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POSITIVE SOLUTIONS FOR SINGULAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS OF FOURTH-ORDER DIFFERENTIAL EQUATIONS

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Abstract. By constructing proper cones and by making use of fixed point theory together with the properties of Green's function, the paper deals with the existence of positive solutions for fourth-order singular nonlinear Sturm-Liouville boundary value problems. The main results which are obtained essentially improve, generalize and unify many well-known results. Examples are given to show the validity of the main results.

1. Introduction

The present paper considers the existence results for the following fourth-order nonlinear singular boundary value problems of the form

$$y^{(4)}(t) - \lambda a(t)F(t, y(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$\begin{cases} a_1 y(0) - b_1 y'(0) = 0 = c_1 y(1) + d_1 y'(1), \\ a_2 y''(0) - b_2 y'''(0) = 0 = c_2 y''(1) + d_2 y'''(1), \end{cases} \tag{1.2}$$

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where $\lambda > 0$, $a(t)$ may be singular at $t = 0$ and/or 1; $F : [0,1] \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous, and $F(t,y)$ is not identically zero on any subinterval of $[0,1]$, for all $0 < y < +\infty$; $a_i, b_i, c_i, d_i \geq 0$, such that

$$\Delta_i = b_i c_i + a_i c_i + a_i d_i > 0, \quad i = 1, 2.$$

The boundary value problems for differential equations arise quite naturally in a variety of mathematical models (see [1, 2, 8] for references along this line and therein). Much more attention has been put to the existence of positive solutions for fourth order non-singular boundary value problems. For example, when $\lambda a(t)F(t,y(t)) = e(t) - g(t,y(t)) + \pi^4 y(t)$, the differential equation (1.1) with the following boundary conditions

$$y(0) = y(1) = y''(0) = y''(1) = 0 \tag{1.3}$$

describes the bending of an elastic beam which is simply supported at both ends and is at resonance. Gupta [2] established the existence and uniqueness results of the nonlinear boundary value problem $(1.1)-(1.3)$, where $g(t,y)$ is strictly increasing on y for every t in $[0,1]$ and $\int_0^1 g(t,0)\sin \pi t dt = 0$. When $\lambda a(t)F(t,y(t)) = e(t) - g(t)y(t), 0 < t < 1$, Usmani [7] presented a uniqueness theorem for the linear boundary value problem $(1.1)-(1.3)$, where $g(t)$ and $e(t)$ are given real-valued continuous function on $[0,1]$. Y. Yang [10] proved an existence theorem for the equation (1.1) with the following general nonlinear boundary condition

$$y(0) = ty_0, \quad y(1) = ty_1, \quad y''(0) = t\overline{y}_0, \quad y''(1) = t\overline{y}_1. \tag{1.4}$$

When $F(t,y) = f(y)$, the differential equation (1.1) with the following boundary conditions

$$y(0) = y'(1) = y''(0) = y'''(1) = 0 \tag{1.5}$$

describes an elastic beam with one of its end simply supported and the other end clamped by sliding clamps. By employing Krasnosel'skii fixed point theorem of norm type cone expansion and compression, Ma and Wang [4] studied the existence of positive solutions for the problem $(1.1)-(1.3)$, and the problem $(1.1)-(1.5)$, where superlinear or sublinear conditions imposed on f.

However, the singular problems have been received much more attention in recent years (see [5, 6, 9] and the references therein). Motivated by [5, 6, 9], the aim of the paper, we consider more general differential equation (1.1) with more general boundary conditions (1.2), under some weaker assumptions imposed on $a(t)$ and $F(t,y)$. Also, we allow $a(t)$ may be singular at $t = 0$ and/or 1. Moreover, the paper is not only to obtain at least one positive solutions for the problem $(1.1)-(1.2)$, but also to derive an explicit interval for λ, and for any λ in this interval. Our results extend, contain and improve many known results in [1, 4, 10].

The paper is organized as follows: In Section 2, we collect some preliminaries and properties of Green's functions. We also construct valid integral operator. In Section 3, The main results will be stated and proved. In Section 4, some examples are given to show the validity of our main results.

2. Some Preliminaries

In this section we shall present some preliminaries which will be used to prove the main results.

We denote by $H(t,s)$ the Green's function for the homogeneous boundary value problem:

$$y^{(4)}(t) = 0, \quad t \in [0,1],$$

subject to the boundary conditions (1.2). We then know that $H(t,s)$ is non-negative on $[0,1]\times[0,1]$, and is expressed by

$$H(t,s) = \int_0^1 H_1(t,v)H_2(v,s)dv,$$

where $H_i(t,s)(i=1,2)$ is the Green's function for the following boundary value problem

$$\begin{cases} y''(t) = 0, \quad 0 < t < 1, \\ a_i y(0) - b_i y'(0) = 0, \\ c_i y(1) + d_i y'(1) = 0, \quad i = 1,2\,; \end{cases}$$

that is

$$H_i(t,s) = \begin{cases} \frac{1}{\Delta_i}\left(b_i + a_i s\right)\left(d_i + c_i(1-t)\right), & \text{if } \quad 0 \le s \le t \le 1, \\ \frac{1}{\Delta_i}\left(b_i + a_i t\right)\left(d_i + c_i(1-s)\right), & \text{if } \quad 0 \le t \le s \le 1, \end{cases}$$

for $i = 1,2$. By some simple calculations, we get

$$H(t,s) = \begin{cases} \frac{1}{6\Delta_1\Delta_2}\left(b_2 + a_2 s\right)\big(\Delta_1\left(c_2 t - 3\left(c_2 + d_2\right)\right)t^2 + \Delta(a_1 t + b_1)\big) \\ -\Delta_2\left(c_1(1-t) + d_1\right)\left(a_1 s + 3b_1\right)s^2, \quad \text{if } \ 0 \le s \le t \le 1, \\ \frac{1}{6\Delta_1\Delta_2}\left(b_1 + a_1 t\right)\big(\Delta_2\left(c_1 s - 3\left(c_1 + d_1\right)\right)s^2 + \Delta(a_2 s + b_2)\big) \\ -\Delta_1\left(c_2(1-s) + d_2\right)\left(a_2 t + 3b_2\right)t^2, \quad \text{if } \ 0 \le t \le s \le 1, \end{cases}$$

where $\Delta = 2c_1c_2 + 3c_1d_2 + 3c_2d_1 + 6d_1d_2$.
For convenience, we list the following assumptions:

(A) $\quad a(t) \in C((0,1),[0,+\infty))$, and $0 < \int_0^1 H_i(s,s)a(s)ds < +\infty$, for $i = 1,2$;

$$F(t,y) \in C([0,1]\times[0,+\infty),[0,+\infty)).$$

Remark 2.1. By (A), there exist $a, b \in (0,1)$ with $a < b$ such that

$$0 < \int_a^b H_i(s,s)a(s)ds < +\infty, \quad i = 1,2.$$

In the rest of the paper, a, b will be taken in this way. It is rather straight forward that

$$\begin{aligned} 0 &\le H_i(t,s) \le H_i(s,s) \\ &\le \frac{1}{\Delta_i}(a_i+b_i)(c_i+d_i) < +\infty, \quad \text{for} \quad i=1,2, \quad t,s \in [0,1]; \end{aligned} \tag{2.1}$$

$$0 < \tau_i H_i(s,s) \le H_i(t,s), \ \text{for} \ \ t \in [a,b] \subset (0,1), s \in (0,1), \ i=1,2; \tag{2.2}$$

where

$$0 < \tau_i = \min\left\{ \frac{d_i + c_i(1-b)}{d_i+c_i}, \frac{b_i + a a_i}{b_i + a_i} \right\} < 1, \quad \text{for} \quad i=1,2. \tag{2.3}$$

Remark 2.2. From Remark 2.1 and (2.1) together with (2.2), we know that

$$0 < \min_{t\in[a,b]} \int_a^b H_i(t,s)a(s)ds < +\infty, \quad \text{for} \quad i=1,2,$$

$$0 < \min_{t\in[a,b]} \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)dv \right] ds < +\infty.$$

Similarly,

$$0 < \max_{t\in[0,1]} \int_0^1 H_i(t,s)a(s)ds < +\infty, \quad \text{for} \quad i=1,2,$$

$$0 < \max_{t\in[0,1]} \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)dv \right] ds < +\infty.$$

Let

$$Q = \left(\max_{t\in[0,1]} \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)dv \right] ds \right)^{-1}. \tag{2.4}$$

$$q = \left(\min_{t\in[a,b]} \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)dv \right] ds \right)^{-1}. \tag{2.5}$$

Note that Q and q are constants and $0 < Q \le q \le +\infty$.

By a positive solution of boundary value problem (1.1) – (1.2), we mean a function $y(t) \in C([0,1], R^+) \cap C^{(4)}((0,1), R^+)$ satisfying the problem (1.1) – (1.2), and with $y(t)$ nonnegative and not identically zero on $[0,1]$.

Now, we denote $E = C[0,1]$ with norm $\|y\| = \max_{0\le t\le 1} |y(t)|, y(t) \in C[0,1]$. Then E is a Banach space. Let

$$K = \left\{ y \mid y \in C^+[0,1], \min_{t\in[a,b]} y(t) \ge \tau \|y\| \right\}, \tag{2.6}$$

where $C^+[0,1] = \{y \in C[0,1] \mid y \geq 0\}$, $0 < \tau = \tau_1\tau_2 < 1$. Then, we know that K is a positive cone and $K \subset C^+[0,1] \subset E$. Now we define an operator $B : C^+[0,1] \to C^+[0,1]$ by

$$\begin{aligned}(By)(t) &= \lambda \int_0^1 H(t,s)a(s)F(s,y(s))ds \\ &= \lambda \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)F(s,y(s))dv\right] ds.\end{aligned} \tag{2.7}$$

It is well known that y is a positive solution of the problem (1.1) – (1.2) if and only if y is a fixed point of operator B in $C[0,1]$.

Lemma 2.1. *Assume that* (A) *holds. Then* $B : K \to K$ *is a completely continuous operator.*

Proof. Let $G = \max\{H_1(t,v) \mid t,v \in [0,1]\}$. Suppose that $y_n \to y_0$ $(n \to \infty), y_n, y_0 \in C^+[0,1]$. Then there exists a constant $d > 0$, such that $\|y_n\| < d < +\infty$, for $n = 1, 2, \cdots$. Since $F(t,y)$ is continuous on $[0,1] \times [0,d]$, it is uniformly continuous. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\mid y' - y'' \mid < \delta$, for $y', y'' \in [0,d]$, implies that

$$\mid F(t,y') - F(t,y'') \mid < \varepsilon \left(\lambda G \int_0^1 H_2(s,s)a(s)ds\right)^{-1}.$$

Since $y_n \to y_0$, there exists a natural number N such that $\|y_n - y_0\| < \delta$ for any $n > N$. Thus, for any $n > N$ and $t \in [0,1]$, we have

$$\mid F(t,y_n(t)) - F(t,y_0(t)) \mid < \varepsilon \left(\lambda G \int_0^1 H_2(s,s)a(s)ds\right)^{-1},$$

which implies that

$$\begin{aligned}&\|By_n(t) - By_0(t)\| \\ &\leq \lambda \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s) \mid F(s,y_n(s)) - F(s,y_0(s)) \mid dv\right] ds \\ &< \lambda G\varepsilon \left(\lambda G \int_0^1 H_2(s,s)a(s)ds\right)^{-1} \left(\int_0^1 H_2(s,s)a(s)ds\right) = \varepsilon\end{aligned}$$

for all $t \in [0,1]$ and $n > N$, and therefore $\|By_n - By_0\| < \varepsilon$ for all $n > N$. Thus B is continuous.

Now, we suppose that $T \subset C^+[0,1]$ is a bounded set, then there exists a constant $l > 0$ such that $\|y\| \leq l$, for all $y \in T$.

Let $L = \max\{\lambda F(t,y) \mid 0 \leq t \leq 1, 0 \leq y \leq l\}$, $\xi = \int_0^1 H_2(s,s)a(s)ds$. Then $B(T)$ is a uniformly bounded subset of $C^+[0,1]$, because we have $\|By\| \leq L\xi G$

for $y \in T$. Since $H_1(t,v)$ is uniformly continuous on $[0,1]\times[0,1]$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$| H_1(t_1,v) - H_1(t_2,v) | < \varepsilon(L\xi)^{-1}$$

for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$ and $v \in [0,1]$. Then, for any $y \in T$, and $t_1, t_2 \in [0,1]$, with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
&\|(By)(t_1) - (By)(t_2)\| \\
&\le \lambda \int_0^1 \left[\int_0^1 | H_1(t_1,v) - H_1(t_2,v) | H_2(v,s)a(s)F(s,y(s))dv \right] ds \\
&< \varepsilon(L\xi)^{-1} L \int_0^1 H_2(s,s)a(s)ds \\
&= \varepsilon\xi^{-1}\xi \\
&= \varepsilon.
\end{aligned}$$

Thus $B(T)$ is a equicontinuous subset of E. It follows from Arzela-Ascoli Theorem that $B(T)$ is relatively compact. Therefore, the operator B is completely continuous. This completes the proof of Lemma 2.1. □

Lemma 2.2. *$BK \subset K$.*

Proof. For all $y \in C^+[0,1], \quad t \in [0,1]$,

$$\begin{aligned}
(By)(t) &= \lambda \int_0^1 H(t,s)a(s)F(s,y(s))ds \\
&= \lambda \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)F(s,y(s))dv \right] ds \\
&\le \lambda \int_0^1 \left[\int_0^1 H_1(v,v)H_2(s,s)a(s)F(s,y(s))dv \right] ds.
\end{aligned}$$

Thus $\|By\| \le \lambda \int_0^1 \left[\int_0^1 H_1(v,v)H_2(s,s)a(s)F(s,y(s))dv \right] ds$.

On the other hand, we know that

$$\begin{aligned}
\min_{t\in[a,b]} (By)(t) &= \min_{t\in[a,b]} \lambda \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)F(s,y(s))dv \right] ds \\
&\ge \tau_1\tau_2\lambda \int_0^1 \left[\int_0^1 H_1(v,v)H_2(s,s)a(s)F(s,y(s))dv \right] ds \\
&= \tau\lambda \int_0^1 \left[\int_0^1 H_1(v,v)H_2(s,s)a(s)F(s,y(s))dv \right] ds.
\end{aligned}$$

which implies that $\min_{t\in[a,b]} (By)(t) \ge \tau\|By\|$, thus $By \in K$. So $BK \subset K$. □

Lemma 2.3. [3] *Let E be a Banach space and let $K(\subset E)$ be a cone. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let $B : K(\overline{\Omega_2} \backslash \Omega_1) \longrightarrow K$ be a continuous and compact operator such that either*

(a) $\|By\| \leq \|y\|, y \in K \cap \partial\Omega_1,$ *and* $\|By\| \geq \|y\|, y \in K \cap \partial\Omega_2,$ *or*

(b) $\|By\| \geq \|y\|, y \in K \cap \partial\Omega_1,$ *and* $\|By\| \leq \|y\|, y \in K \cap \partial\Omega_2.$

Then B has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

3. The Main Results

In this section, we give our main results.

Theorem 3.1. *Suppose that (A) holds. In addition, assume that*

$$(A_1) \quad \begin{cases} 0 \leq F^0 = \limsup\limits_{y \to 0+} \max\limits_{t \in [0,1]} \dfrac{F(t,y)}{y} < Q, & (I) \\[2ex] 0 < q < F_\infty = \liminf\limits_{y \to +\infty} \min\limits_{t \in [a,b]} \dfrac{F(t,y)}{y} \leq +\infty. & (II) \end{cases}$$

Then the problem (1.1) – (1.2) *has at least one positive solution in K for any*

$$\lambda \in \left(\frac{q}{\tau F_\infty}, \frac{Q}{F^0} \right), \tag{3.1}$$

where Q and q are defined as (2.4) *and* (2.5).

Proof. Let λ satisfy (3.1) and $\varepsilon > 0$ be a number such that $F_\infty - \varepsilon > 0$ on $[0,1]$ and

$$\frac{q}{\tau(F_\infty - \varepsilon)} \leq \lambda \leq \frac{Q}{F^0 + \varepsilon}. \tag{3.2}$$

From $(A_1)(I)$, there exists $r > 0$ such that

$$F(t,y) \leq (F^0 + \varepsilon)y \leq (F^0 + \varepsilon)r, \tag{3.3}$$

for any $0 < y \leq r$ and $t \in [0,1]$. Let $\Omega_1 = \{y \in E \mid \|y\| < r\}$. For any $y \in K \cap \partial\Omega_1$, it follows from (3.3) and (2.4) that,

$$\begin{aligned} \|By\| &= \max_{t \in [0,1]} \lambda \int_0^1 H(t,s)a(s)F(s,y(s))ds \\ &= \lambda \max_{t \in [0,1]} \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)F(s,y(s))dv \right] ds \\ &\leq \lambda(F^0 + \varepsilon)r \max_{t \in [0,1]} \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)dv \right] ds \\ &\leq r = \|y\|. \end{aligned}$$

Thus, $\|By\| \leq \|y\|$, for $y \in K \cap \partial\Omega_1$.

Again by virtue of $(A_1)(II)$, there exists $r_0 > r > 0$ such that

$$F(t,y) \geq (F_\infty - \varepsilon)y, \quad \text{for } \ y \geq r_0, 0 \leq t \leq 1. \tag{3.4}$$

Let $R > \max\{2r, r_0\tau^{-1}\}$ and $\Omega_2 = \{y \in E \mid \|y\| < R\}$. Then

$$\min_{t\in[a,b]} y(t) \geq \tau\|y\| \geq r_0$$

for any $y \in K \cap \partial\Omega_2$. Thus, from (2.5) and (2.6) together with (3.4), for any $a \leq t \leq b, y \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}
(By)(t) &= \lambda \int_0^1 H(t,s)a(s)F(s,y(s))ds \\
&= \lambda \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)F(s,y(s))dv\right] ds \\
&\geq \lambda \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)F(s,y(s))dv\right] ds \\
&\geq \lambda(F_\infty - \varepsilon) \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)y(s)dv\right] ds \\
&\geq \lambda(F_\infty - \varepsilon)\tau\|y\| \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)dv\right] ds \\
&\geq \lambda(F_\infty - \varepsilon)\tau\|y\| \min_{t\in[a,b]} \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)dv\right] ds \\
&\geq \|y\|.
\end{aligned}$$

Therefore $\|By\| \geq \|y\|$ for $y \in K \cap \partial\Omega_2$.

From Lemma 2.3, B has a fixed point y^* in $K \cap (\overline{\Omega_2}\backslash\Omega_1)$. Then y^* is a positive solution of the problem (1.1) – (1.2). This completes the proof. $\square$

Remark 3.1. From Theorem 3.1, we can see that $F(t,y)$ need not be superlinear or sublinear. So our conclusions extend and improve the corresponding results in [1, 4, 5, 10]. In fact, the conclusion of Theorem 3.1 still holds if one of the following conditions hold:

(i) If $F_\infty = +\infty$, $F^0 > 0$, then for each $\lambda \in \left(0, \frac{Q}{F^0}\right)$;

(ii) If $F_\infty = +\infty$, $F^0 = 0$, then for each $\lambda \in (0, +\infty)$;

(iii) If $F_\infty > q > 0$, $F^0 = 0$, then for each $\lambda \in \left(\frac{q}{\tau F_\infty}, +\infty\right)$.

Theorem 3.1 yields the following corollary.

Corollary 3.1. *Suppose that* (A) *holds. In addition, assume that* $F^0 \in [0,+\infty), F_\infty \in (0,+\infty)$. *Then for any* $\lambda \in \left(\frac{q}{\tau F_\infty}, \frac{Q}{F^0}\right) \subset (0,+\infty)$, *the problem* $(1.1)-(1.2)$ *has nonnegative solutions.*

Theorem 3.2. *Suppose that* (A) *holds. In addition, assume that*

$$(A_2) \qquad \begin{cases} 0 \le F^\infty = \limsup\limits_{y\to+\infty} \max\limits_{t\in[0,1]} \dfrac{F(t,y)}{y} < Q, & (I) \\ 0 < q < F_0 = \liminf\limits_{y\to 0+} \min\limits_{t\in[a,b]} \dfrac{F(t,y)}{y} \le +\infty. & (II) \end{cases}$$

Then the problem $(1.1)-(1.2)$ *has at least one positive solution for any*

$$\lambda \in \left(\frac{q}{\tau F_0}, \frac{Q}{F^\infty}\right), \tag{3.5}$$

where Q *and* q *are defined as in* (2.4) *and* (2.5).

Proof. Let λ satisfy (3.5) and $\varepsilon > 0$ be chosen such that $F_0 - \varepsilon > 0, t \in [0,1]$, and

$$\frac{q}{\tau(F_0-\varepsilon)} \le \lambda \le \frac{Q}{F^\infty + \varepsilon}. \tag{3.6}$$

By virtue of $(A_2)(II)$, there exists $r > 0$ such that $F(t,y) \ge (F_0-\varepsilon)y$, for $0 < y < r, a \le t \le b$. Let $\Omega_1 = \{y \in E \mid \|y\| < r\}$. Then

$$0 < \tau\|y\| < \min_{t\in[a,b]} y(t) \le \max_{t\in[a,b]} y(t) \le \|y\| = r.$$

Therefore, by making use of (2.6) and (3.6), for any $y \in K \cap \partial\Omega_1$ and $a \le t \le b$, we obtain

$$\begin{aligned}
(By)(t) &= \lambda \int_0^1 H(t,s)a(s)F(s,y(s))ds \\
&= \lambda \int_0^1 \left[\int_0^1 H_1(t,v)H_2(v,s)a(s)F(s,y(s))dv\right] ds \\
&\ge \lambda \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)F(s,y(s))dv\right] ds \\
&\ge \lambda(F_0-\varepsilon) \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)y(s)dv\right] ds \\
&\ge \lambda(F_0-\varepsilon)\tau\|y\| \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)dv\right] ds \\
&\ge \lambda(F_0-\varepsilon)\tau\|y\| \min_{t\in[a,b]} \int_0^1 \left[\int_a^b H_1(t,v)H_2(v,s)a(s)dv\right] ds \ge \|y\|.
\end{aligned}$$

Thus $\|By\| \geq \|y\|$, for $y \in K \cap \partial\Omega_1$.

Let $k(t, y) = \sup_{x \in [0,y]} F(t, x)$. Then $F(t, y) \leq k(t, y)$ and k is increasing for $y \in [0, +\infty)$. By virtue of $(A_2)(I)$, there exist $R_0 > 0$ such that

$$F(t, y) \leq (F_\infty + \varepsilon)y,$$

for $y \geq R_0, \quad 0 \leq t \leq 1$. Then

$$F(t, y) \leq M_0 + (F_\infty + \varepsilon)y$$

for $y \geq 0, \quad 0 \leq t \leq 1$, where $M_0 = \max\{F(t, y) \mid (t, y) \in [0, 1] \times [0, R_0]\}$. Thus

$$\limsup_{y \to +\infty} \max_{t \in [0,1]} \frac{k(t, y)}{y} \leq F^\infty.$$

Consequently, by the fact that $F(t, y) \leq k(t, y)$, we have

$$\limsup_{y \to +\infty} \max_{t \in [0,1]} \frac{k(t, y)}{y} = F^\infty.$$

Let $R > 2r$ such that $k(t, y) \leq (F^\infty + \varepsilon)y$, for $y \geq R, 0 \leq t \leq 1$. Now let $\Omega_2 = \{y \in E \mid \|y\| < R\}$. Then for any $y \in K \cap \partial\Omega_2$, and $0 \leq t \leq 1$, we have

$$\begin{aligned}
(By)(t) &= \lambda \int_0^1 H(t, s)a(s)F(s, y(s))ds \\
&\leq \lambda \max_{t \in [0,1]} \int_0^1 \left[\int_0^1 H_1(t, v)H_2(v, s)a(s)F(s, y(s))dv\right] ds \\
&\leq \lambda \max_{t \in [0,1]} \int_0^1 \left[\int_0^1 H_1(t, v)H_2(v, s)a(s)k(s, y(s))dv\right] ds \\
&\leq \lambda \max_{t \in [0,1]} \int_0^1 \left[\int_0^1 H_1(t, v)H_2(v, s)a(s)k(s, R)dv\right] ds \\
&\leq \lambda(F^\infty + \varepsilon)R \max_{t \in [0,1]} \int_0^1 \left[\int_0^1 H_1(t, v)H_2(v, s)a(s)dv\right] ds \\
&\leq R = \|y\|.
\end{aligned}$$

Thus $\|By\| \leq \|y\|$ for $y \in K \cap \partial\Omega_2$. By virtue of Lemma 2.3, we know that B has a fixed point y^* in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$, and so y^* is a positive solution of the problem (1.1) – (1.2). This completes the proof. $\square$

Remark 3.2. From the proof of Theorem 3.2, we can see the same conclusion of Theorem 3.2 remains valid if one of the following conditions holds:

(i) If $F^\infty < Q$, $F_0 = +\infty$, then for each $\lambda \in \left(0, \frac{Q}{F^\infty}\right)$;

(ii) If $F^\infty = 0$, $F_0 = +\infty$, then for each $\lambda \in (0, +\infty)$;

(iii) If $F^\infty = 0$, $F_0 > q > 0$, then for each $\lambda \in \left(\frac{q}{\tau F_0}, +\infty\right)$.

The following corollary is obtained from Theorem 3.2.

Corollary 3.2. *Suppose that (A) holds. In addition, assume that* $F^\infty \in [0, +\infty), F_0 \in (0, +\infty)$. *Then for any* $\lambda \in \left(\frac{q}{\tau F_0}, \frac{Q}{F^\infty}\right) \subset (0, +\infty)$, *the problem* $(1.1) - (1.2)$ *has nonnegative solution.*

Remark 3.3. Theorem 3.1 and Theorem 3.2 extend, contain and improve the main results in [1, 2, 4, 6, 10] from the following aspects:

(i) We allow $a(t)$ to be singular at $t = 0$ and/or 1. Also, $a(t)$ is permitted to be vanished at some subinterval of $[0, 1]$.

(ii) The boundary conditions in our problem is more general than that of in [1, 2, 10].

(iii) F need not to be superlinear or sublinear. Note that if F is superlinear (i.e. $F^0 = 0, F_\infty = +\infty$) or sublinear (i.e. $F_0 = +\infty, F^\infty = 0$), then for each $\lambda \in (0, +\infty)$, the problem $(1.1) - (1.2)$ has at least one positive solution. And our results still hold for the non-singular cases as in [1, 7, 10].

4. Examples

In this section, we present some examples to illustrate the validity of our main results.

Example 4.1. Consider the following singular boundary value problem

$$\begin{cases} y^{(4)}(t) - \lambda \dfrac{1}{\sqrt{1-t}}\Big(0.001(1-t)y^2 + 11879\dfrac{y^3}{1+y} \\ \qquad\qquad + 0.009 \mid \sin y^2 \mid \Big)^{\frac{1}{2}} = 0, \quad 0 < t < 1, \\ y(0) = y(1) = y''(0) = y''(1) = 0. \end{cases} \tag{4.1}$$

Then the problem (4.1) has at least one positive solution.

It is obviously that $a(t) = \frac{1}{\sqrt{1-t}}$ is singular at $t = 1$. The problem (4.1) describes the bending of an elastic beam both of whose ends simply supported at 0 and 1. Because of the singularity and the form of the problem (4.1), it seems to be difficult that the problem is solved by making use of the results obtained by [1, 2, 4, 5, 7, 10] as well as their extension. Now we study the problem (4.1) by making use of Theorem 3.1 (where $a_i = c_i = 1, b_i = d_i = 0$).

Let $a(t) = \frac{1}{\sqrt{1-t}}$,

$$F(t, y) = \lambda \left[0.001(1-t)y^2 + 11879\frac{y^3}{1+y} + 0.009 \mid \sin y^2 \mid\right]^{\frac{1}{2}}, \quad 0 < t < 1.$$

It is easy to see that the condition (A) holds.

Now the Green's function $H_1(t,v)$ and $H_2(t,s)$ are same, that is

$$H_2(t,s)=\begin{cases} t(1-s), & if \quad 0\le t\le s\le 1,\\ s(1-t), & if \quad 0\le s\le t\le 1.\end{cases}$$

By taking subinterval $[a,b]=[\frac{1}{4},\frac{3}{4}]$, then we have

$$\limsup_{y\to 0+}\max_{t\in[0,1]}\frac{F(t,y)}{y}=0.1\lambda, \tag{4.2}$$

$$\liminf_{y\to +\infty}\min_{t\in[\frac{1}{4},\frac{3}{4}]}\frac{F(t,y)}{y}=\sqrt{11879.00025}\lambda. \tag{4.3}$$

Now we compute Q and q in Theorem 3.1. Since

$$\begin{aligned}
\varphi(t) &= \int_0^1\left[\int_0^1 H_1(t,v)H_2(v,s)a(s)dv\right]ds\\
&= \int_0^1 H_1(t,v)\left[\int_0^v s(1-v)\frac{ds}{\sqrt{1-s}}+\int_v^1(1-s)v\frac{ds}{\sqrt{1-s}}\right]dv\\
&= \frac{4}{3}\int_0^1 H_1(t,v)\left(1-\sqrt{1-v}\right)(1-v)dv\\
&= \frac{4}{3}\Big(\int_0^t v(1-t)(1-v)(1-\sqrt{1-v})dv\\
&\quad +\int_t^1 t(1-v)^2(1-\sqrt{1-v})dv\Big)\\
&= \frac{2}{315}(1-t)\Big(24\left(-1+\sqrt{1-t}\right)+\left(70-48\sqrt{1-t}\right)t\\
&\quad +\left(-35+24\sqrt{1-t}\right)t^2\Big)\\
&\le \frac{2}{315}(1-t)\left(70t-11t^2\right).
\end{aligned}$$

So

$$\varphi(t)\le\frac{2}{315}(1-t)(70t-11t^2),$$

then

$$
\begin{aligned}
\max_{t\in[0,1]} \varphi(t) &\le \frac{2}{315}\left(\max_{t\in[0,1]}(1-t)\left(70t-11t^2\right)\right)\\
&= \frac{2}{315}\left(1-\frac{81-\sqrt{4251}}{33}\right)\left(\frac{70\left(81-\sqrt{4251}\right)}{33}-\frac{11\left(81-\sqrt{4251}\right)^2}{33^2}\right)\\
&= \frac{4\left(-83592+1417\sqrt{4251}\right)}{343035},
\end{aligned}
$$

and thus

$$
Q=\left(\max_{t\in[0,1]}\varphi(t)\right)^{-1} \ge \frac{343035}{4\left(-83592+1417\sqrt{4251}\right)}. \tag{4.4}
$$

On the other hand,

$$
\begin{aligned}
\phi(t) &= \int_0^1\left[\int_{\frac14}^{\frac34} H_1(t,v)H_2(v,s)a(s)dv\right]ds\\
&= \int_0^1 H_1(t,v)\left[\int_{\frac14}^{v} s(1-v)\frac{ds}{\sqrt{1-s}}+\int_v^{\frac34}(1-s)v\frac{ds}{\sqrt{1-s}}\right]dv\\
&= \frac{1}{12}\int_0^1 H_1(t,v)\left(9\sqrt{3}-16\sqrt{1-v}+\left(16\sqrt{1-v}-1-9\sqrt{3}\right)v\right)dv\\
&= \frac{1}{12}\int_0^t v(1-t)\left(9\sqrt{3}-16\sqrt{1-v}+\left(16\sqrt{1-v}-1-9\sqrt{3}\right)v\right)dv\\
&\quad+\frac{1}{12}\int_t^1 t(1-v)\left(9\sqrt{3}-16\sqrt{1-v}+\left(16\sqrt{1-v}-1-9\sqrt{3}\right)v\right)dv\\
&= \frac{1}{2520}(1-t)\Big(384\left(-1+\sqrt{1-t}\right)+\left(-35+630\sqrt{3}-768\sqrt{1-t}\right)t\\
&\quad+\left(-35-315\sqrt{3}+384\sqrt{1-t}\right)t^2\Big).
\end{aligned}
$$

It is easy to verify that

$$
\begin{aligned}
\min_{t\in[\frac14,\frac34]}\phi(t) &= \min\left\{\phi(\tfrac14),\phi(\tfrac34)\right\}=\phi(\tfrac34)\\
&= \frac{1}{2520}\left(1-\frac34\right)\left(384\left(-1+\frac12\right)+\left(-35+630\sqrt{3}-768\times\frac12\right)\times\frac34\right.\\
&\quad\left.+\left(-35-315\sqrt{3}+384\times\frac12\right)\times\frac{9}{16}\right)\\
&= \frac{-743+525\sqrt{3}}{17920}.
\end{aligned}
$$

Thus

$$q = \left(\min_{t \in [\frac{1}{4}, \frac{3}{4}]} \phi(t) \right)^{-1} = \frac{17920}{-743 + 525\sqrt{3}}. \tag{4.5}$$

It follows from (A_1) in Theorem 3.1 and $(4.2)-(4.5)$ that the problem (4.1) has at least one positive solution if

$$0.1\lambda < 9.74978, \quad \sqrt{11879.00025}\lambda > \frac{17920}{-743 + 525\sqrt{3}},$$

that is

$$\frac{17920}{\left(-743 + 525\sqrt{3}\right) \sqrt{11879.00025}} < \lambda < \frac{9.74978}{0.1}.$$

Note that

$$\frac{17920}{\left(-743 + 525\sqrt{3}\right) \sqrt{11879.00025}} \approx 0.988522,$$

$$\frac{9.74978}{0.1} \approx 97.4978.$$

Then we obtain that the approximate subinterval about λ is $(0.98853, 97.4978)$. In particular, we can see that the problem (4.1) has at least one positive solution, for $\lambda = 1$.

Example 4.2. Now we consider the following singular boundary value problem

$$\begin{cases} y^{(4)}(t) - \dfrac{\lambda}{t(1-t)} \Big(10^9 y^2 \mid \sin y \cos y^2 \mid + \dfrac{y^4}{1+y} \\ \qquad\qquad + 10^9 (1-t) t y^2 |\sin y| \Big)^{\frac{1}{3}} = 0, \quad 0 < t < 1, \\ y(0) = y(1) = y'(0) = y'(1) = 0. \end{cases} \tag{4.6}$$

Then the problem (4.6) has at least one positive solution.

It is obviously that $a(t) = \frac{1}{t(1-t)}$ is singular at $t = 1$ and $t = 0$. The boundary value problem (4.6) describes the deflection of an elastic beam rigidly fixed at both ends. Because of the singularity and the form of the problem (4.6), it seems to be difficult that the problem is solved by using the results obtained by [1, 4, 7, 10] as well as their extension. Now we study the problem (4.6) by making use of Theorem 3.2.

Let $a(t) = \frac{1}{t(1-t)}$,

$$F(t, y) = \lambda \Big(10^9 y^2 \mid \sin y \cos y^2 \mid + \frac{y^4}{1+y} + 10^9 (1-t) t y^3 \Big)^{\frac{1}{3}}, \quad 0 < t < 1.$$

It is easy to see that the condition (A) holds.

Now the Green's function of homogeneous linear problem $y^{(4)}(t) = 0, \quad 0 \leq t \leq 1$, $y(0) = y(1) = y'(0) = y'(1) = 0$, that is

$$H(t,s) = \begin{cases} \frac{1}{6}t^2(1-s)^2[(s-t)+2(1-t)s], & if \quad 0 \leq t \leq s \leq 1, \\ \frac{1}{6}s^2(1-t)^2[(t-s)+2(1-s)t], & if \quad 0 \leq s \leq t \leq 1. \end{cases}$$

By taking subinterval $[a,b] = [\frac{1}{4}, \frac{3}{4}]$, then we have

$$\limsup_{y \to 0+} \max_{t \in [0,1]} \frac{F(t,y)}{y} = 10^3 \sqrt[3]{\frac{5}{4}}\lambda, \tag{4.7}$$

$$\liminf_{y \to +\infty} \min_{t \in [\frac{1}{8}, \frac{7}{8}]} \frac{F(t,y)}{y} = \lambda. \tag{4.8}$$

Now we compute Q and q in Theorem 3.2. Since

$$\begin{aligned}
\varphi_1(t) &= \int_0^1 H(t,s)a(s)ds \\
&= \int_0^t \frac{1}{6}s^2(1-t)^2[(t-s)+2(1-s)t]\frac{1}{s(1-s)}ds \\
&+ \int_t^1 \frac{1}{6}t^2(1-s)^2[(s-t)+2(1-t)s]\frac{1}{s(1-s)}ds \\
&\leq \frac{1}{6}(1-t)^2 \int_0^t s^2((1-s)+2(1-s))\frac{1}{s(1-s)}ds \\
&+ \frac{1}{2}t^2 \int_t^1 (1-s)^2(s+2s)\frac{1}{s(1-s)}ds \\
&= \frac{1}{2}(1-t)^2 \int_0^t sds + \frac{1}{2}t^2 \int_t^1 3(1-s)ds \\
&= \frac{1}{2}(1-t)^2t^2 + \frac{3}{4}(1-t)^2t^2 = \frac{5}{4}(1-t)^2t^2.
\end{aligned}$$

So

$$\max_{t \in [0,1]} \varphi_1(t) \leq \frac{5}{4} \max_{t \in [0,1]}[(1-t)^2t^2] = \frac{5}{4} \times \frac{1}{16} = \frac{5}{64},$$

and thus

$$Q = \left(\max_{t \in [0,1]} \varphi_1(t)\right)^{-1} \geq 12.8. \tag{4.9}$$

On the other hand,

$$\begin{aligned}\phi_1(t) &= \int_0^1 H(t,s)a(s)ds \\ &= \int_0^t \frac{1}{6}s^2(1-t)^2[(t-s)+2(1-s)t]\frac{1}{s(1-s)}ds \\ &+ \int_t^1 \frac{1}{6}t^2(1-s)^2[(s-t)+2(1-t)s]\frac{1}{s(1-s)}ds \\ &= \frac{1}{6}(1-t)^2[t^3-\frac{t^2}{2}+t+(1-t)\ln(1-t)] \\ &+ \frac{1}{6}t^2[\frac{1}{2}(1-t)^2(3-2t)+t(1-t)+t\ln t].\end{aligned}$$

It is easy to verify that

$$\min_{t\in[\frac{1}{8},\frac{7}{8}]} \phi_1(t) = \phi_1(\frac{7}{8}) = \phi_1(\frac{1}{8}) = 0.002643$$

Thus

$$q = \left(\min_{t\in[\frac{1}{8},\frac{7}{8}]} \phi_1(t)\right)^{-1} = 378.358. \tag{4.10}$$

It follows from (A_1) in Theorem 3.1 and $(4.7) - (4.10)$ that the problem (4.6) has at least one positive solution if

$$\lambda < 12.8, \quad 10^3\sqrt[3]{\frac{5}{4}}\lambda > 378.358,$$

that is

$$0.351236 < \lambda < 11.8825.$$

Then we obtain that the approximate subinterval about λ is $(0.351236, 12.8)$. In particular, we can see that the problem (4.6) has at least one positive solution, for $\lambda = 1$.

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References

[1] P. W. Eloe and J. Henderson, *Positive solutions for higher order ordinary differential equations*, Electron J. Differential Equations, **1995** (1995), 1–8.

[2] C. P. Gupta, *Existence and uniquences results for bending of an elastic beam equation at resonance*, J. Math. Anal. Appl., **135 (1)** (1988), 208 – 225.

[3] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[4] R. Ma and H. Wang, *On the existence of positive solutions of fourth-order ordinary differential equations*, Appl. Anal., **59** (1995), 225 – 231.

[5] D. O'Regan, *Solvability of some fourth (and higher) order singular boundary value problems*, J. Math. Anal. Appl., **161(1)** (1991), 78–116.

[6] D. O'Regan, *Fourth (and higher) order singular boundary value problems*, Nonlinear Anal., **14(12)** (1990) 1001–1038.

[7] R. A. Usmani, *A uniqueness theorem for a boundary value problem*, Proc. Amer. Math. Soc., **77** (1979), 327-335.

[8] P. J. Y. Wong and R. Agarwal, *Eigenvalue of Lidstone boundary value problems*, Appl. Math. Comput., **104** (1999), 15-31.

[9] Z. L. Wei, *Positive solution for fourth order singular boundary value problems*, Acta Mathematic Sincia, **42** (1999), 715 - 722. (in Chinese)

[10] Y. Yang, *Fourth-order two-point boundary value problems*, Proc. Amer. Math. Soc., **104** (1988), 175-180.