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C-CLASS FUNCTIONS ON COUPLED FIXED POINT THEOREM FOR MIXED MONOTONE MAPPINGS ON PARTIALLY ORDERED DISLOCATED QUASI METRIC SPACES

Arslan Hojat Ansari[1] and Ali Mutlu[2]

[1]Department of Mathematics
Karaj Branch, Islamic Azad, University, Karaj, Iran
e-mail: analsisamirmath2@gmail.com

[2]Department of Mathematics
Faculty of Arts and Science, Celal Bayar University
Muradiye Campus, 45030, Manisa, Turkey
e-mail: abgamutlu@gmail.com

Abstract. In this paper, we prove a coupled fixed point theorem for mixed monotone mapping F-φ of C-class function in partial ordered dislocated quasi metric space. The introduced theorem extend and unify various known fixed point conclusion.

1. Introduction

In [5], some fixed point theorems were proved on complete and compact metric spaces and also given some examples. Fixed points of a generalised weakly contractive map was established the existence in T- orbitally completed metric spaces with respect to boundary spaces, besides some examples were present in [3]. Some fixed point results were attained for generalised weakly contractive mappings with some auxiliary functions with respect to b-metric spaces and examples were given suitable for this construction [2].

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[0]Corresponding author: Ali Mutlu.

The first author described f is a continuous function of C-class and also some theorems, consequences and examples in [1]. A common fixed point theorem was testified for multi-valued and single-valued mappings in a dislocated metric spaces verifying a weak contractive condition with function of C with respect to F-φ-weak contractively condition in [4].

The presence of a coupled fixed point theorem is proved for a mixed monotone mapping $T : X \times X \to X$ under a universalized contraction and build the exclusiveness under a supplementary assumption on partially ordered complete dislocated quasi metric space in [8].

In this article using C-class function, we prove a coupled fixed point theorem for mixed monotone mapping F-φ-of C-class function in partial ordered dislocated quasi metric space. The proffered theorem extend and unify various known fixed point conclusion.

2. Mathematical preliminaries

We state primary definitions and notations to be used throughout the article, where N is the non-negative set of integers. Dislocated quasi metric spaces are designated as below:

Definition 2.1. ([1, 8]) Let X be a nonempty set. If $d : X \times X \to [0, \infty)$ is a map which satisfies the conditions for all $x, y, z \in X$:

(1) (dm) $d(x, y) = d(y, x) = 0$ implies $x = y$,
(2) (dm) $d(x, y) \leq d(x, z) + d(z, y)$,

then the d is named a dislocated quasi metric (d_q-metric) on X. If d verifies $d(x, x) = 0$, it is named a quasi-metric on X. If d verifies $d(x, y) = d(y, x)$, so it is named dislocated metric.

Definition 2.2. ([1, 8]) A sequence $\{x_n\}$ in $dq-$metric space (dislocated quasi-metric space) (X, d) is named Cauchy if given $\varepsilon > 0$, $n_0 \in N$ with for all $n, m \geq n_0$, implies $d(x_n, x_m) < \varepsilon$ or $d(x_m, x_n) < \varepsilon$, namely

$$\min\{d(x_n, x_m), d(x_m, x_n)\} < \varepsilon.$$

In this instance x is named the dq limit of $\{x_n\}$.

Definition 2.3. ([1, 8]) A dq metric space space (X, d) is named complete if every Cauchy sequence in it is dq convergent.

Lemma 2.4. ([1, 8]) *Assume (X, d) is a $dq-$metric space. If $f : X \to X$ is a contraction map, at the time $\{f^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.*

Definition 2.5. [8] Assume $(X, \preceq)$ is a partially ordered set and $T : X \times X \to X$. If T possesses the mixed monotone property, then for any $x, y \in X, T(x, y)$ is monotone non-decreasing with respect to x, and also it is monotone nonincreasing with respect to y for all $x, y \in X$ such that

$$x_1, x_2, y \in X, \;\; x_1 \preceq x_2 \quad \Longrightarrow \quad T(x_1, y) \preceq T(x_1, y)$$

and

$$y_1, y_2, x \in X, \;\; y_1 \preceq y_2 \quad \Longrightarrow \quad T(x, y_1) \succeq T(x, y_2).$$

Definition 2.6. ([8]) If $T(x, y) = x$ and $T(y, x) = y$ then a member $(x, y) \in X \times X$ is named a coupled fixed point of the mapping T.

For any $(x, y), (u, v) \in X \times X$, the multiply space $X \times X$ is equipped with the metric ρ described by

$$\rho((x, y), (u, v)) = \frac{d(x, u) + d(y, v)}{2}.$$

So ρ is a $dq-$metric.

Theorem 2.7. ([8]) *Assume (X, d) is a complete $dq-$metric space and $f : X \to X$ is a continuous contraction map. In this case f possesses an individual fixed point.*

In 2014, the concept of $C-$class functions (see Definition 2.8) was introduced by Ansari in [1] and is important,for example see numbers (1), (2), (9) and (15) from Example 2.9. Also see [2], [4], [6] and [7].

Definition 2.8. ([1]) A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called C-function if for any $s, t \in [0, \infty)$, the following conditions hold:

(1) $F(s, t) \leq s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0, 0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ will denote the class of all C-functions.

PROBLEM: This is an open problem, can we say that for all $F \in \mathcal{C}, F(0, 0) = 0$?

Example 2.9. ([1]) Following examples show that the class $\mathcal{C}$ is nonempty:

(1) $F(s, t) = s - t$.
(2) $F(s, t) = ms$, $0 < m < 1$
(3) $F(s, t) = \frac{s}{(1+t)^r}$ for some $r \in (0, \infty)$.
(4) $F(s, t) = \log(t + a^s)/(1 + t)$ for some $a > 1$.

(5) $F(s,t) = \ln(1+a^s)/2$ for $a > e$. Indeed $F(s,1) = s$ implies that $s = 0$.
(6) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$ for $r \in (0,\infty)$.
(7) $F(s,t) = s\log_{t+a} a$ for $a > 1$.
(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.
(9) $F(s,t) = s\beta(s)$, where $\beta : [0,\infty) \to [0,1)$.
(10) $F(s,t) = s - \frac{t}{k+t}$.
(11) $F(s,t) = s - \varphi(s)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
(12) $F(s,t) = sh(s,t)$, where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$.
(13) $F(s,t) = s - (\frac{2+t}{1+t})t$.
(14) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$.
(15) $F(s,t) = \phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.
(16) $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$.

Definition 2.10. ([5]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 2.11. We denote altering distance function as Ψ.

Definition 2.12. ([1]) An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

Remark 2.13. We denote ultra altering distance function as Φ_u.

Lemma 2.14. ([3]) *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ and*

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$;
(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;
(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$.

Remark 2.15. We note that also can see $\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon$ and $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon$.

3. Main results

Now the main conclusions are certified.

Theorem 3.1. *Supposing $(X, \preceq)$ is a partially ordered set and (X, d) complete dq-metric space. Assuming there is a function $\varphi \in \Phi_u$, $F \in C$ and also assume $T : X \times X \to X$ is a continuous mapping possessing the mixed monotone property on X,*

$$d(T(x,y), T(u,v)) \leq F(\rho((x,y),(u,v)), \varphi(\rho((x,y),(u,v)))), \tag{3.1}$$

for all $x, y, u, v \in X$ for which $x \preceq u$, $y \succeq v$. If there exists $(x_0, y_0) \in X \times X$ with $x_0 \preceq T(x_0, y_0)$, $y_0 \preceq T(y_0, x_0)$, at the time T possesses an individual coupled fixed point.

Proof. Let us suppose $(x_0, y_0) \in X \times X$ with $x_0 \preceq T(x_0, y_0)$, $y_0 \succeq T(y_0, x_0)$. Besides that let us assume $x_2 = T(x_1, y_1)$, $y_2 = T(y_1, x_1)$ so $x_1 \preceq x_2$ and $y_1 \succeq y_2$. To continue to do so through, two sequences $\{x_n\}, \{y_n\}$ are built in X with $x_{n+1} = T(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$ and

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots \tag{3.2}$$

and

$$y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots \tag{3.3}$$

define

$$\delta_n = \rho((x_n, y_n), (x_{n+1}, y_{n+1})) = \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2}. \tag{3.4}$$

Now, enforcing if the inequality (3.1) is implemented with $(x, y) = (x_n, y_n)$, then $(u, v) = (x_{n+1}, y_{n+1})$ for $n \geq 0$. Utilizing properties of ϕ, we possess

$$\begin{aligned} &d(x_n, x_{n+1}) \\ &= d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\ &\leq F(\rho((x_{n-1}, y_{n-1}), (x_n, y_n)), \varphi(\rho((x_{n-1}, y_{n-1}), (x_n, y_n)))) \\ &\leq \rho((x_{n-1}, y_{n-1}), (x_n, y_n)). \end{aligned} \tag{3.5}$$

Similarly, we can obtain

$$d(y_n, y_{n+1}) \leq \rho((x_{n-1}, y_{n-1}), (x_n, y_n)). \tag{3.6}$$

Thus, we acquire

$$\begin{aligned} &\rho((x_n, y_n), (x_{n+1}, y_{n+1})) = d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\ &\leq F(\rho((x_{n-1}, y_{n-1}), (x_n, y_n)), \varphi(\rho((x_{n-1}, y_{n-1}), (x_n, y_n)))) \end{aligned} \tag{3.7}$$

and

$$\delta_n = \rho((x_n, y_n), (x_{n+1}, y_{n+1})) \leq \rho((x_{n-1}, y_{n-1}), (x_n, y_n)) = \delta_{n-1}.$$

Thus $\{\delta_n\}$ is monotone decreasing bounded to the bottom. Hereby, there is some $\delta \geq 0$ with $\lim_{n\to\infty} \delta_n = \delta$. Using by equation (3.7) with $n \to \infty$ we get

$$\delta \leq F(\delta, \varphi(\delta)) \leq \delta, \tag{3.8}$$

so, $\delta = 0$ or $\varphi(\delta) = 0$. Therefore $\delta = 0$.

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. By Lemma 2.14, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$\begin{aligned} &\lim_{k\to\infty} \rho((x_{n_k}, y_{n_k}), (x_{m_k}, y_{m_k})) = \lim_{k\to\infty} \tfrac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k})}{2} = \varepsilon, \\ &\lim_{k\to\infty} \rho((x_{n_k-1}, y_{n_k-1}), (x_{m_k-1}, y_{m_k-1})) \\ &= \lim_{k\to\infty} \tfrac{d(x_{n_k-1}, x_{m_k-1}) + d(y_{n_k-1}, y_{m_k-1})}{2} = \varepsilon. \end{aligned} \tag{3.9}$$

Now, enforcing if the inequality (3.1) is implemented with $(x, y) = (x_{n_k}, y_{n_k})$ then $(u, v) = (x_{m_k}, y_{m_k})$ for $n_k > m_k > k$. We possess

$$\begin{aligned} &d(x_{n_k}, x_{m_k}) \\ &= d(T(x_{n_k-1}, x_{m_k-1}), T(y_{n_k-1}, y_{m_k-1})) \\ &\leq F(\rho(x_{n_k-1}, y_{n_k-1}), (x_{m_k-1}, y_{m_k-1})), \varphi(\rho((x_{n_k-1}, y_{n_k-1}), (x_{m_k-1}, y_{m_k-1})))) \\ &\leq \rho((x_{n_k-1}, y_{n_k-1}), (x_{m_k-1}, y_{m_k-1})). \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} d(y_{n_k}, y_{m_k}) \leq F(&\rho((x_{n_k-1}, y_{n_k-1}), (x_{m_k-1}, y_{m_k-1})), \\ &\varphi(\rho((x_{n_k-1}, y_{n_k-1}), (x_{m_k-1}, y_{m_k-1})))). \end{aligned}$$

Thus we acquire

$$\begin{aligned} &\rho((x_{n_k}, y_{n_k}), (x_{m_k}, y_{m_k})) \\ &\leq F(\rho((x_{n_k-1}, y_{n_k-1}), (x_{m_k-1}, y_{m_k-1})), \\ &\qquad \varphi(\rho((x_{n_k-1}, y_{n_k-1}), (x_{m_k-1}, y_{m_k-1})))) \end{aligned} \tag{3.10}$$

with $n \to \infty$. We get

$$\varepsilon \leq F(\varepsilon, \varphi(\varepsilon)),$$

so $\varepsilon = 0$ or $\phi(\varepsilon) = 0$. Therefore $\varepsilon = 0$. This is a contraction, therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and rest proof is similar [8]. □

With choice $F(s, t) = \phi(s)$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$ in Theorem 3.1 we get the following corollary.

Corollary 3.2. *Supposing $(X, \preceq)$ is a partially ordered set and (X, d) complete dq-metric space. Assuming there is $\phi : [0, \infty) \to [0, \infty)$ a continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$ and also assume $T : X \times X \to X$ is a continuous mapping possessing the mixed monotone property on X,*

$$d(T(x, y), T(u, v)) \leq \phi(\rho((x, y), (u, v))),$$

for all $x, y, u, v \in X$ for which $x \preceq u$, $y \succeq v$. If there exists $(x_0, y_0) \in X \times X$ with $x_0 \preceq T(x_0, y_0)$, $y_0 \preceq T(y_0, x_0)$, at the time T possesses an individual coupled fixed point.

Theorem 3.3. *Let $(X, \preceq)$ be partially ordered set and (X, d) complete dq-metric space. Let $T : X \times X \to X$ be a mapping having mixed monotone property. Assume there is a function φ like in Theorem 3.1. Assume that X has the following property:*

(a) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
(b) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$, then there exist $x, y \in X$ such that

$$x = T(x, y) \quad \text{and} \quad y = T(y, x),$$

that is, T has a unique coupled fixed point.

Proof. Following the proof of Theorem 3.1, we construct a non-decreasing sequence $\{x_n\}$ in X and a non-increasing sequence $\{y_n\}$ in X such that $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$ for all $n \geq 0$ and the following equations are satisfied

$$\begin{aligned} \lim_{n\to\infty} d(x_n, x) &= \lim_{n\to\infty} d(x, x_n) = 0, \\ \lim_{n\to\infty} d(y_n, y) &= \lim_{n\to\infty} d(y, y_n) = 0. \end{aligned} \tag{3.11}$$

Thus by properties of X, we have $x_n \preceq u$ and $y_n \succeq v$ for all $n \geq 0$. By (3.1), we have

$$\begin{aligned} d(x_{n+1}, T(x, y)) &= d(T(x_n, y_n), T(x, y)) \\ &\leq F(\rho((x_n, y_n), (x, y))), \varphi(\rho((x_n, y_n), (x, y)))) \\ &\leq \rho((x_n, y_n), (x, y))). \end{aligned}$$

On letting $n \to \infty$, using (3.11), we get that

$$\lim_{n\to\infty} d(x_{n+1}, T(x, y)) = 0. \tag{3.12}$$

Again from (3.1), we can get

$$\begin{aligned} d(T(x,y),x_{n+1}) &= d(T(x,y),T(x_n,y_n)) \\ &\leq F(\rho((x,y),(x_n,y_n)),\varphi(\rho((x,y),(x_n,y_n)))) \\ &\leq \rho((x,y),(x_n,y_n)). \end{aligned}$$

On letting $n \to \infty$, using (3.11), we get that

$$\lim_{n\to\infty} d(T(x,y),x_{n+1}) = 0. \tag{3.13}$$

On the other hand, from (ii), we get

$$d(x,T(x,y)) \leq d(x,x_{n+1}) + d(x_{n+1},T(x,y)). \tag{3.14}$$

Letting $n \to \infty$ in the above inequality, using (3.11) and (3.12), we have $d(x,T(x,y)) = 0$. Similarly, we can show that

$$d(T(x,y),x) \leq d(T(x,y),x_{n+1}) + d(x_{n+1},x).$$

Letting $n \to \infty$ in the above inequality, using (3.13) and (3.11), we have $d(T(x,y),x) = 0$. Then from property of d_q-metric $x = T(x,y)$. Similarly we can show $y = T(y,x)$. □

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