Nonlinear Functional Analysis and Applications
Vol. 22, No. 1 (2017), pp. 77-86
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

COUPLED FIXED POINT THEOREMS FOR SINGLE-VALUED MAPPINGS IN COMPLETE b-METRIC SPACES

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Abstract. In this note, certain common fixed point results for single-valued mappings in complete b-metric spaces are obtained. Some illustrative examples are also given which demonstrate the validity of the hypotheses of our results. In process, a host of previously known results in the context of complete b-metric spaces are generalized and improved.

1. Introduction and preliminaries

In 1922, Banach proved contraction principle [2] which provides a technique for solving existence problems in many branches of mathematical sciences. Further Banach contraction principle was generalized and improved by many authors in different ways. The concept of b-metric space was introduced and

[0]Received March 23, 2016 Revised June 21, 2016.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Coupled fixed point, complete b-metric space, weakly related mappings.

studied by Bakhtin [1] and Czerwik [5] as a generalization of metric spaces. Since then, several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces ([7],[8],[9]). In this paper, we give some fixed point results in such spaces. Our fixed point theorems, in the case of b-metric spaces generalize and improve some well-known results in the literature. Moreover, some examples are provided to illustrate the usability of the obtained results.

Definition 1.1. Let X be a non empty set and $s \geq 1$ a given real number. A function $d : X \times X \to C$ is called a b-metric space on x if d satisfies the following conditions:

(BM1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(BM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(BM3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a b-metric on X, and (X, d) is called b-metric space. It is obvious that a b-metric space with base $s = 1$ is a metric space.

The notions of a Cauchy sequence and a convergent sequence in b-metric spaces are defined by Boriceanu [4].

Definition 1.2. Let $\{x_n\}$ be a sequence in a b-metric space (X, d).

(i) A sequence $\{x_n\}$ is called convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$, when $n \to +\infty$.
(ii) $\{x_n\}$ is a Cauchy sequence if and only if

$$d(x_n, x_m) \to 0, \quad \text{when} \quad n, m \to +\infty.$$

In general, a b-metric space is said to be complete if and only if each Cauchy sequence in this space is convergent. From the properties of a b-metric space, we recall that if the limit of a convergent sequence exists, then it is unique. Also, each convergent sequence is a Cauchy sequence. But note that a b-metric, in the general case, is not continuous (see Roshan et al. [7]). The continuity of a mapping with respect to a b-metric is defined as follows.

Definition 1.3. Let (X, d) and (X', d') be two b-metric spaces with constant s and s', respectively. A mapping $T : X \to X'$ is called continuous if for each sequence $\{x_n\}$ in X, which converges to $x \in X$ with respect to d, then Tx_n converges to Tx with respect to d'.

If $(X, \precsim)$ is a partially ordered set and $T : X \to X$ such that for $x, y \in X$, $x \precsim y$ implies $Tx \precsim Ty$ then a mapping T is said to be non-decreasing. Similarly, mapping $T : X \times X \to X$ is said to be non-decreasing, if for

$(x_1, y_1), (x_2, y_2) \in X \times X \to X$ and $x_1 \precsim x_2$, $y_1 \precsim y_2$ implies $T(x_1, y_1) \precsim T(x_2, y_2)$. Here we state some useful definitions.

Definition 1.4. ([3], [6]) An element $(x, y) \in X \times X$ is called:

(C1) A coupled fixed point of mappings $T : X \times X \to X$ if

$$x \leq T(x, y),\ y \leq T(y, x).$$

(C2) A coupled coincidence point of mappings $T : X \times X \to X$ and $f : X \to X$ if $f(x) = T(x, y)$ and $f(y) = T(y, x)$ and in this case (f_x, f_y) is called coupled point of coincidence.

(C3) A coupled common fixed point of mapping $T : X \times X \to X$ and $f : X \to X$ if $x = f(x) = T(x, y)$ and $y = f(y) = T(y, x)$.

Definition 1.5. An element $(x, y) \in X \times X \to X$ is called:

(CC1) a common coupled coincidence point of the mapping $T : X \times X \to X$ and $f, g : X \to X$ if $T(x, y) = fx = gx$ and $T(y, x) = fy = gy$;

(CC2) a common coupled fixed point of mappings $T : X \times X \to X$ and $f, g : X \to X$ if $T(x, y) = fx = gx = x$ and $T(y, x) = fy = gy = y$.

Our technique of proof is simpler and essentially different from the ones used in the numerous papers devoted to coupled fixed point as we have used more natural mappings(non-decreasing) than that of in other papers.

Concept of weakly related mappings on ordered spaces as follows.

Definition 1.6. Let $(X, \precsim)$ be a partially ordered space, $F : X \times X \to X$ and $g : X \to X$. Then the pair F, g is said to be weakly related if $F(x, y) \precsim gF(x, y)$ and $gx \precsim F(gx, gy)$ also $F(y, x) \precsim gF(y, x)$ and $gy \precsim F(gy, gx)$ for all $(x, y) \in X \times X$.

Now we prove the common coupled fixed point existence theorem for the weakly related mappings.

We first prove the following lemma.

Lemma 1.7. *Let (X, d) be a b-metric space and $\phi : X \to R$. Define the relation, $\precsim$ on X as follows: $x \precsim y \Leftrightarrow d(x, y) \leq s[\phi(y) - \phi(x)], s \geq 1$ then $\precsim$ is partial order on X, called the partial order induced by ϕ.*

Proof. For all $x \in X, d(x, x) = 0 = s[\phi(x) - \phi(x)]$ then $x \precsim x$ that is $\precsim$ is reflexive. Now for $x, y \in X$ satisfy $x \precsim y$ and $y \precsim x$ then,

$$d(x, y) \leq s[\phi(y) - \phi(x)]$$

and

$$d(y, x) \leq s[\phi(x) - \phi(y)]$$

this shows that $d(x,y)=0$ *i.e.*, $x=y$. Thus $\precsim$ is antisymmetric.
Again for $x,y,z\in X$ satisfy $x\precsim y$ and $y\precsim z$, then

$$d(x,y)\leq s[\phi(y)-\phi(x)]$$

and

$$d(y,z)\leq s[\phi(z)-\phi(y)]$$

we have

$$\begin{aligned}d(x,z)&\leq d(x,y)+d(y,z)\\&\leq s[\phi(y)-\phi(x)]+s[\phi(z)-\phi(y)]\\&=s[\phi(z)-\phi(x)],\end{aligned}$$

then $x\precsim z$. Thus $\precsim$ is transitive and so the relation $\precsim$ is a partial order on X. □

Example 1.8. Let $X=[0,1]$ and $d(x,y)=\frac{2}{3}(x-y)^2$ then (X,d) is a b-metric space on $\mathbb{R}$ with $s=2$. Let $\phi:X\to R$, $\phi(x)=4x$. Therefore, X is a partially ordered space.

2. Main results

Now we prove our main theorems.

Theorem 2.1. *Let (X,d) is a complete b-metric space, $\phi:X\to R$ be a bounded from above function and $\precsim$ the partial order induced by ϕ. Let $F:X\times X\to X$ be a non decreasing continuous mapping on X such that there exist two elements $x_0,y_0\in X$ with $x_0\precsim F(x_0,y_0)$ and $y_0\precsim F(y_0,x_0)$. Then F has coupled fixed point in X.*

Proof. Let $x_0,y_0\in X$ such that $x_0\precsim F(x_0,y_0)$ and $y_0\precsim F(y_0,x_0)$. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1}=F(x_n,y_n)\quad and\quad y_{n+1}=F(y_n,x_n)\quad for\ \ all\quad n\geq 0.\tag{2.1}$$

We shall show that

$$x_n\precsim x_{n+1}\quad for\ \ all\quad n\geq 0,\tag{2.2}$$

$$y_n\precsim y_{n+1}\quad for\ \ all\quad n\geq 0.\tag{2.3}$$

We shall use the mathematical induction.

Let $n=0$, since $x_0\precsim F(x_0,y_0)$ and $y_0\precsim F(y_0,x_0)$ and as $x_1=F(x_0,y_0)$ and $y_1=F(y_0,x_0)$ we have $x_0\precsim x_1$ and $y_0\precsim y_1$. Thus (2.2) and (2.3) hold for $n = 0$. Suppose now that (2.2) and (2.3) hold for some fixed $n \geq 0$. Then since $x_n \precsim x_{n+1}$ and $y_n \precsim y_{n+1}$ and since F is non decreasing, we have

$$\begin{aligned} x_{n+2} &= F(x_{n+1}, y_{n+1}) \succsim F(x_n, y_n) = x_{n+1}, \\ y_{n+2} &= F(y_{n+1}, x_{n+1}) \succsim F(y_n, x_n) = y_{n+1}. \end{aligned} \tag{2.4}$$

Thus by mathematical induction we conclude that (2.2) and (2.3) hold for all $n \geq 0$. Therefore,

$$\begin{aligned} &x_0 \precsim x_1 \precsim x_2 \precsim x_3 \precsim \cdots \precsim x_n \precsim x_{n+1} \precsim \cdots, \\ &y_0 \precsim y_1 \precsim y_2 \precsim y_3 \precsim \cdots \precsim y_n \precsim y_{n+1} \precsim \cdots. \end{aligned}$$

That is the sequences $\{x_n\}$ and $\{y_n\}$ are non decreasing in X. By the definition of $\precsim$, we have

$$\begin{aligned} &\phi(x_0) \precsim \phi(x_1) \precsim \phi(x_2) \precsim \phi(x_3) \precsim \cdots, \\ &\phi(y_0) \precsim \phi(y_1) \precsim \phi(y_2) \precsim \phi(y_3) \precsim \cdots. \end{aligned}$$

In other words, the sequences $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are non-decreasing sequences in the set of real numbers. Since ϕ is bounded from above, $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are convergent and hence b-Cauchy. So, for all $\epsilon \geq 0$, there exists $n_0 \in N$ such that for all $m > n > n_0$. We have $|\phi(x_m) - \phi(x_n)| = \phi(x_m) - \phi(x_n) \leq \frac{\epsilon}{s}$ and $|\phi(y_m) - \phi(y_n)| = \phi(y_m) - \phi(y_n) \leq \frac{\epsilon}{s}$. Since $x_n \precsim x_m$, it follows that

$$d(x_n, x_m) \leq s[\phi(x_m) - \phi(x_n)] < \epsilon$$

and since $y_n \precsim y_m$, it follows that

$$d(y_n, y_m) \leq s[\phi(y_m) - \phi(y_n)] < \epsilon.$$

This shows that the sequences $\{x_n\}$ and $\{y_n\}$ are b-Cauchy in X and since X is complete, there exist points $x, y \in X$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$. Consequently, taking the limit as $n \to \infty$ in (2.1) and using the continuity of F, we get

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}) = F(y, x).$$

Thus we proved that $x = F(x, y)$ and $y = F(y, x)$. Hence (x, y) is a coupled fixed point of F. □

Example 2.2. Let $X = [0, 1], d(x, y) = \frac{2}{3}(x - y)^2$, then (X, d) is a complete b-metric space for $s = 2$. We define $\phi : X \to R$ by $\phi(x) = -2x$ and define the relation $\precsim$ on X as $x \precsim y$ iff $d(x,y) \leq s[\phi(y) - \phi(x)]$. Then $\precsim$ is partial order induced by ϕ. Also let $F : X \times X \to X$ as follows:

$$F(x,y) = \frac{x(1+y)}{2},$$

which is obviously a non-decreasing function on X. If we let $x_0 = 0$ and $y_0 = 1$, then $F(x_0, y_0) = \frac{x_0(1+y_0)}{2} = 0$ and $F(y_0, x_0) = \frac{y_0(1+x_0)}{2} = \frac{1}{2}$. So we see that $x_0 \precsim F(x_0, y_0)$ and $y_0 \precsim F(y_0, x_0)$. Hence all conditions of Theorem 2.1 are satisfied. Thus (0,0) is a coupled fixed point of F.

2.1. **Common coupled fixed point for weakly related single-valued mappings in complete b-metric spaces.** Concept of weakly related mappings on ordered spaces is as follows:

Definition 2.3. Let $(X, \precsim)$ be a partially ordered space, $F : X \times X \to X$ and $g : X \to X$. Then the pair $\{F, g\}$ is said to be weakly related if $F(x,y) \precsim gF(x,y)$ and $gx \precsim F(gx, gy)$ also $F(y,x) \precsim gF(y,x)$ and $gy \precsim F(gy, gx)$ for all $x, y \in X \times X$.

Now we prove the common coupled fixed point existence theorem for the weakly related mappings.

Theorem 2.4. *Let (X,d) is a complete b-metric space, $\phi : X \to R$ be a bounded from above function and $\precsim$ the partial order induced by ϕ. Let $F : X \times X \to X$ and $G : X \to X$ are two continuous mappings such that the pair $\{F, G\}$ is weakly related on X. If there exist two elements $x_0, y_0 \in X$ with $x_0 \precsim F(x_0, y_0)$ and $y_0 \precsim F(y_0, x_0)$ then F and G have a common coupled fixed point in X.*

Proof. Let $x_0, y_0 \in X$ such that $x_0 \precsim F(x_0, y_0)$ and $y_0 \precsim F(y_0, x_0)$. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{2n+1} = F(x_{2n}, y_{2n}) \quad \text{and} \quad x_{2n+2} = Gx_{2n+1} \tag{2.5}$$

and

$$y_{2n+1} = F(y_{2n}, x_{2n}) \quad \text{and} \quad y_{2n+2} = Gy_{2n+1} \tag{2.6}$$

for all $n \geq 0$. We shall show that

$$x_n \precsim x_{n+1}, \;\; y_n \precsim y_{n+1} \quad \text{for all} \quad n \geq 0.$$

Since $x_0 \precsim F(x_0, y_0)$, using(2.5) we have $x_0 \precsim x_1$. Again since the pair $\{F, G\}$ is weakly related, we have $F(x_0, y_0) \precsim GF(x_0, y_0)$ *i.e.*, $x_1 \precsim Gx_1$ and using (2.6) we get $x_1 \precsim x_2$. Also, since $Gx_1 \precsim F(Gx_1, Gy_1)$, using (2.5) we have $x_2 \precsim x_3$. Similarly using weakly related property for $\{F, G\}$ and repeated use of (2.3) gives

$$x_0 \precsim x_1 \precsim x_2 \precsim x_3 \precsim \cdots \precsim x_n \precsim x_{n+1} \precsim \cdots .$$

Also, since $y_0 \precsim F(y_0, x_0)$, using (2.5) we have $y_0 \precsim y_1$. Again since the pair $\{F, G\}$ is weakly related, we have $F(y_0, x_0) \precsim GF(y_0, x_0)$ *i.e.*, $y_1 \precsim Gy_1$ and using (2.5) we get $y_1 \precsim y_2$. Also, since $Gy_1 \precsim F(Gy_1, Gx_1)$, using (2.5) we have $y_2 \precsim y_3$. Similarly using the weakly related property for $\{F, G\}$ and repeated use of (2.5) gives

$$y_0 \precsim y_1 \precsim y_2 \precsim y_3 \precsim \cdots \precsim y_n \precsim y_{n+1} \precsim \cdots .$$

That is the sequences $\{x_n\}$ and $\{y_n\}$ non decreasing in X. By the definition of $\precsim$, we have

$$\phi(x_0) \precsim \phi(x_1) \precsim \phi(x_2) \precsim \phi(x_3) \precsim \cdots ,$$
$$\phi(y_0) \precsim \phi(y_1) \precsim \phi(y_2) \precsim \phi(y_3) \precsim \cdots .$$

In other words, the sequences $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are non-decreasing sequences in the set of real numbers. Since ϕ is bounded from above, $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are b-convergent and hence are b-Cauchy. So, for all $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m > n > n_0$, we have

$$|\phi(x_m) - \phi(x_n)| = \{\phi(x_m)\} - \{\phi(x_n)\} < \frac{\epsilon}{s}$$

and

$$|\phi(y_m) - \phi(y_n)| = \phi(y_m) - \phi(y_n) < \frac{\epsilon}{s}.$$

Since $x_n \precsim x_m$, it follows that $d(x_n, x_m) \leq s[\phi(x_m) - \phi(x_n)] < \epsilon$. Since $y_n \precsim y_m$, it follows that

$$d(y_n, y_m) \leq s[\phi(y_m) - \phi(y_n)] < \epsilon.$$

This shows that the sequence $\{x_n\}$ and $\{y_n\}$ are b-Cauchy in X and since X is b-complete, there exist points $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Since the sequences $\{x_{2n}\}$,$\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ are sub sequences of $\{x_n\}$, therefore $x_{2n} \to x$, $x_{2n+1} \to x$ and $x_{2n+2} \to x$. Also the sequence $\{y_{2n}\}$, $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ are sub sequences of $\{y_n\}$, therefore $y_{2n} \to y$, $y_{2n+1} \to y$ and $y_{2n+2} \to y$.

Consequently, taking the limit as $n \to \infty$ in (2.5) and using the continuity of F and G, we get

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} F(x_{2n}, y_{2n}) = F(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}) = F(x, y)$$

and

$$x = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Gx_{2n+1} = G \lim_{n\to\infty} x_{2n+1}) = Gx.$$

Similarly, taking the limit as $n \to \infty$ in (2.6) and using the continuity of F and G, we get

$$y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} F(y_{2n}, x_{2n}) = F(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}) = F(y, x)$$

and

$$y = \lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} Gy_{2n+1} = G(\lim_{n\to\infty} y_{2n+1}) = Gy.$$

Thus we proved that $Gx = x = F(x, y)$ and $Gy = y = F(y, x)$. Hence (x, y) is a common coupled fixed point of F and G. $\square$

Theorem 2.5. *Let (X, d) is a complete b-metric space, $\phi : X \to R$ be a bounded from above function and $\precsim$ the partial order induced by ϕ. Let $F : X \times X \to X$ and $G, H : X \to X$ are three continuous mappings such that the pair $\{F, G\}$ and $\{F, H\}$ are weakly related on X. Then F, G and H have a common coupled fixed point in X.*

Proof. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{3n} = Gx_{3n-1}, x_{3n-1} = F(x_{3n-2}, y_{3n-2}) \text{ and } \quad x_{3n-2} = Hx_{3n-3} \tag{2.7}$$

and

$$y_{3n} = (Gy_{3n-1}, y_{3n-1}) = F(y_{3n-2}, x_{3n-2}) \quad \text{and} \quad y_{3n-2} = Hy_{3n-3} \tag{2.8}$$

for all $n \geq 0$. We shall show that

$$x_n \precsim x_{n+1}, \quad y_n \precsim y_{n+1} \quad \text{for all} \quad n \geq 0.$$

We have $x_1 = Hx_0$. Since the pair $\{F, H\}$ is weakly related, we have $Hx_0 \precsim F(Hx_0, Hy_0)$ *i.e.*, $x_1 \precsim F(x_1, y_1)$ and using (2.7), we get $x_1 \precsim x_2$. Again since the pair $\{F, G\}$ is weakly related, we have $F(x_1, y_1) \precsim GF(x_1, y_1)$. Using (2.7), we get $x_2 \precsim x_3$. Similarly using the weakly related property for $\{F, G\}$ and $\{F, H\}$ repeated use of (2.7) gives

$$x_1 \precsim x_2 \precsim x_3 \precsim \cdots \precsim x_n \precsim x_{n+1} \precsim \cdots .$$

Also, we have $y_1 = Hy_0$. Since the pair $\{F, H\}$ is weakly related, we have $Hy_0 \precsim F(Hy_0, Hx_0)$ *i.e.*, $y_1 \precsim F(y_1, x_1)$. Using (2.8), we get $y_1 \precsim y_2$. Again since the pair $\{F, G\}$ is weakly related, we have $F(y_1, x_1) \precsim GF(y_1, x_1)$ *i.e.*, $y_2 \precsim Gy_2$ and using (2.8) we get $y_2 \precsim y_3$.

Similarly using the weakly related property for $\{F, G\}$ and $\{F, H\}$ and repeated use of (2.7) and (2.8) gives

$$y_1 \precsim y_2 \precsim y_3 \precsim \cdots \precsim y_n \precsim y_{n+1} \precsim \cdots .$$

That is the sequence $\{x_n\}$ and $\{y_n\}$ are non decreasing in X. By the definition of $\precsim$, we have

$$\phi(x_0) \precsim \phi(x_1) \precsim \phi(x_2) \precsim \phi(x_3) \precsim \cdots$$

and

$$\phi(y_0) \precsim \phi(y_1) \precsim \phi(y_2) \precsim \phi(y_3) \precsim \cdots .$$

In other words, the sequence $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ are non decreasing sequences in the set of real numbers. Since ϕ is bounded from above, $\{\phi(x_n)\}$

and $\{\phi(y_n)\}$ are b-convergent and hence are b-Cauchy. So, for all $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m > n > n_0$. We have $|\phi(x_m) - \phi(x_n)| = \phi(x_m) - \phi(x_n) < \frac{\epsilon}{s}$ and $|\phi(y_m) - \phi(y_n)| = \phi(y_m) = \phi(y_n) < \frac{\epsilon}{s}$. Since $x_n \precsim x_m$, it follows that

$$d(x_n, x_m) \leq s[\phi(x_m) - \phi(x_n)] < \epsilon$$

and since $y_n \precsim y_m$, it follows that

$$d(y_n, y_m) \leq s[\phi(y_m) - \phi(y_n)] < \epsilon.$$

This shows that the sequences $\{x_n\}$ and $\{y_n\}$ are b-Cauchy in X and since X is b-complete, there exist points $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Since the sequences $\{x_{3n}\}$,$\{x_{3n-1}\}$ and $\{x_{3n-2}\}$ are subsequences of $\{x_n\}$, therefore $x_{3n} \to x$, $x_{3n-1} \to x$ and $x_{3n-2} \to x$. Also the sequences $\{y_{3n}\}$, $\{y_{3n-1}\}$ and $\{y_{3n-2}\}$ are subsequences of $\{y_n\}$, therefore $y_{3n} \to y$, $y_{3n-1} \to y$ and $y_{3n-2} \to y$.

Consequently, taking the limit as $n \to \infty$ in (2.7) and using the continuity of F, G and H, we get

$$\begin{aligned} x &= \lim_{n\to\infty} x_{3n-1} = \lim_{n\to\infty} F(x_{3n-2}, y_{3n-2}) \\ &= F(\lim_{n\to\infty} x_{3n-2}, \lim_{n\to\infty} y_{3n-2}) = F(x, y), \\ x &= \lim_{n\to\infty} x_{3n} = \lim_{n\to\infty} Gx_{3n-1} = G(\lim_{n\to\infty} x_{3n-1}) = Gx \end{aligned}$$

and also

$$x = \lim_{n\to\infty} x_{3n-2} = \lim_{n\to\infty} Hx_{3n-3} = H(\lim_{n\to\infty} x_{3n-3}) = Hx.$$

Similarly, taking the limit as $n \to \infty$ in (2.8) and using the continuity of F and G, we get

$$y = \lim_{n\to\infty} y_{3n-1} = \lim_{n\to\infty} F(y_{3n-2}, x_{3n-2}) = F(\lim_{n\to\infty} y_{3n-2}, \lim_{n\to\infty}) = F(y, x),$$

$$y = \lim_{n\to\infty} y_{3n} = \lim_{n\to\infty} Gy_{3n-1} = G(\lim_{n\to\infty} y_{3n-1}) = Gy$$

and also

$$y = \lim_{n\to\infty} y_{3n-2} = \lim_{n\to\infty} Hy_{3n-3} = H(\lim_{n\to\infty} y_{3n-3}) = Hy.$$

Thus we proved that $Hx = Gx = x = F(x, y)$ and $Hy = Gy = y = F(y, x)$. Hence (x, y) is a common coupled fixed point of H, G and F. $\square$

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