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POSITIVE SOLUTIONS FOR ASYMPTOTICALLY LINEAR BIHARMONIC PROBLEMS

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Abstract. In this article, we consider the elliptic problem $\Delta^2 u = \lambda f(u)$, where $\lambda > 0$ and f is asymptotically linear, in the unit ball of $\mathbb{R}^N$, $N \geq 2$, with Dirichlet Boundary conditions. We introduce the notion of semi-stability for a solution. We determine the existence and nonexistence results for positive solutions, uniqueness and regularity. We also study a type of semi-stable solutions called extremal solutions. Moreover, we characterize the nonexistence of singular extremal solutions.

1. Introduction

Let $B \subset \mathbb{R}^N$ be the unit ball of $\mathbb{R}^N$ (centered at the origin), $N \geq 2$. We consider the following biharmonic problem

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u = \lambda f(u) & \text{in } B, \\ u > 0 & \text{in } B, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

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where λ is a positive parameter, $\dfrac{\partial u}{\partial n}$ denotes the outer normal derivative of u on ∂B and the nonlinearity f satisfies the following assumptions:

$$f \text{ is in } C^1[0,+\infty), \text{ positive, nondecreasing and convex function.} \tag{1.1}$$

The biharmonic equations are an important class of equations in both physics and engineering. In fluid dynamics, the so-called stream function satisfies the biharmonic equation and many problems in elasticity can be formulated in terms of the biharmonic equation. The choice of the boundary conditions is usually related to the physical problem described by the equation, see [16]. Also the equation (P_λ) is the natural fourth order analogue of the classical Gelfand problem

$$\begin{cases} -\Delta u &= \lambda f(u) & \text{in} & \Omega, \\ u &> 0 & \text{in} & \Omega, \\ u &= 0 & \text{on} & \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, for which a vast literature exists, we can cite e.g. [4, 5, 7, 8, 10, 11, 13, 14] and references therein. One of the basic tools in the analysis of (1.2) is the maximum principle. As pointed out in [3], in general domain the maximum principle for bi-Laplacian with Dirichlet boundary condition is not valid any more. One of the reasons to study (P_λ) in a ball is that a maximum principle holds, see [6].

As a generalization, Sanchon [15] has treated the case of a super nonlinearity for the p-Laplacian under Dirichlet boundary condition on a smooth domain. In this work and in order to generalize the results of [14] and [1], we suppose that the nonlinearity f satisfies

$$\lim_{t \longrightarrow \infty} \frac{f(t)}{t} = a \in (0,+\infty). \tag{1.3}$$

By (1.3), we call f asymptotically linear at infinity.

We will prove the existence of the bifurcation phenomena and study the regularity of the extremal solutions. Note that the role of bifurcation problems in applied mathematics has been synthesized by Kielhöfer [12], who observed that the buckling of the Euler rod, the appearance of Taylor vortices, and the onset of oscillations in an electric circuit, for instance, all have a common cause: a specific physical parameter crosses a threshold, and that event forces the system to the organization of a new state that differs considerably from that observed before.

In the sequel,we say that u is a solution of (P_λ) if $u \in C^4(\overline{B})$ and satisfies (P_λ) in the classical sense. By a weak solution, we mean the following.

Definition 1.1. $u \in L^1(B)$ is a weak solution of (P_λ) if $f(u) \in L^1(B)$ and

$$\int_B u\Delta^2\zeta = \lambda \int_B f(u)\zeta \tag{1.4}$$

for all $\zeta \in C^4(\overline{B})$ with $\dfrac{\partial \zeta}{\partial n} = \zeta = 0$ on ∂B.

Moreover, if $u \in L^\infty(B)$ we say that u is regular while if u is not in $L^\infty(B)$ we say that u is singular. By elliptic regularity, we know that regular solutions are smooth and solve (P_λ) in the classical sense.

We say that a solution u_λ of (P_λ) is minimal if $u_\lambda \le u$ a.e. in B for any other solution of (P_λ).

In order to state our results, we have to introduce the stability condition for regular solutions.

Definition 1.2. A regular solution u of (P_λ) is said to be stable if the first eigenvalue $\eta_1(\lambda, u)$ of the linearized operator $L_{\lambda,u} = \Delta^2 - \lambda f'(u)$ given by

$$\eta_1(\lambda, u) := \inf_{\varphi \in H_0^2(B)\setminus\{0\}} \frac{\displaystyle\int_B |\Delta\varphi|^2 - \lambda \int_B f'(u)\varphi^2}{\|\varphi\|_2^2}$$

is positive in $H_0^1(B)$. In other words,

$$\lambda \int_B f'(u)\varphi^2 \le \int_B |\Delta\varphi|^2 \qquad \text{for any} \quad \varphi \in H_0^2(B). \tag{1.5}$$

If $\eta_1(\lambda, u) < 0$, the solution u is said to be *unstable*.

Our first result is the following.

Proposition 1.3. *Assume that f is a function satisfying* (1.1) *and* (1.3). *If (P_λ) has a weak solution u, then u is a regular solution and hence a classical solution.*

Before stating the other results, we give some notations. Throughout this articles, we denote $\|.\|_p$ the $L^p(B)$-norm for $1 \le p \le \infty$ and $\|.\|$ the H^2-norm given by

$$\|u\|^2 = \int_B |\Delta u|^2.$$

We denote by λ_1 the first eigenvalue of $L = \Delta^2$ in B with Dirichlet boundary condition and φ_1 the positive normalized eigenfunction associated, that is, such that

$$\left\{\begin{array}{rlll} \Delta^2\varphi_1 & = \lambda_1\,\varphi_1 & \text{in} & B,\\ \varphi_1 & > 0 & \text{in} & B,\\ \varphi_1 = \dfrac{\partial\varphi_1}{\partial n} & = 0 & \text{on} & \partial B,\\ \|\varphi_1\|_2 & = 1. & & \end{array}\right. \tag{1.6}$$

Let

$$\Lambda := \{\lambda > 0 \ \text{ such that } (P_\lambda) \text{ admits a solution}\} \text{ and } \lambda^* := \sup\Lambda \le +\infty.$$

We denote

$$r_0 := \inf_{t>0}\frac{f(t)}{t} \quad \text{ and } \quad l := \lim_{t\to\infty}\Big(f(t) - at\Big). \tag{1.7}$$

The following results hold.

Theorem 1.4. *Let f a positive function satisfying (1.1) and (1.3). Then there exists a critical value $\lambda^* \in (0,\infty)$ such that the following properties hold true.*

- (i) *For any $\lambda \in (0,\lambda^*)$, problem (P_λ) has a minimal solution u_λ, which is the unique stable solution of (P_λ) and the mapping $\lambda \mapsto u_\lambda$ is increasing.*
- (ii) *For any $\lambda \in (0, \frac{\lambda_1}{a})$, u_λ is the unique solution of problem (P_λ).*
- (iii) *If problem (P_{λ^*}) has a solution u, then*
$$u = u^* = \lim_{\lambda\to\lambda^*} u_\lambda \quad \textit{and} \quad \eta_1(\lambda^*, u^*) = 0.$$
- (iv) *For $\lambda > \lambda^*$, the problem (P_λ) has no solution even in the weak sense.*

In the next results, we study the existence and the non existence of the extremal solution (for $\lambda = \lambda^*$). We prove the following results.

Theorem 1.5. *Assume that $l \ge 0$. The following results hold.*

- (i) $\lambda^* = \dfrac{\lambda_1}{a}\cdot$
- (ii) *Problem (P_{λ^*}) has no solution.*
- (iii) $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of B.*

Theorem 1.6. *Assume that $l < 0$. Then we have the following results.*

- (i) *The critical value λ^* belongs to $(\dfrac{\lambda_1}{a}, \dfrac{\lambda_1}{r_0})$.*
- (ii) *(P_{λ^*}) has a unique solution u^*.*
- (iii) *The problem (P_λ) has an unstable solution v_λ for any $\lambda \in (\dfrac{\lambda_1}{a}, \lambda^*)$ and the sequence $(v_\lambda)_\lambda$ satisfies*
  - (a) $\lim_{\lambda\to\frac{\lambda_1}{a}} v_\lambda = \infty$ *uniformly on compact subsets of B,*

(b) $\lim_{\lambda \to \lambda^*} v_\lambda = u^*$ *uniformly in* B.

Before the proof of the Theorem 1.5, we treat the exemplar of $f(t) = at + b$ in the Proposition 3.2.

This paper is organized as follows: in the next section we prove Theorem 1.4, in section 3 we give the proof of Propositions 1.3 and 3.2, and Theorem 1.5. Finally, in section 4 we prove Theorem 1.6.

2. Proof of theorem 1.4

In the proof of this Theorem, we shall make use of the following auxiliary results. We begin by the following result due to [4] for the case of Laplacian.

Lemma 2.1. *Given* $g \in L^1(B)$, *there exists an unique* $v \in L^1(B)$ *which is a weak solution of*

$$\begin{cases} \Delta^2 v = g & \text{in } B, \\ v = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial B, \end{cases} \tag{2.1}$$

in the sense that

$$\int_B v \Delta^2 \zeta = \int_B g\, \zeta \tag{2.2}$$

for all $\zeta \in C^4(\overline{B})$ *with* $\dfrac{\partial \zeta}{\partial n} = \zeta = 0$ *on* ∂B. *Moreover, there exists a constant* c_0 *independents of* g *such that*

$$\|v\|_1 \le c_0 \, \|g\|_1.$$

In addition, if $g \ge 0$ *a.e. in* B, *then* $v \ge 0$ *a.e. in* B.

Proof. For the uniqueness, let v_1 and v_2 be two solutions of (2.1). Then $v = v_1 - v_2$ satisfies

$$\int_B v(\Delta^2 \zeta) = 0$$

for all $\zeta \in C^4(\overline{B})$ with $\dfrac{\partial \zeta}{\partial n} = \zeta = 0$. Given $\varphi \in \mathcal{D}(B)$, there exist a ζ solution of

$$\begin{cases} \Delta^2 \zeta = \varphi & \text{in } B, \\ \zeta = \dfrac{\partial \zeta}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

it follows that

$$\int_B v\varphi = 0.$$

Since φ is arbitrary, we deduce that $v = 0$.

For the existence, since $g = g^+ - g^-$, we can assume that $g \ge 0$. Let $g_n(x) = \min\{g(x), n\}$, then the family $(g_n)_n$ converge to g in $L^1(B)$.

Now let v_n the solution of

$$\begin{cases} \Delta^2 v_n &= g_n \quad \text{in} \quad B, \\ v_n = \dfrac{\partial v_n}{\partial n} &= 0 \quad \text{on} \quad \partial B. \end{cases} \tag{2.3}$$

The sequence $(v_n)_n$ is monotone nondecreasing.

On the other hand, we have

$$\int_B (v_k - v_l)1 = \int_B (g_k - g_l)\zeta_0,$$

where ζ_0 is defined by

$$\begin{cases} \Delta^2 \zeta_0 &= 1 \quad \text{in} \quad B, \\ \zeta_0 = \dfrac{\partial \zeta_0}{\partial n} &= 0 \quad \text{on} \quad \partial B. \end{cases} \tag{2.4}$$

So

$$\int_B |v_k - v_l| \leq c_0 \int_B |g_k - g_l| dx$$

and then $(v_n)_n$ is a Cauchy sequence in $L^1(B)$. Passing to the limit in (2.3), after multiplication by ζ, we have that $v = \lim v_n$ is a weak solution of equation (2.1). If we take $\zeta = \zeta_0$ in (2.2), we obtain

$$\|v\|_1 = \int_B v = \int_B g\zeta_0 \leq c_0 \|g\|_1.$$

□

Lemma 2.2. *If (P_λ) has a weak super solution $\overline{u}$, then there exists a weak solution u of (P_λ) such that $0 \leq u \leq \overline{u}$ and u does not depend on $\overline{u}$.*

Proof. We use a standard monotone iteration argument and maximum principle for the operator Δ^2 with Dirichlet boundary condition in the ball. Let $u_0 = 0$ and u_{n+1} the solution of

$$\begin{cases} \Delta^2 u_{n+1} &= \lambda f(u_n) \quad \text{in} \quad B, \\ u_{n+1} = \frac{\partial u_{n+1}}{\partial n} &= 0 \quad \text{on} \quad \partial B, \end{cases}$$

which exists by Lemma 2.1. We prove that $0 = u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq \overline{u}$ and $(u_n)_n$ converge to $u \in L^1(\Omega)$ which is a weak solution of (P_λ). Moreover u is independent of $\overline{u}$ by construction. □

Lemma 2.3. *The problem (P_λ) has no solution for any $\lambda > \lambda_1/r_0$, but has at least one solution provided λ is positive and small enough.*

Proof. To show that (P_λ) has a solution for λ small enough, we use the Lemma 2.1. To this aim, let $\zeta_0 \in C^4(\overline{B}))$ given by (2.4). We have ζ_0 is a super solution of (P_λ) for $\lambda \le 1/f(\|\zeta_0\|_\infty)$. Then by Lemma 2.2, the equation (P_λ) has a weak solution u for $\lambda \le 1/f(\|\zeta_0\|_\infty)$. Because $\zeta_0 \in L^\infty(B)$, u is a regular solution and then it is a classical one by elliptic regularity as mentioned in the introduction.

Assume now that u is a solution of (P_λ) for some $\lambda > 0$. Using φ_1 given by (1.6) as a test function, we get

$$\int_B u\Delta^2\varphi_1 = \lambda \int_B f(u)\varphi_1.$$

Using the definition of r_0, we have

$$(\lambda_1 - \lambda r_0)\int_B \varphi_1 u \ge 0.$$

Since $\varphi_1 > 0$ and $u > 0$, we conclude that the parameter λ should belong to $(0, \lambda_1/r_0]$. This completes the proof. □

Corollary 2.4. $\lambda^* = \sup\Lambda < \infty$.

Lemma 2.5. *Assume that the problem (P_λ) has a solution for some $\lambda \in (0,\lambda^*)$. Then there exists a minimal solution denoted by u_λ for the problem (P_λ). Moreover, for any $\lambda' \in (0,\lambda)$, the problem $(P_{\lambda'})$ has a solution.*

Proof. Fix $\lambda \in (0,\lambda^*)$ and let u be a solution of (P_λ). By Lemma 2.2, we obtain a solution u_λ of (P_λ) which is independent of u used as super solution. Since u_λ is independent of the choice of u, then it is a minimal solution.

Now, if u is a solution of (P_λ), then u is a super solution of the problem $(P_{\lambda'})$ for any λ' in $(0,\lambda)$ and the Lemma 2.2 completes the proof. □

Proof of Theorem 1.4 (i). First, we claim that u_λ is stable. Indeed, arguing by contradiction, we suppose that the first eigenvalue $\eta_1 = \eta_1(\lambda, u_\lambda)$ is non-positive. Then, there exists an eigenfunction

$$\psi \in C^4(\overline{B}) \quad \text{and} \quad \frac{\partial\psi}{\partial n} = \psi = 0 \quad \text{on} \quad \partial B$$

such that

$$\Delta^2\psi - \lambda f'(u_\lambda)\psi = \eta_1\psi \ \text{ in } \ B \quad \text{and} \quad \psi > 0 \ \text{ in } \ B.$$

Consider $u^\varepsilon := u_\lambda - \varepsilon\psi$. Hence

$$\begin{aligned}\Delta^2 u^\varepsilon - \lambda f(u^\varepsilon) &= -\eta_1\varepsilon\psi + \lambda\Big[f(u_\lambda) - f(u_\lambda - \varepsilon\psi) - \varepsilon f'(u_\lambda)\psi\Big] \\ &= \varepsilon\psi(-\eta_1 + o_\varepsilon(1)).\end{aligned}$$

Since $\eta_1 \leq 0$ for $\varepsilon > 0$ small enough, we have

$$\Delta^2 u^\varepsilon - \lambda f(u^\varepsilon) \geq 0 \qquad \text{in} \quad B.$$

Then, for $\varepsilon > 0$ small enough, we use the strong maximum principle (Hopf's Lemma) to deduce that $u^\varepsilon \geq 0$. u^ε is a super solution of (P_λ), so by the Lemma 2.2, we obtain a solution u such that $u \leq u^\varepsilon$ and since $u^\varepsilon < u_\lambda$, then we contradict the minimality of u_λ.

Now, we show that (P_λ) has at most one stable solution. Assume the existence of another stable solution $v \neq u_\lambda$ of problem (P_λ). Let $\varphi := v - u_\lambda$, then by maximum principle $\varphi > 0$ and from (1.5) taking φ as a test function, we have

$$\lambda \int_B f'(v)\,\varphi^2 \leq \int_B |\Delta\varphi|^2 = \int_B \varphi\Delta^2\varphi = \lambda \int_B \Big[f(v) - f(u_\lambda)\Big]\varphi.$$

Therefore

$$\int_B \Big[f(v) - f(u_\lambda) - f'(v)(v-u_\lambda)\Big]\varphi \geq 0.$$

Thanks to the convexity of f, the term in the brackets is non positive, hence

$$f(v) - f(u_\lambda) - f'(v)(v-u_\lambda) = 0 \ \text{ in } B,$$

which implies that f is affine over $[u_\lambda, v]$ in B. So, there exists two real numbers $\bar{a}$ and b such that

$$f(x) = \bar{a}x + b \quad \text{ in } \quad [0, \max_B v].$$

Finally, since u_λ and v are two solutions to $\Delta^2 w = \bar{a}w + b$, we obtain that

$$0 = \int_B (u_\lambda \Delta^2 v - v\Delta^2 u_\lambda) = b\int_B (v - u_\lambda) = b\int_B \varphi.$$

This is impossible since $b = f(0) > 0$ and φ is positive in B. Finally, by Lemma 2.5 and the definition of u_λ, we have that the function $\lambda \to u_\lambda$ is an increasing mapping.

Proof of Theorem 1.4 (ii). In this stage, we give the proof of the Proposition 1.3. By convexity of f, we have $a = \sup_{t\geq 0} f'(t)$ and

$$f(t) \leq at + f(0) \quad \text{ for all } \quad t \geq 0. \tag{2.5}$$

Let u a weak solution of (P_λ), $f(u) \in L^1(B)$. By elliptic regularity, $u \in L^p(B)$, for all $p \geq 1$ such that

$$p < \frac{N}{N-4} \qquad (p \leq \infty \text{ if } N = 2,3 \text{ and } p < \infty \text{ if } N = 4). \tag{2.6}$$

Again by (2.5), $f(u) \in L^p$ for all p satisfying (2.6) so $u \in W^{4,r}(B)$ for all $r \geq 1$ such that

$$r < \frac{N}{N-8} \quad (r \leq \infty \ \text{ if } N = 2,3,4,5,6,7 \ \text{ and } r < \infty \ \text{ if } N = 8). \tag{2.7}$$

By iteration and after $k(n) = [\frac{N}{4}] + 1$ operations, the solution u belongs to $L^\infty(B)$. By elliptic regularity and standard bootstrap argument, $u \in C^4(\overline{B})$.

Next, we treat the case of $f(t) = f_0(t) = at + b$ and we prove the following result.

Proposition 2.6. *Let $B \subset \mathbb{R}^N$ the unit ball of $\mathbb{R}^N$, $N \geq 2$. Assume that $f(t) = f_0(t) = at + b$, where $a, b > 0$. Then*

(i) $\lambda^* = \frac{\lambda_1}{a}$.

(ii) *The problem (P_λ) has no weak solution for $\lambda = \lambda^*$.*

Proof. Let $0 < \lambda < \dfrac{\lambda_1}{a}$, the problem (P_λ), given by

$$\begin{cases} \Delta^2 u - \lambda a u &= \lambda b \quad \text{in} \quad B, \\ \dfrac{\partial u}{\partial n} = u &= 0 \quad \text{on} \quad \partial B, \end{cases} \tag{2.8}$$

has a unique solution in $C^4(\overline{B})$. Since $\lambda a < \lambda_1$, by Maximum principle $u > 0$. Now let $\lambda = \dfrac{\lambda_1}{a}$. If the problem (2.8) has a solution u, then by multiplication (2.8) by φ_1 the positive function associated to λ_1 introduced by (1.6) and integration by parts, it follows that $\int_B \varphi_1 = 0$ which is impossible since $\varphi_1 > 0$ in B. So for $f_0(t) = at + b$, a and $b > 0$, we have $\lambda^* = \dfrac{\lambda_1}{a}$ and the equation (P_{λ^*}) has no solution. □

For the proof of Theorem 1.4 (ii), let $\lambda \in (0, \frac{\lambda_1}{a})$, $b = f(0)$ and w a solution for the problem (2.8) when $f_0(t) = at + b$. Since we have for the function f in Theorem 1.4, $f(w) \leq aw + f(0)$, then w is a super-solution of (P_λ) and hence by Lemma 2.2 and Proposition 1.3, the equation (P_λ) has a solution.

For the uniqueness, let u a solution of (P_λ) for a reel $\lambda \in (0, \frac{\lambda_1}{a})$. We denote $\lambda_1(L)$ the first eigenvalue of an operator L, that is $\lambda_1(\Delta^2) = \lambda_1$. Because $a = \sup_{t \geq 0} f'(t)$, we have $\Delta^2 - \lambda f'(u) \geq \Delta^2 - \lambda a$ and so

$$\lambda_1(\Delta^2 - \lambda f'(u)) \geq \lambda_1(\Delta^2 - \lambda a),$$

that is

$$\eta_1(\lambda, u) \geq \lambda_1 - \lambda a > 0.$$

The solution u is stable then, by Theorem 1.4 (i), we obtain $u = u_\lambda$.

Proof of Theorem 1.4 (iii). Suppose that (P_λ) has a solution u. then, for every $\lambda \in (0, \lambda^*)$, we have $u_\lambda \leq u$ and so $u^* = \lim\limits_{\lambda \to \lambda^*} u_\lambda$ is well defined in $L^1(B)$ and furthermore u^* is a weak then classical solution for (P_{λ^*}). Since $0 \leq u^* \leq u$, u^* is a minimal solution and also satisfies (1.5) for $\lambda = \lambda^*$. We have then $\eta_1(\lambda^*, u^*) \geq 0$.

Now, consider the nonlinear operator

$$\begin{array}{rcl} G : (0, +\infty) \times C^{4,\alpha}(\bar{B}) \cap E & \longrightarrow & C^{0,\alpha}(\bar{B}) \\ (\lambda, u) & \longmapsto & \Delta^2 u - \lambda f(u), \end{array}$$

where $\alpha \in (0, 1)$ and E the function space defined by

$$E := \left\{ u \in H^4(B) / \frac{\partial u}{\partial n} = u = 0 \text{ on } \partial B \right\}. \tag{2.9}$$

Assuming that the first eigenvalue $\eta_1(\lambda^*, u^*)$ is positive. By the implicit function theorem applied to the operator G, it follows that problem (P_λ) has a solution for λ in a neighborhood of λ^*. But this contradicts the definition of λ^* so $\eta_1(\lambda^*, u^*) = 0$. Furthermore, u^* is a the unique solution for (P_{λ^*}) and we can proceed as in the proof of Theorem 1.4 (ii).

Proof of Theorem 1.4 (iv). If the problem (P_λ) has a weak solution u for $\lambda > \lambda^*$, then by Proposition 1.3, u is a classical solution for (P_λ) and this contradicts the definition of λ^*. This completes the Theorem 1.4. □

3. Proof of theorem 1.5

In the proof of Theorem 1.5, we shall use the following auxiliary result which is a reformulation of Theorem due to Hörmander [9] and maximum principle.

Lemma 3.1. *Let $B \subset \mathbb{R}^N$ the unit ball of $\mathbb{R}^N$, $N \geq 2$. Let (u_n) be a sequence of nonnegative functions defined on B and satisfying $\Delta^2 u_n \geq 0$. Then the following alternative holds.*

(i) $\lim\limits_{n \to \infty} u_n = \infty$ *uniformly on compact subsets of* B

or

(ii) (u_n) *contains a subsequence which converges in* $L^1_{loc}(B)$ *to some function* u.

We first prove the following result.

Proposition 3.2. *Let f be a positive function satisfying* (1.1) *and* (1.3). *Then the following assertions are equivalent.*

(i) $\lambda^* = \dfrac{\lambda_1}{a}\cdot$

(ii) (P_{λ^*}) *has no solution.*

(iii) $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of* B.

Proof. (i)⇒(ii) By contradiction. Assume that (P_{λ^*}) has a solution u. By Theorem 1.4 (ii), $u = u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ and $\eta_1(\lambda^*, u^*) = 0$. Thus there exists $\psi \in C^4(\overline{B})$ satisfying

$$\Delta^2\psi - \lambda^* f'(u^*)\psi = 0,\ \psi > 0 \ \text{ in } \ B \ \text{ and } \ \psi = \frac{\partial\psi}{\partial n} = 0 \ \text{ on } \ \partial B, \qquad (3.1)$$

(see Remark 1.1 in [17]). Using φ_1 given by (1.6) as a test function, we obtain

$$\int_B (\lambda_1 - \lambda^* f'(u^*))\varphi_1\psi = 0. \qquad (3.2)$$

Since $\varphi_1 > 0$, $\psi > 0$, $\lambda^* = \dfrac{\lambda_1}{a}$ and $a = \sup_{t>0} f'(t)$, we have $\lambda_1 - \lambda^* f'(u^*) \geq 0$. Then equality (3.2) gives $f'(u^*) = a$ in B. This implies that $f(t) = at + b$ in $[0, max_B u^*]$ for some scalar $b > 0$ and this impossible by Proposition 2.6. Hence (P_{λ^*}) has no solution.

(ii)⇒(iii) By contradiction, suppose that (iii) doesn't hold. By Lemma 3.1 and up to subsequence, u_λ converges locally in $L^1(B)$ to the function u^* as $\lambda \to \lambda^*$.
Claim: u_λ is bounded in $L^2(B)$. Indeed, if not, we may assume that

$$u_\lambda = k_\lambda w_\lambda$$

with

$$\int_B {w_\lambda}^2 dx = 1 \quad \text{and} \quad \lim_{\lambda\to\lambda^*} k_\lambda = \infty. \qquad (3.3)$$

We have

$$\frac{\lambda}{k_\lambda} f(u_\lambda) \to 0 \quad \text{in} \quad L^1_{loc}(B) \quad \text{as} \quad \lambda \to \lambda^*$$

and then

$$\Delta^2 w_\lambda \to 0 \quad \text{in} \quad L^1_{loc}(B). \qquad (3.4)$$

We have

$$\int_B |\Delta w_\lambda|^2 = \int_B \Delta^2 w_\lambda\, w_\lambda = \int_B (\frac{\lambda f(u_\lambda)}{k_\lambda}),$$

then

$$\begin{aligned}\int_B |\Delta w_\lambda|^2 &\leq \int_B \frac{\lambda f(u_\lambda)}{k_\lambda} w_\lambda \leq \lambda^* \int_B a\, w_\lambda^2 + \frac{f(0)}{k_\lambda} w_\lambda \\ &\leq a\lambda^* + c_0 \int_B w_\lambda \leq a\lambda^* + c_0\sqrt{|B|}\end{aligned}$$

for some $c_0 > 0$ independent of λ. Then (w_λ) is bounded in $H^4(B)$ and up to a subsequence, we obtain

$$\begin{aligned} w_\lambda \rightharpoonup w \quad &\text{weakly in} \quad H^4(B) \quad \text{and} \\ w_\lambda \to w \quad &\text{strongly in} \quad L^2(B) \quad \text{as} \quad \lambda \to \lambda^*. \end{aligned} \tag{3.5}$$

Moreover, by the trace Theorem

$$\frac{\partial w_\lambda}{\partial n} = w_\lambda = 0 \quad \text{on} \quad \partial B.$$

It follows from (3.4) and (3.5), that $w = 0$ in B and this contradicts (3.3). This complete the proof of the claim. Thus u_λ is bounded in $L^2(B)$ and with the same argument above, u_λ is bounded in $H^4(B)$ and up to a subsequence, we have

$$u_\lambda \rightharpoonup u^* \quad \text{weakly in} \quad H^4(B),$$

$$u_\lambda \to u^* \quad \text{in} \quad L^2(B) \quad \text{as} \quad \lambda \to \lambda^*,$$

$$\begin{cases} \Delta^2 u^* &= \lambda^* f(u^*) \quad \text{in} \quad B, \\ \dfrac{\partial u^*}{\partial n} &= u^* = 0 \quad \text{on} \quad \partial B, \end{cases}$$

and this impossible by the hypothesis (ii). We remark clearly that (iii)⇒(ii) and hence (ii)⇔(iii).

(iii)⇒(i) If (iii) occurs, that (ii) also is true and we have $\lim\limits_{\lambda \to \lambda^*} \|u_\lambda\|_2 = \infty$. Let

$$u_\lambda = k_\lambda w_\lambda \quad \text{with} \quad \|w_\lambda\|_2 = 1. \tag{3.6}$$

Up to subsequence, we obtain

$$w_\lambda \rightharpoonup w \quad \text{weakly in} \quad H^4(B)$$

and

$$w_\lambda \to w \quad \text{in} \quad L^2(B) \quad \text{as} \quad \lambda \to \lambda^*. \tag{3.7}$$

We have also

$$\frac{\lambda}{k_\lambda} f(u_\lambda) \to \lambda^* a w \quad \text{as} \quad \lambda \to \lambda^* \tag{3.8}$$

and

$$\Delta^2 w_\lambda \to \Delta^2 w \quad \text{in} \quad L^2(B),$$

then

$$\begin{cases} \Delta^2 w &= a\lambda^* w \quad \text{in} \quad B, \\ w &= 0 \quad \text{on} \quad \partial B, \\ \frac{\partial w}{\partial n} &= 0 \quad \text{on} \quad \partial B. \end{cases} \tag{3.9}$$

Taking φ_1 as a test function in (3.9), we obtain

$$\lambda_1 \int_B w\varphi_1 = \int_B w(\Delta^2 \varphi_1) = \int_B a\lambda^* w\varphi_1.$$

Since $\varphi_1 > 0$ and $w > 0$ in B, we have $\lambda^* = \dfrac{\lambda_1}{a}$ and this complete the proof of Proposition 3.2. □

To finish the proof of Theorem 1.5, we need only to show that $(P_{\frac{\lambda_1}{a}})$ has no solution. Assume that u is a solution of $(P_{\frac{\lambda_1}{a}})$. Since

$$l := \lim_{t\to\infty}\Big(f(t) - at\Big) \geq 0 \quad \text{and} \quad a = \sup_{t\geq 0} f'(t),$$

we have $l \in (0,\infty)$ and $f(t) - at \geq 0$ and

$$\Delta^2 u = \frac{\lambda_1}{a} f(u) \ \text{ in } \ B. \tag{3.10}$$

Taking φ_1 as a test function in (3.10), we get $f(u) = a\,u$ in B, which contradicts $f(0) > 0$. This concludes the proof of Theorem 1.5. □

4. Proof of theorem 1.6

Proof of Theorem 1.6 (i). We have shown that

$$\frac{\lambda_1}{a} \leq \lambda^* \leq \frac{\lambda_1}{r_0}.$$

Suppose that $\lambda^* = \frac{\lambda_1}{a}$. By Proposition 3.2, we have

$$\lim_{\lambda\to\lambda^*} u_\lambda = \infty \text{ uniformly on compact subsets of } B.$$

Let u_λ be the minimal solution of (P_λ) for $\frac{\lambda_1}{a} < \lambda < \lambda^*$. Then multiplying (P_λ) by φ_1 and integrating by parts, we obtain

$$\int_B \varphi_1\Big(\lambda_1\, u_\lambda - \lambda\, f(u_\lambda)\Big) = \int_B \varphi_1\Big((\lambda_1 - a\lambda)u_\lambda - \lambda(f(u_\lambda) - au_\lambda)\Big) = 0 \tag{4.1}$$

and then

$$\lambda \int_B \varphi_1\Big(f(u_\lambda) - au_\lambda\Big) \geq 0. \tag{4.2}$$

Passing to the limit in the inequality (4.2) as λ tends to λ^*, we find

$$0 \leq l\lambda^* \int_B \varphi_1 < 0,$$

which is impossible and then $\lambda^* \neq \frac{\lambda_1}{a}$.

If $\lambda^* = \frac{\lambda_1}{r_0}$, let u be a solution of problem (P_{λ^*}) which exists by Proposition 3.2. Multiplying (P_{λ^*}) by φ_1 and integrating by parts, we obtain

$$\lambda_1 \int_B u\varphi_1 = \frac{\lambda_1}{r_0}\int_B f(u)\varphi_1,$$

that is

$$\int_B (f(u) - r_0 u)\varphi_1 = 0,$$

then $f(u) = r_0\, u$ in B, and this contradicts the fact that $f(0) > 0$.

Proof of Theorem 1.6 (ii). Since $\lambda^* > \frac{\lambda_1}{a}$, the existence of a solution to (P_{λ^*}) is assured by Proposition 3.2 and the uniqueness is given by Theorem 1.4.

Proof of Theorem 1.6 (iii). In this stage, we will use the mountain pass Theorem of Ambrosetti and Rabinowitz ([2]).

Theorem 4.1. ([2]) *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. Assume that J satisfies the Palais-Smale condition and the following geometric assumptions.*

(i) *There exist positive constants R and ρ such that*

$$J(u) \geq J(u_0) + \rho, \quad \textit{for all} \quad u \in E \quad \textit{with} \quad \|u - u_0\| = R.$$

(ii) *There exists $v_0 \in E$ such that $\|v_0 - u_0\| > R$ and $J(v_0) \leq J(u_0)$, then the functional J possesses at least a critical point. The critical value is characterized by*

$$\alpha := \inf_{g\in\Gamma} \max_{u\in g([0,1])} J(u),$$

where

$$\Gamma := \Big\{ g \in C([0,1], E) \,|\, g(0) = u_0,\, g(1) = v_0 \Big\}$$

and satisfies

$$\alpha \geq J(u_0) + \rho.$$

Let

$$\begin{array}{cccc} J: & E & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \displaystyle\frac{1}{2}\int_B |\Delta u|^2 - \int_B F(u), \end{array}$$

where E is the function space defined by (2.9) and

$$F(t) = \lambda \int_0^t f(s)ds, \quad \text{for all} \quad t \geq 0.$$

We take u_0 as the stable solution u_λ for each $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$.

The energy functional J belongs to $C^1(E, \mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_B \Delta u \cdot \Delta v - \lambda \int_B f(u)v,$$

for all $u, v \in E$. Since $\eta_1(\lambda, u_\lambda) \geq 0$, the function u_λ is a local minimum for J. In order to transform it into a local strict minimum, consider the perturbed functional J_ε defined by

$$\begin{array}{rccl} J_\varepsilon : & E & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & J(u) + \dfrac{\varepsilon}{2} \displaystyle\int_B |\Delta(u - u_\lambda)|^2, \end{array} \tag{4.3}$$

for all $\varepsilon \in [0,\ \varepsilon_0]$, where

$$\varepsilon_0 := \frac{3}{4}\ \frac{\lambda a - \lambda_1}{\lambda_1}.$$

We observe that J_ε is also in $C^1(E, \mathbb{R})$ and

$$\langle J_\varepsilon'(u), v\rangle = \int_B \Delta u \Delta v - \lambda \int_B f(u)v + \varepsilon \int_B \Delta(u - u_\lambda)\Delta v,$$

for all $u, v \in E$. Using the same arguments of Mironescu and Rădulescu in [14, Lemma 9], we show that J_ε satisfies the Palais-Smale condition and so we have the next lemma.

Lemma 4.2. *Let $(u_n) \subset E$ be a Palais-Smale sequence, that is,*

$$\sup_{n \in \mathbb{N}} |J_\varepsilon(u_n)| < +\infty \tag{4.4}$$

and

$$\|J_\varepsilon'(u_n)\|_{E^*} \to 0 \quad as \quad n \to \infty. \tag{4.5}$$

Then (u_n) is relatively compact in E.

Now, we need only to check that the two geometric assumptions are fulfilled. First, since u_λ is a local minimum of J, there exists $R > 0$ such that for all $u \in E$ satisfying $\|u - u_\lambda\| = R$, we have $J(u) \geq J(u_\lambda)$. Then

$$J_\varepsilon(u) \geq J_\varepsilon(u_\lambda) + \frac{\varepsilon}{2} \int_B |\Delta(u - u_\lambda)|^2.$$

Since $u - u_\lambda$ is not harmonic, we can choose

$$\rho := \frac{\varepsilon\, R^2}{4} > 0$$

and u_λ becomes a strict local minimal for J_ε, which proves (1).
Also, we have

$$\begin{aligned} J_\varepsilon(t\varphi_1) = \frac{\lambda_1}{2} t^2 + \frac{\varepsilon}{2}\lambda_1 t^2 - \varepsilon \lambda_1 t \int_B \varphi_1 u_\lambda + \frac{\varepsilon}{2}\lambda \int_B f(u_\lambda) u_\lambda \\ - \int_B F(t\varphi_1), \quad \forall\, t > 0. \end{aligned} \tag{4.6}$$

Recall that $\lim_{t\mapsto+\infty}(f(t)-a\,t)$ is finite, then there exists $\beta\in\mathbb{R}$ such that

$$f(t)\geq a\,t+\beta,\quad \forall\, t>0.$$

Hence

$$F(t)\geq \frac{a\,\lambda}{2}t^2+\beta\,\lambda\,t,\quad \forall\, t>0.$$

This yields

$$\frac{J_\varepsilon(t\varphi_1)}{t^2}\leq\Big(\frac{\lambda_1}{2}+\frac{\varepsilon\lambda_1}{2}-\frac{a\,\lambda}{2}\Big)+\frac{\varepsilon}{2t^2}\int_B f(u_\lambda)u_\lambda,$$

which implies that

$$\limsup_{t\to+\infty}\frac{1}{t^2}\,J_\varepsilon(t\varphi_1)\leq\Big(\frac{\lambda_1}{2}+\frac{\varepsilon_0\lambda_1}{2}-\frac{a\,\lambda}{2}\Big)<0,\quad \forall\,\varepsilon\in[0,\ \varepsilon_0].$$

Therefore

$$\lim_{t\to+\infty}J_\varepsilon(t\varphi_1)=-\infty$$

and so, for all $\varepsilon\in[0,\ \varepsilon_0]$, there exists $v_0\in E$ such that

$$J_\varepsilon(v_0)\leq J_\varepsilon(u_\lambda)$$

and (2) is proved. Finally, let v_ε (respectively, c_ε) be the critical point (respectively, critical value) of J_ε.

Remark 4.3. The fact that J_ε increases with ε implies that for all $\varepsilon\in[0,\varepsilon_0]$, $c_\varepsilon\in[c_0,c_{\varepsilon_0}[$. Then, c_ε is uniformly bounded. Thus, for all $\varepsilon\in[0,\ \varepsilon_0]$, the critical point v_ε satisfies $\|v_\varepsilon-u_\lambda\|\geq R$.

Recall that for any $\varepsilon\in[0,\varepsilon_0]$, the function v_ε belongs to E and satisfies

$$\Delta^2 v_\varepsilon=\frac{\lambda}{1+\varepsilon}f(v_\varepsilon)+\frac{\lambda\varepsilon}{1+\varepsilon}f(u_\lambda)\quad\text{in}\quad B \tag{4.7}$$

and

$$J_\varepsilon(v_\varepsilon)=c_\varepsilon. \tag{4.8}$$

By Lemma 4.2, Remark 4.3, (4.7) and (4.8), there exists $v\in E$ such that

$$v_\varepsilon\to v\quad\text{in}\quad E\quad\text{as}\quad\varepsilon\to 0$$

satisfying

$$\Delta^2 v=\lambda f(v)\quad\text{in}\quad B.$$

From Remark 4.3, we see that $v\neq u_\lambda$.

Proof of Theorem 1.6 (iii)-(a). By contradiction, suppose that (a) doesn't hold. By Lemma 2.5 there is a sequence of positives scalars (μ_n) and a sequence (v_n) of unstable solutions to (P_{μ_n}) such that $v_n\to v$ in $L^1_{loc}(B)$ as $\mu_n\to\lambda_1/a$ for some function v.

We first claim that (v_n) cannot be bounded in E. Otherwise, let $w \in E$ be such that, up to a subsequence,

$$v_n \rightharpoonup w \quad \text{weakly in} \quad E \quad \text{and} \quad v_n \to w \quad \text{strongly in} \quad L^2(B).$$

Therefore,

$$\Delta^2 v_n \to \Delta^2 w \quad \text{in} \quad \mathcal{D}'(B) \quad \text{and} \quad f(v_n) \to f(w) \quad \text{in} \quad L^2(B),$$

which implies that $\Delta^2 w = \frac{\lambda_1}{a} f(w)$ in B. It follows that $w \in E$ and solves $(P_{\lambda_1/a})$. From Lemma 3.1, we deduce that

$$\eta_1 \left(\frac{\lambda_1}{a}, w \right) \leq 0. \tag{4.9}$$

Relation (4.9) shows that $w \neq u_{\lambda_1/a}$ which contradicts the fact that $(P_{\lambda_1/a})$ has a unique solution. Now, since $\Delta^2 v_n = \mu_n f(v_n)$, the unboundedness of (v_n) in E implies that this sequence is unbounded in $L^2(B)$, too. To see this, let

$$v_n = k_n\, w_n, \quad \text{where} \quad k_n > 0, \quad \|w_n\|_2 = 1 \quad \text{and} \quad k_n \to \infty.$$

Then

$$\Delta^2 w_n = \frac{\mu_n}{k_n} f(v_n) \to 0 \quad \text{in} \quad L^1_{loc}(B).$$

So, we have convergence also in the sense of distributions and (w_n) is seen to be bounded in E with standard arguments. We obtain

$$\Delta^2 w = 0 \quad \text{and} \quad \|w\|_2 = 1.$$

The desired contradiction is obtained since $w \in E$.

Proof of Theorem 1.6 (iii)-(b). As before, it is enough to prove the $L^2(B)$ boundless of v_λ near λ^* and to use the uniqueness property of u^*. Assume that $\|v_n\|_2 \to \infty$ as $\mu_n \to \lambda^*$, where v_n is a solution to (P_{μ_n}). We write again $v_n = l_n\, w_n$. Then,

$$\Delta^2 w_n = \frac{\mu_n}{l_n} f(v_n). \tag{4.10}$$

The fact that the right-hand side of (4.10) is bounded in $L^2(B)$ implies that (w_n) is bounded in E. Let (w_n) be such that (up to a subsequence)

$$w_n \rightharpoonup w \quad \text{weakly in} \quad E \quad \text{and} \quad w_n \to w \quad \text{strongly in} \quad L^2(B).$$

A computation already done shows that

$$\Delta^2 w = \lambda^* a w, \quad w \geq 0 \quad \text{and} \quad \|w\|_2 = 1,$$

which forces λ^* to be λ_1/a. This contradiction concludes the proof. □

References

[1] I. Abid, M. Dammak and I. Douchich, *Stable solutions and bifurcation problem for asymptotically linear Helmholtz equations*, Nonl. Funct. Anal. and Appl., **21** (2016), 15–31.

[2] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), 349–381.

[3] G. Arioli, F. Gazzoula, H.C. Grunau and E. Mitidieri, *A semilinear fourth order elliptic problem with exponential nonlinearity*, Siam J. Math. Anal., **36** (2005), 1226–1258.

[4] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow up for $u_t - \Delta u = g(u)$ revisited*, Adv. Diff. Eq., **1** (1996), 73–90.

[5] H. Brezis and J.L. Vásquez, *Blow up solutions of some nonlinear elliptic problems*, Rev. Math. Univ. Compl. Madrid, **10** (1997), 443–468.

[6] T. Boggio, *Sulle funzioni di Ffreen d'ordine m*, Rend. Circ. Mat. Palermo, **20** (1905), 97–135.

[7] I.M. Gelfand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Translations Heidelberg, **29** (1963), 295–381.

[8] M.G. Grandall and P. Rabionowitz, *Some continuation and variational methods for positive solutions of non linear elliptic eigenvalue problems*, Arch. Ration. Mech. Anal., **58** (1975), 207–218.

[9] L. Hörmander, *The Analysis of Linear Differential Operators I*, Springer-Verlag, Berlin, 1983.

[10] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet driven by positive sources*, Arch. Rachinal Mech. Anal., **49** (1973), 241–269.

[11] D.D. Joseph and E.M. Sparrow, *Nonlinear diffusion induced by nonlinear sources*, Quar. J. Appl. Math., **28** (1970), 327–342 .

[12] H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 2003.

[13] Y. Martel, *Uniqueness of weak solution for nonlinear elliptic problems*, Houston J. Math., **23** (1997), 161–168.

[14] P. Mironescu and V. Rădulescu, *The study of a bifurcation problem associated to an asymtotically linear function*, Nonlinear Anal., **26** (1996), 857–875.

[15] M. Sanchón, *Boundedness of the extremal solution of some p-Laplacian problems*, Nonlinear Anal., **67** (2007), 281–294.

[16] V. Villaggio, *Mathematical Models for Elastic Stuctures,* Cambridge Univ. Press Cambridge 1997.

[17] L. Yue and W. Zhengping, *Biharmonic equations with asymptotically linear nonlinearities*, Acta Mathematica Scientia, **27** (2007), 549–560.