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INEQUALITIES DESCRIBING THE GROWTH OF POLYNOMIALS

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Abstract. If $p(z) = \sum_{i=1}^{n} a_i z^i$ is a polynomial of degree n, then from the theorem of Maximum modulus $\max_{|z|=1} |p(Rz)| \leq R^n \max_{|z|=1} |p(z)|$. Dewan and Hans [Anal. Theory and Appl., **26** (2010), 1–6] obtained the bound of polynomial $\left|p(Rz) + \beta \left(\frac{R+1}{2}\right)^n p(z)\right|$, for some β with $|\beta| \leq 1$ and $R \geq 1$, in more general form of principle of maximum modulus of polynomial. The aim of this paper is to generalized the bound of above polynomial and some related inequalities by extending them to the class of polynomial having restricted zeros except $s-$fold zeros at the origin, where $0 \leq s < n$.

1. Introduction

Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n and denoted as P_n. According to well known result on the derivative of polynomial

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

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Inequality (1.1) known as Bernstein's Inequality [6] and a simple deduction from Maximum modulus principle [6]

$$\max_{|z|=R>1}|p(z)| \le R^n \max_{|z|=1}|p(z)|. \tag{1.2}$$

The inequalities (1.1) and (1.2) are best possible and equality holds for the polynomial having all its zeros at the origin.

It was shown by Ankeny and Rivlin [1] that if $p \in P_n$ and $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=R>1}|p(z)| \le \frac{R^n+1}{2} \max_{|z|=1}|p(z)|. \tag{1.3}$$

Aziz and Dawood [3] obtained a refinement of inequality (1.3) by considering $m = \min_{|z|=1}|p(z)|$ and show that, if $p \in P_n$ and $p(z) \neq 0$ in $|z| < 1$, then $R \ge 1$

$$\max_{|z|=R}|p(z)| \le \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}|p(z)|. \tag{1.4}$$

The inequality (1.3) and (1.4) are best possible and equality holds for $p(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta| = 1$.

Inequality (1.3) was generalized by Jain [5], who proved that if $p(z)$ is polynomial of degree n and $p(z) \neq 0$ in $|z| < 1$, then for any $|\beta| \le 1, R \ge 1$ and $|z| = 1$

$$\begin{aligned}&\left|p(Rz) + \beta\left(\frac{R^n+1}{2}\right)p(z)\right| \\ &\le \frac{1}{2}\left\{\left|R^n + \beta\left(\frac{R^n+1}{2}\right)\right| + \left|1 + \beta\left(\frac{R^n+1}{2}\right)\right|\right\}\max_{|z|=1}|p(z)|. \end{aligned}\tag{1.5}$$

As a generalization of inequality (1.4) and a refinement of inequality (1.5), Dewan and Hans [4] proved that for $p(z) \neq 0$ in $|z| < 1$ and for any $|\beta| \le 1$

$$\begin{aligned}&\left|p(Rz) + \beta\left(\frac{R^n+1}{2}\right)p(z)\right| \\ &\le \frac{1}{2}\left[\left\{\left|R^n + \beta\left(\frac{R^n+1}{2}\right)\right| + \left|1 + \beta\left(\frac{R^n+1}{2}\right)\right|\right\}\max_{|z|=1}|p(z)| \right.\\ &\quad \left. - \left\{\left|R^n + \beta\left(\frac{R^n+1}{2}\right)\right| - \left|1 + \beta\left(\frac{R^n+1}{2}\right)\right|\right\}\min_{|z|=1}|p(z)|\right], \end{aligned}\tag{1.6}$$

for $|z| = 1$ and $R \ge 1$. The result is best possible and equality holds for $p(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta|$.

Aziz and Dawood [3] obtained following result concerning the minimum modulus of polynomial $p(z)$ analogous to (1.2) on $|z| = 1$ by applying a restriction on $p \in P_n$. Basically, they proved that if $p \in P_n$ and having all its zeros in $|z| \leq 1$, then

$$\min_{|z|=R\geq 1} |p(z)| \geq R^n \min_{|z|=1} |p(z)|. \tag{1.7}$$

Dewan and Hans [4] generalized above inequality (1.7) due to Aziz and Dawood [3] and proved that if $p \in P_n$ and having all its zeros in $|z| \leq 1$, then for any $|\beta| \leq 1, R \geq 1$ and $|z| = 1$

$$\left| p(Rz) + \beta \left(\frac{R^n + 1}{2} \right) p(z) \right| \geq \left| R^n + \beta \left(\frac{R^n + 1}{2} \right) \right| \min_{|z|=1} |p(z)|. \tag{1.8}$$

The inequality (1.7) and (1.8) are best possible and equality holds for $p(z) = ae^{i\alpha} z^n, a > 0$.

In this paper, we obtained an extension of inequality (1.6) and (1.8) for the class of polynomial $p(z)$ of degree n, defined as

$$p(z) = z^s \left\{ \sum_{\nu=0}^{n-s} a_\nu z^\nu \right\}, \quad 0 \leq s < n$$

with s number of zeros at origin and we denote it as P_n^s. By taking $s = 0$, we get $p \in P_n$, *i.e.*, $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ and some other generalizations.

2. Lemmas

For the proof of results, we require following Lemmas. The following first lemma is due to Aziz [2].

Lemma 2.1. *If $p \in P_n$ and $p(z) \neq 0, |z| < k, k \geq 1$, then for $0 \leq r \leq 1$,*

$$\max_{|z|=r} |p(z)| \geq \left(\frac{r + k}{1 + k} \right)^n \max_{|z|=1} |p(z)|. \tag{2.1}$$

Lemma 2.2. *If $p \in P_n^s$ and having $n - s$ zeros in $|z| \leq k, k \leq 1$, then for $R \geq 1$*

$$\max_{|z|=R} |p(z)| \geq R^s \left(\frac{R + k}{1 + k} \right)^{n-s} \max_{|z|=1} |p(z)|. \tag{2.2}$$

Proof. Let $p(z) = z^s \left\{ \sum_{\nu=0}^{n-s} a_\nu z^\nu \right\}, 0 \leq s < n$ is a n^{th} degree polynomial and having all its zeros in $|z| \leq k, k \leq 1$ with an $s-$fold zeros at origin $0 \leq s < n$. Define

$$q(z) = z^n \overline{p(1/\overline{z})}.$$

Clearly $q(z)$ is of degree $n-s$ with $q(0) \neq 0$ and having no zeros in $|z| < (1/k)$, then from Lemma 2.1 for $0 \leq r \leq 1$

$$\max_{|z|=r} |q(z)| \geq \left(\frac{r+1/k}{1+1/k}\right)^{n-s} \max_{|z|=1} |q(z)|. \tag{2.3}$$

Equivalently

$$r^n \max_{|z|=1/r} |p(z)| \geq \left(\frac{r+1/k}{1+1/k}\right)^{n-s} \max_{|z|=1} |q(z)|. \tag{2.4}$$

By taking $1/r = R$, we have

$$\max_{|z|=R} |p(z)| \geq R^s \left(\frac{R+k}{1+k}\right)^{n-s} \max_{|z|=1} |p(z)|,$$

where $R \geq 1$. This prove Lemma. □

Lemma 2.3. *Let $f \in P_n^s$ and having $n-s$ zeros $|z| \leq k$, $k \leq 1$ and $p(z)$ is a polynomial of degree no exceeding that $f(z)$ with $s-$fold zeros at origin $0 \leq s < n$. If $|p(z)| \leq |f(z)|$ for $|z| = k$, then for any β with $|\beta| \leq 1$ and for $|z| = 1$*

$$\left| f(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} f(z) \right| \geq \left| p(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z) \right|, \tag{2.5}$$

where $R \geq 1$.

Proof. Since $|p(z)| \leq |f(z)|$ for $|z| = k$, therefore from Rouche's Theorem $f(z) + \alpha p(z)$ has all its zeros in $|z| \leq k, k \leq 1$ with $s-$fold zeros at origin, for some α with $|\alpha| < 1$. Applying Lemma 2.2 for $f(z) + \alpha p(z)$, we have for $|z| = 1$ and $R \geq 1$

$$|f(Rz) + \alpha p(Rz)| \geq R^s \left(\frac{R+k}{1+k}\right)^{n-s} |f(z) + \alpha p(z)|. \tag{2.6}$$

For any β with $|\beta| \leq 1$, we get for $|z| = 1$,

$$\{f(Rz) + \alpha p(Rz)\} + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} \{f(z) + \alpha p(z)\} \neq 0,$$

i.e.,

$$\begin{aligned} S(z) &= \left\{ f(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} f(z) \right\} \\ &\quad + \alpha \left\{ p(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z) \right\} \neq 0 \end{aligned} \tag{2.7}$$

on $|z| = k$. For appropriate choice of the argument of α, we have for $|z| = 1$,

$$\left| f(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} f(z) \right| \geq \left| p(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z) \right|. \quad (2.8)$$

If the above inequality (2.8) is not true for some, then there exist a point $z = z_0$ with $|z_0| = 1$ such that for $R \geq 1$

$$\left| f(Rz_0) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} f(z_0) \right| < \left| p(Rz_0) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z_0) \right|. \quad (2.9)$$

By taking α as

$$\alpha = -\frac{f(Rz_0) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} f(z_0)}{p(Rz_0) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z_0)}. \quad (2.10)$$

Therefore, from inequality (2.9), $|\alpha| < 1$. With this choice of α, we have $S(z_0) = 0$ for $|z_0| = 1$ from (2.7). But this contradict (2.7), *i.e.*, $S(z) \neq 0$ for all $|z| = 1$. On taking $|\alpha| \to 1$ in inequality (2.8), we have for $|z| = 1$

$$\left| f(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} f(z) \right| \geq \left| p(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z) \right|. \quad (2.11)$$

Which complete the proof of Lemma. □

If we take $f(z) = (z/k)^n M, M = \max_{|z|=k} |p(z)|$, then following result has been obtained.

Lemma 2.4. *If $p \in P_n^s$, then for any β with $|\beta| \leq 1$ and $|z| = 1$*

$$\left| p(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z) \right| \leq k^{-n} \left| R^n + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} \right| M, \quad (2.12)$$

where $R \geq 1$ and $M = \max_{|z|=k} |p(z)|$.

Lemma 2.5. *If $p \in P_n^s$, then for any β with $|\beta| \leq 1$ and $|z| = 1$*

$$\left| p(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z) \right| + \left| q(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} q(z) \right|$$
$$\leq \left\{ k^{-n} \left| R^n + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} \right| + k^{-s} \left| R^s + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} \right| \right\} M, \quad (2.13)$$

where $q(z) = (z/k)^{n+s}\overline{p(k^2/\overline{z})}$ and $M = \max_{|z|=k} |p(z)|$.

Proof. Let $p \in P_n^s$. For α with $|\alpha| > 1$, it follows from Rouches Theorem that the polynomial $S(z) = p(z) - \alpha(z/k)^s M$ does not vanish in $|z| < k$, then the polynomial

$$\begin{aligned} T(z) &= (z/k)^{n+s}\overline{S(k^2/\overline{z})} = (z/k)^{n+s}\overline{p(k^2/\overline{z})} - \overline{\alpha}(z/k)^n M \\ &= q(z) - \overline{\alpha}(z/k)^n M \end{aligned}$$

of degree n and has all its zeros in $|z| \leq k$ except $s-$fold zeros at origin, for $|z| = 1, |S(z)| = T(z)$. Therefore, using Lemma 2.2, for $T(z)$

$$\left|S(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}S(z)\right| \leq \left|T(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}T(z)\right| \quad (2.14)$$

for any β with $|\beta| \leq 1$ and $|z| = 1$. This implies that for $|z| = 1$,

$$\begin{aligned} &\left|(p(Rz) - \alpha R^s z^s M) + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}(p(z) - \alpha(z/k)^s M)\right| \\ &\leq \left|(q(Rz) - \overline{\alpha} R^n (z/k)^n M) + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}(q(z) - \overline{\alpha}(z/k)^n M)\right|, \end{aligned}$$

i.e., for $|z| = 1$ and $R \geq 1$

$$\begin{aligned} &\left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right| - |\alpha|k^{-s}\left|R^s+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|M \\ &\leq \left|\left\{q(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right\} - \overline{\alpha}k^{-n}\left\{R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right\}z^n M\right|. \quad (2.15) \end{aligned}$$

Since $|p(z)| = |q(z)|$ for $|z| = k$, therefore $M = \max_{|z|=k}|p(z)| = \max_{|z|=k}|q(z)|$ and $q(z)$ has $s-$fold zeros at origin. Now, applying Lemma 2.4 for $q(z)$, we have for $|z| = k$ and $R \geq 1$

$$\left|q(Rz) + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}q(z)\right| \leq k^{-n}\left|R^n + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|\max_{|z|=k}|q(z)|,$$

i.e., for $|\alpha| > 1$

$$\left|q(Rz) + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}q(z)\right| \leq |\alpha|k^{-n}\left|R^n + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|M.$$

Therefore inequality (2.15) implies that

$$\begin{aligned} &\left|p(Rz) + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right| - |\alpha|k^{-s}\left|R^s + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|M \\ &\leq |\alpha|k^{-n}\left|R^n + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|M - \left|q(Rz) + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|. \quad (2.16) \end{aligned}$$

By making $|\alpha| \to 1$, the Lemma follows. □

We have following subsequent result by taking $\beta = 0$ in Lemma 2.5 and Lemma 2.4 respectively.

Corollary 2.6. *If $p \in P_n^s$, then for $R \geq 1$ and $|z| = 1$*

$$|p(Rz)| + |q(Rz)| \leq \left(\frac{R^n}{k^n} + \frac{R^s}{k^s}\right) \max_{|z|=k} |p(z)|, \tag{2.17}$$

where $q(z) = (z/k)^{n+s}\overline{p(k^2/\overline{z})}$ and $k \leq 1$.

Corollary 2.7. *If $p \in P_n$, then for $R \geq 1$ and $k \leq 1$*

$$k^n \max_{|z|=1} |p(Rz)| \leq R^n \max_{|z|=k} |p(z)|. \tag{2.18}$$

3. Main results

Theorem 3.1. *If $p \in P_n^s$ and having $n - s$ zeros in $|z| \leq k, k \leq 1$ and $R \geq 1$, then for β with $|\beta| \leq 1$ and $|z| = 1$*

$$\left|p(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z)\right| \geq k^{-n} \left|R^n + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s}\right| m, \tag{3.1}$$

where $m = \min_{|z|=k} |p(z)|$. The result is best possible and equality holds for $p(z) = \alpha\,(z/k)^n, \alpha \geq 0$.

Proof. Let $p \in P_n^s$, having $n-s$ zeros in $|z| \leq k, k \leq 1$ and $m = \min_{|z|=k} |p(z)|$. If $p(z)$ has a zeros on $|z| = k$, then result is obvious. So, we suppose all the zeros of $p(z)$ in $|z| < k$ with $s-$fold zeros at origin, $0 \leq s < n$, *i.e.*, $m > 0$. Therefore, it follows from Rouche's Theorem, for any α with $|\alpha| < 1$, the polynomial $p(z) - \alpha(z/k)^n m$ of degree n has all its zeros in $|z| \leq k$ with $s-$fold zeros at origin. Applying Lemma 2.2, for $p(z) - \alpha(z/k)^n m$, we have for $R \geq 1$ and $|z| = k$

$$|p(Rz) - \alpha R^n (z/k)^n m| \geq R^s \left(\frac{R+k}{1+k}\right)^{n-s} |p(z) - \alpha(z/k)^n m|. \tag{3.2}$$

Therefore, for any β with $|\beta| \leq 1$, it is obvious that

$$(p(Rz) - \alpha R^n (z/k)^n m) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} (p(z) - \alpha(z/k)^n m) \neq 0$$

on $|z| = k$, *i.e.*,

$$\left\{p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right\}-\alpha\left\{R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right\}(z/k)^n m\neq 0. \quad (3.3)$$

As $|\alpha| < 1$, we have for $|\beta| < 1$ and $R \geq 1$

$$\left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right| > k^{-n}\left|R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|m, \quad (3.4)$$

for $|z| = 1$. For $|\beta| = 1$, the result follow by continuity of zeros. This completes the proof of Theorem. □

Remark 3.2. We have an inequality due to Zireh, Khojastehnejhad and Musawi [7] by putting $s = 0$ in inequality (3.1) and if we take $k = 1$ and $s = 0$ in Theorem 3.1, inequality (3.1) reduces to inequality (1.8).

As for $k = 1$, we also have following extension of inequality (1.8) due to Dewan and Hans [4] on the class of $p(z) = z^s\left\{\sum_{\nu=0}^{n-s} a_\nu z^\nu\right\}$, $0 \leq s < n$ with an $s-$fold zeros at origin.

Corollary 3.3. *If $p \in P_n^s$ and having $n - s$ zeros in $|z| \leq 1$, then for β with $|\beta| \leq 1$, $R \geq 1$ and $|z| = 1$*

$$\left|p(Rz)+\beta R^s\left(\frac{R+1}{2}\right)^{n-s}p(z)\right| \geq \left|R^n+\beta R^s\left(\frac{R+1}{2}\right)^{n-s}\right|m, \quad (3.5)$$

where $m = \min_{|z|=1}|p(z)|$. The result is best possible and equality holds for $p(z) = \alpha z^n, \alpha \geq 0$.

From Lemma 2.2 and with suitable choice of β, we get for $|z| = 1$ and $R \geq 1$

$$\left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right| = |p(Rz)| - |\beta|R^s\left(\frac{R+k}{1+k}\right)^{n-s}|p(z)|. \quad (3.6)$$

Combining above inequality (3.6) with inequality (3.1), we obtained for $|z| = 1$ and $m = \min_{|z|=k}|p(z)|$,

$$\begin{aligned}
&|p(Rz)| - |\beta|R^s\left(\frac{R+k}{1+k}\right)^{n-s}|p(z)| \\
&= \left||p(Rz)| + \beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right| \geq k^{-n}\left|R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|m \\
&\geq k^{-n}\left\{R^n - |\beta|R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right\}m \qquad (3.7)
\end{aligned}$$

and by making $\beta \to 1$, we deduced the following result.

Corollary 3.4. *If $p \in P_n^s$ and having $n-s$ zeros in $|z| \le k, k \le 1$, then for β with $|\beta| \le 1$ and $R \ge 1$*

$$\max_{|z|=R} |p(z)| \ge R^s \left(\frac{R+k}{1+k}\right)^{n-s} M + k^{-n} \left\{ R^n - R^s \left(\frac{R+k}{1+k}\right)^{n-s} \right\} m, \quad (3.8)$$

where $M = \max_{|z|=1} |p(z)|$ and $m = \min_{|z|=k} |p(z)|$. The equality in above holds for $p(z) = \alpha\,(z/k)^n$, $\alpha \ge 0$.

We next prove following extension of inequality (1.6) by using above Theorem 3.1.

Theorem 3.5. *If $p \in P_n^s$ and having $n-s$ zeros in $|z| \ge k, k \le 1$, then for β with $|\beta| \le 1$ and $R \ge 1$*

$$\begin{aligned}
&\left| p(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} p(z) \right| \\
&\le \frac{1}{2}\left[\left\{ k^{-n} \left| R^n + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} \right| + k^{-s} \left| R^s + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} \right| \right\} M \right. \\
&\quad \left. - \left\{ k^{-n} \left| R^n + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} \right| - k^{-s} \left| R^s + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} \right| \right\} m \right], \quad (3.9)
\end{aligned}$$

for $|z| = 1$, where $M = \max_{|z|=k} |p(z)|$ and $m = \min_{|z|=k} |p(z)|$. The result is sharp and equality holds for $p(z) = (z/k)^n + (z/k)^s$, $0 \le s < n$.

Proof. Since $p \in P_n^s$ and having $n-s$ zeros in $|z| \ge k$, therefore if $m = \min_{|z|=k} |p(z)|$, then $m \le |p(z)|$ for $|z| < k$. So, for λ with $|\lambda| < 1$, it follows from Rouche's Theorem that the polynomial $F(z) = p(z) - \lambda (z/k)^s m$ has no zeros in $|z| < k$ except $s-$fold zeros at origin, $0 \le s < n$. Define

$$\begin{aligned}
G(z) = (z/k)^{n+s}\overline{F(k^2/\overline{z})} &= (z/k)^{n+s}\overline{p(k^2/\overline{z})} - \overline{\lambda}(z/k)^n m \\
&= q(z) - \overline{\lambda}(z/k)^n m
\end{aligned}$$

such as $|F(z)| = G(z)$ for $|z| = 1$. It is clear that all the zeros of $G(z)$ lies in $|z| \le k$ with $s-$fold zeros at origin, therefore from Lemma 2.3 , we have for $|\beta| \le 1$, $R \ge 1$ and $|z| = 1$

$$\left| F(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} F(z) \right| \le \left| G(Rz) + \beta R^s \left(\frac{R+k}{1+k}\right)^{n-s} G(z) \right|, \quad (3.10)$$

which implies that,

$$\left|\left\{p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right\}-\lambda\left\{R^s+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right\}(z/k)^s m\right|$$
$$\leq\left|\left\{q(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}q(z)\right\}-\overline{\lambda}\left\{R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right\}(z/k)^n m\right|, \quad (3.11)$$

i.e., for $|z|=1$,

$$\left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right|-|\lambda|k^{-s}\left|R^s+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|m$$
$$\leq\left|\left\{q(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}q(z)\right\}-\overline{\lambda}\left\{R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right\}z^n m\right|. \quad (3.12)$$

Since all the zeros of $q(z)$ lies in $|z|\leq k$, from Theorem 3.1, therefore we have, for $|\beta|\leq 1$ and $|z|=1$

$$\left|q(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}q(z)\right|\geq k^{-n}\left|R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|m,$$

where $R\geq 1$ and $m=\min_{|z|=k}|p(z)|$.

Now, by suitable choice of argument of λ, we have from inequality (3.12) for $|z|=1$

$$\left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right|-|\lambda|k^{-s}\left|R^s+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|m$$
$$\leq\left|q(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}q(z)\right|-|\lambda|k^{-n}\left|R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|m. \quad (3.13)$$

and taking $|\lambda|\to 1$

$$\left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right|$$
$$\leq\left|q(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}q(z)\right|$$
$$-\left\{k^{-n}\left|R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|-k^{-s}\left|R^s+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|\right\}m. \quad (3.14)$$

Equivalently,

$$\begin{aligned} &2\left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right| \\ &\leq \left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right|+\left|q(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}q(z)\right| \\ &\quad -\left\{k^{-n}\left|R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|-k^{-s}\left|R^s+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|\right\}m. \end{aligned} \tag{3.15}$$

Using Lemma 2.5 in last inequality, we get for $|\beta|\leq 1$ and $|z|=1$

$$\begin{aligned} &2\left|p(Rz)+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}p(z)\right| \\ &\leq \left\{k^{-n}\left|R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|+k^{-s}\left|R^s+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|\right\}M \\ &\quad -\left\{k^{-n}\left|R^n+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|-k^{-s}\left|R^s+\beta R^s\left(\frac{R+k}{1+k}\right)^{n-s}\right|\right\}m, \end{aligned} \tag{3.16}$$

which follows the inequality (3.9). □

We obtained another generalization of inequality (1.6) by taking $k=1$ in above Theorem 3.5, which is as follows.

Corollary 3.6. *If $p\in P_n^s$ and having $n-s$ zeros in $|z|\geq 1$, then for β with $|\beta|\leq 1$ and $R\geq 1$*

$$\begin{aligned} &\left|p(Rz)+\beta R^s\left(\frac{R+1}{2}\right)^{n-s}p(z)\right| \\ &\leq \frac{1}{2}\left[\left\{\left|R^n+\beta R^s\left(\frac{R+1}{2}\right)^{n-s}\right|+\left|R^s+\beta R^s\left(\frac{R+1}{2}\right)^{n-s}\right|\right\}M\right. \\ &\quad \left.-\left\{\left|R^n+\beta R^s\left(\frac{R+1}{2}\right)^{n-s}\right|-\left|R^s+\beta R^s\left(\frac{R+k}{2}\right)^{n-s}\right|\right\}m\right], \end{aligned} \tag{3.17}$$

for $|z|=1$, where $M=\max_{|z|=k}|p(z)|$ and $m=\min_{|z|=1}|p(z)|$. The result is sharp and equality holds for $p(z)=p(z)=(z/k)^n+(z/k)^s$, $0\leq s<n$.

Remark 3.7. By taking $s = 0$ in inequality (3.9), we get an inequality due to Zireh, Khojastehnejhad and Musawi [7] and on putting $s = 0$ in Corollary 3.6, we have inequality (1.6) due to Dewan and Hans [4].

If we take $\beta = 0$ in above Theorem 3.5, we have following generalization of inequality (1.4) due to Aziz and Dawood [3].

Corollary 3.8. *If $p \in P_n^s$ and having $n - s$ zeros in $|z| < k, k \leq 1$, then for $R \geq 1$*

$$\max_{|z|=1} |p(Rz)| \leq \frac{1}{2}\left[\left(\frac{R^n}{k^n} + \frac{R^s}{k^s}\right) M - \left(\frac{R^n}{k^n} - \frac{R^s}{k^s}\right) m\right], \tag{3.18}$$

where $M = \max_{|z|=1} |p(z)|$ and $m = \min_{|z|=k} |p(z)|$. Equality in above holds for $p(z) = z^s + z^n$, $0 \leq s \leq n$.

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