_1} - \frac{e^{z_1}-e^{v_1}}{z_1-v_1} & 0 & 0 \\ 0 & x_2 + y_2 - z_2 - v_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.9}$$

In view of

$$\begin{aligned}
&|\tfrac{e^{x_1}-e^{y_1}}{x_1-y_1}-\tfrac{e^{z_1}-e^{v_1}}{z_1-v_1}| = |\int_0^1 (e^{y_1+t(x_1-y_1)}-e^{v_1+t(z_1-v_1)})dt|\\
&= |\int_0^1\int_0^1 e^{v_1+t(z_1-v_1)+\theta(y_1+t(x_1-y_1)-v_1-t(z_1-v_1))}\\
&\quad \times\big(y_1+t(x_1-y_1)-v_1-t(z_1-v_1)\big)d\theta dt|\\
&\leq \int_0^1\int_0^1 e|t(x_1-z_1)+(1-t)(y_1-v_1)|d\theta dt\\
&\leq \tfrac{e}{2}(|x_1-z_1|+|y_1-v_1|),
\end{aligned}$$

we have

$$\begin{aligned}
&\|A_0^{-1}([x,y;F]-[z,v;F])\|\\
&\leq \max(\tfrac{e\times 0.895819205}{2}(|x_1-z_1|+|y_1-v_1|), 0.819672131(|x_2-z_2|+|y_2-v_2|))\\
&\leq \max(\tfrac{e\times 0.895819205}{2}(\|x-z\|+\|y-v\|), 0.819672131(\|x-z\|+\|y-v\|))\\
&= \tfrac{e\times 0.895819205}{2}(\|x-z\|+\|y-v\|).
\end{aligned} \tag{4.10}$$

In particular, set $z = 2\overline{x}_0 - \overline{x}_{-1}$ and $v = \overline{x}_{-1}$ in (4.10), we have

$$\begin{aligned}
&\|A_0^{-1}([x,y;F]-A_0)\|\\
&\leq \tfrac{e\times 0.895819205}{2}(\|x-(2\overline{x}_0-\overline{x}_{-1})\|+\|y-\overline{x}_{-1}\|).
\end{aligned} \tag{4.11}$$

That is, we can choose constants $L_0 = L_1 = L_2 = L_3 = \bar{L}_2 = \bar{L}_3 \approx \frac{e\times 0.895819205}{2} \approx 1.217544533$ in Theorem 2.3.

By the definition of A_1, we have

$$A_1 \approx \begin{pmatrix} 1.007672865 & 0 & 0 \\ 0 & 1.019836066 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\|A_0^{-1}(A_1-A_0)\| \approx 0.384068799.$$

Since $\|2\overline{x}_1 - \overline{x}_0 - (2\overline{x}_0 - \overline{x}_{-1})\| = 0.23$ and $\|\overline{x}_0 - \overline{x}_{-1}\| = 0.01$, the following inequlity holds

$$\begin{aligned}
\|A_0^{-1}(A_1-A_0)\| &\approx 0.384068799\\
&\leq L_{-1}\|2\overline{x}_1-\overline{x}_0-(2\overline{x}_0-\overline{x}_{-1})\|+L\|\overline{x}_0-\overline{x}_{-1}\|
\end{aligned} \tag{4.12}$$

provided that we choose $L_{-1} = L \approx \frac{0.384068799}{0.24} \approx 1.600286661$. So, the second inequlity of (2.3) is true. Using method (1.2), we get that $t_2 \approx 0.148267584$, $t_3 \approx 0.161640408$, $t_4 \approx 0.163517484$, $t_5 \approx 0.163626179$ and $t_6 \approx 0.163627029$. That is to say, we have $t^\star \approx 0.163627029$. Then, we have $r_0 = \max(2(t_1 - t_0), t^\star - t_0) = 0.22$, and $\overline{U}(x_0, r_0) = (-0.11, 0.33)^3 \subset \Omega$.

Next, we verify that all conditions of Lemma 2.1 hold. In fact, by the definition of polynomial p, we get that $\alpha \approx 0.722714177$. We also have

$$0 < \frac{L_0(t_1-t_{-1})+L_1(t_0-t_{-1})}{1-(L_{-1}(2t_1-t_0)+Lt_0)} \approx 0.256978034 \leq \alpha,$$

$$0 < \frac{L_2(t_2 - t_0) + L_3(t_1 - t_0)}{1 - (L_0(2t_2 - t_1) + L_1 t_1)} \approx 0.473079844 \leq \alpha$$

and

$$0 < \alpha \leq 1 - \frac{(L_0 + L_1)(t_1 - t_0)}{1 - (L_0 + L_1)t_0} \approx 0.725454782.$$

By now, we see that all conditions of Theorem 2.3 are satisfied, so Theorem 2.3 applies.

Example 4.4. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^6$, $\Omega_0 = \Omega = (-1, 1)^6$ and define $F = (F_1, F_2, \cdots, F_6)^t$ on Ω by

$$\begin{aligned} F(x) &= F(x_1, x_2, \cdots, x_6) \\ &= \Bigg(\sum_{k=1}^{6} dx_k - dx_1 + e^{x_1} - 1,\ \sum_{k=1}^{6} dx_k - dx_2 + e^{x_2} - 1, \\ &\qquad \cdots,\ \sum_{k=1}^{6} dx_k - dx_6 + e^{x_6} - 1 \Bigg)^t, \end{aligned} \tag{4.13}$$

where, $d \in \mathbb{R}$ is a constant. For the points $u = (u_1, u_2, \cdots, u_6)^t$, $v = (v_1, v_2, \cdots, v_6)^t \in \Omega$, we get

$$[u, v; F] = \begin{pmatrix} \frac{e^{u_1}-e^{v_1}}{u_1-v_1} & d & d & d & d & d \\ d & \frac{e^{u_2}-e^{v_2}}{u_2-v_2} & d & d & d & d \\ d & d & \frac{e^{u_3}-e^{v_3}}{u_3-v_3} & d & d & d \\ d & d & d & \frac{e^{u_4}-e^{v_4}}{u_4-v_4} & d & d \\ d & d & d & d & \frac{e^{u_5}-e^{v_5}}{u_5-v_5} & d \\ d & d & d & d & d & \frac{e^{u_6}-e^{v_6}}{u_6-v_6} \end{pmatrix}.$$

Let $\overline{x}_{-1} = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)^t$, $\overline{x}_0 = (0.12, 0.12, 0.12, 0.12, 0.12, 0.12)^t$ be two initial points for the Kurchatov method (1.2). Here, we use $\overline{x}_n$ instead of x_n to distinct iterative points with its component for some integer $n \geq -1$. Then, we have

$$2\overline{x}_0 - \overline{x}_{-1} = (0.14, 0.14, 0.14, 0.14, 0.14, 0.14), \quad t_0 = 0.02,$$

$$A_0 = \begin{pmatrix} c_1 & d & d & d & d & d \\ d & c_1 & d & d & d & d \\ d & d & c_1 & d & d & d \\ d & d & d & c_1 & d & d \\ d & d & d & d & c_1 & d \\ d & d & d & d & d & c_1 \end{pmatrix},\ A_0^{-1} = c_3 \begin{pmatrix} c_2 & d & d & d & d & d \\ d & c_2 & d & d & d & d \\ d & d & c_2 & d & d & d \\ d & d & d & c_2 & d & d \\ d & d & d & d & c_2 & d \\ d & d & d & d & d & c_2 \end{pmatrix},$$

where, $c_1 = \frac{e^{0.14}-e^{0.1}}{0.14-0.1} \approx 1.12757202$, $c_2 = -4d - c_1, c_3 = \frac{1}{(d-c_1)(5d+c_1)}$. Set $d = 3$, we have $c_2 \approx -13.12757202$ and $c_3 \approx 0.033115086$. Hence, we have

$$\begin{aligned} A_0^{-1}F(\overline{x}_0) &\approx (0.119515625, 0.119515625, 0.119515625, \\ &\qquad 0.119515625, 0.119515625, 0.119515625), \\ t_1 &= t_0 + \|A_0^{-1}F(\overline{x}_0)\| \approx 0.139515625, \\ \overline{x}_1 &\approx (0.000484375, 0.000484375, 0.000484375, \\ &\qquad 0.000484375, 0.000484375, 0.000484375). \end{aligned}$$

Note that, for any $x = (x_1, x_2, \cdots, x_6)^t, y = (y_1, y_2, \cdots, y_6)^t, z = (z_1, z_2, \cdots, z_6)^t$, $v = (v_1, v_2, \cdots, v_6)^t \in \Omega$, we have

$$[x, y; F] - [z, v; F] = \begin{pmatrix} c_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \tag{4.14}$$

where

$$c_{kk} = \frac{e^{x_k} - e^{y_k}}{x_k - y_k} - \frac{e^{z_k} - e^{v_k}}{z_k - v_k}, \quad k = 1, 2, \cdots, 6.$$

Similarily as the last example, we have

$$|c_{kk}| \leq \tfrac{e}{2}(|x_k - z_k| + |y_k - v_k|) \leq \tfrac{e}{2}(\|x - z\| + \|y - v\|), \quad k = 1, 2, \cdots, 6$$

and

$$\|A_0^{-1}([x, y; F] - [z, v; F]) \leq \tfrac{e|c_3|(|c_2|+5|d|)}{2}(\|x - z\| + \|y - v\|). \tag{4.15}$$

That is, we can choose constants

$$L_0 = L_1 = L_2 = L_3 = \bar{L}_2 = \bar{L}_3 \approx \frac{e|c_3|(|c_2| + 5|d|)}{2} \approx 1.265967683$$

in Theorem 2.3. By the definition of A_1, we have

$$A_1 \approx \begin{pmatrix} c_4 & d & d & d & d & d \\ d & c_4 & d & d & d & d \\ d & d & c_4 & d & d & d \\ d & d & d & c_4 & d & d \\ d & d & d & d & c_4 & d \\ d & d & d & d & d & c_4 \end{pmatrix},$$

where, $c_4 \approx 1.002868011$. Then we have

$$\|A_0^{-1}(A_1 - A_0)\| \approx 0.11615517.$$

Since $\|2\overline{x}_1 - \overline{x}_0 - (2\overline{x}_0 - \overline{x}_{-1})\| \approx 0.25903125$ and $\|\overline{x}_0 - \overline{x}_{-1}\| = 0.02$, the following inequlity holds

$$\begin{aligned} \|A_0^{-1}(A_1 - A_0)\| &\approx 0.11615517 \\ &\leq L_{-1}\|2\overline{x}_1 - \overline{x}_0 - (2\overline{x}_0 - \overline{x}_{-1})\| + L\|\overline{x}_0 - \overline{x}_{-1}\| \end{aligned} \tag{4.16}$$

provided that we choose $L_{-1} = L \approx \frac{0.11615517}{0.27903125} \approx 0.416280148$. So, the second inequlity of (2.3) is true.

Using method (1.2), we get that $t_2 \approx 0.139034507$, $t_3 \approx 0.138810276$, $t_4 \approx 0.138810795$ and $t_5 \approx 0.138810795$. That is to say, we have $t^\star \approx 0.138810795$. Then we have $r_0 = \max(2(t_1 - t_0), t^\star - t_0) \approx 0.23903125$, and $\overline{U}(x_0, r_0) \approx (-0.11903125, 0.35903125)^6 \subset \Omega$.

Next, we verify that all conditions of Lemma 2.1 hold. In fact, by the definition of polynomial p, we get that $\alpha \approx 0.722714177$. We also have

$$0 < \frac{L_0(t_1 - t_{-1}) + L_1(t_0 - t_{-1})}{1 - (L_{-1}(2t_1 - t_0) + Lt_0)} \approx 0.196138868 \leq \alpha,$$

$$0 < \frac{L_2(t_2 - t_0) + L_3(t_1 - t_0)}{1 - (L_0(2t_2 - t_1) + L_1 t_1)} \approx 0.42698869 \leq \alpha$$

and

$$0 < \alpha \leq 1 - \frac{(L_0 + L_1)(t_1 - t_0)}{1 - (L_0 + L_1)t_0} \approx 0.734593002.$$

By now, we see that all conditions of Theorem 2.3 are satisfied, so Theorem 2.3 applies.

Note that, we can verify that other choices of parameter d such as $d = 5$ and $d = 10$ are also suitable for this example.

References

[1] S. Amat, S. Busquier and M. Negra, *Adaptive approximation of nonlinear operators*, Numer. Funct. Anal. Optim., **25** (2004), 397–405.

[2] I.K. Argyros, *Computational theory of iterative methods*, Series: Studies in Computational Mathematics 15, Editors, C.K. Chui and L. Wuytack, Elservier Publ. Co. New York, USA, 2007.

[3] I.K. Argyros, *On a two point Newton-like methods of convergent $R-$order two*, Intern. J. Comput. Math., **82** (2005), 219–233.

[4] I.K. Argyros, *A Kantororvich-type analysis for a fast iterative method for solving nonlinear equations*, J. Math. Anal. Applic., **332** (2007), 97–108.

[5] I.K. Argyros, *On a quadratically convergent iterative method using divided differences of order one*, J. Korean. Math. S.M.E. Ser. B, **14**(3) (2007), 203–221.

[6] I.K. Argyros and S. Hilout, *Computational methods in nonlinear analysis. Efficient algorithms, fixed point theory and applications*, World Scientific, 2013.

[7] I.K. Argyros and S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity, **28** (2012), 364–387.

[8] I.K. Argyros and H. Ren, *On the Kurchatov method for solving equations under weak conditions*, Journal Applied Mathematics and Computation, **273** (2016), 98–113.

[9] J.A. Ezquerro, A. Grau, M. Grau-Sanchez and M.A. Hernández, *On the efficiency of two variants of Kurchatov's method for solving nonlinear equations*, Numer. Algor., **64** (2013), 685–698.

[10] L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[11] V.A. Kurchatov, *On a method of linear interpolation for the solution of functional equations*. Dokl. Akad. Nauk. SSSR (Russian) **198**(3) (1971), 524–526. Translation in Soviet Math. Dokl., **12** (1974), 835–838.

[12] H. Ren, Q. Wu and W. Bi, *On convergence of a new secant like method for solving nonlinear equations*, Appl. Math. Comput., **217** (2010), 583–589.

[13] W.C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations*, Polish Academy of Science, Banach Ctr. Publ., **3** (1977), 129–142.

[14] S.M. Shakhno, *On a Kurchatov's method of linear interpolation for solving nonlinear equations*, PAMM-Proc. Appl. Math. Mech., **4** (2004), 650–651.

[15] J.F. Traub, *Iterative Methods for the Solution of Equations*, Englewood Cliffs, Prentice Hull, 1984.