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EXISTENCE OF NONOSCILLATORY SOLUTIONS TO FIRST ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS WITH MIXED ARGUMENTS

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Abstract. In this paper, the authors present some sufficient conditions for the existence of nonoscillatory solutions to following first order neutral type difference equation

$$\Delta(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0,\ n \geq n_0$$

having both delay and advanced terms via Banach contraction principle. Examples are provided to illustrate the results. The results obtained in this paper generalized some of the existing results.

1. Introduction

In this paper, we are concerned with the existence of nonoscillatory solutions of first order neutral type difference equation of the form

$$\Delta(x_n + a_n x_{n-l} + b_n x_{n+m}) + p_n x_{n-k} - q_n x_{n+r} = 0,\ n \geq n_0 \tag{1.1}$$

where n_0 a nonnegative integer, subject to the following conditions:

(H_1) $\{a_n\}$ and $\{b_n\}$ are real sequences;

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(H_2) $\{p_n\}$ and $\{q_n\}$ are non-negative real sequences;
(H_3) l and m are positive integers and k and r are non-negative integers.

Let $\theta = \max\{l, k\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfying the equation (1.1) for all $n \geq n_0$. A nontrivial solution of equation (1.1) is said to be nonoscilltory if it is either eventually positive or eventually negative, and oscillatory otherwise.

Recently many authors investigated the existence of nonoscillatory solutions of first order neutral type difference equations, see for example [1, 2, 5], and the references cited therein. In [8, 13], the authors discussed the existence of nonoscillatory solutions for the equation

$$\Delta(x_n + px_{n-l}) + q_n x_{n-m} = 0, \quad n \geq n_0,$$

when $p \geq 0$ or $p \leq 0$. In [11], the authors investigated the existence of nonoscillatory solutions to the equation

$$\Delta(x_n + c_n x_{n-l}) + p_n x_{n-k} - q_n x_{n-m} = 0, \quad n \geq n_0.$$

They established sufficient conditions for the existence of nonoscillatory solutions depending on the different ranges of $\{c_n\}$. In [8], existence of nonoscillatory solutions of first order neutral type difference equation of the form

$$\Delta(x_n - c_n x_{n-l}) = p_n x_{n-k}$$

was studied.

On the other hand, there has been great interest in studying the oscillatory behavior of first and higher order neutral type difference equations with delay and advanced terms, see for example [3, 4, 10, 12, 14], and the references cited therein.

Motivated by the above studies, in this paper we obtain sufficient conditions for the existence of nonoscillatory solutions for the equation (1.1). The results obtained in this paper generalize and complement to some of the known results in [8, 11, 13].

2. Existence of nonoscillatory solutions

In this section, we present sufficient conditions for the existence of nonoscillatory solutions of equations (1.1) using Banach contraction principle. We begin with the following theorem.

Theorem 2.1. (Banach's Contraction Mapping Principle) *A contraction mapping on a complete metric space has a unique fixed point.*

Theorem 2.2. *With respect to the difference equation* (1.1), *assume that* $0 \leq a_n \leq a < 1$ *and* $0 \leq b_n \leq b \leq 1 - a$ *for all* $n \geq n_0$. *If*

$$\sum_{n=n_0}^{\infty} p_n < \infty \quad \textit{and} \quad \sum_{n=n_0}^{\infty} q_n < \infty, \tag{2.1}$$

then equation (1.1) *has a bounded nonoscillatory solution.*

Proof. Inview of condition (2.1), we may choose an integer $N > n_0$ so that

$$N \geq n_0 + \theta \tag{2.2}$$

sufficiently large such that

$$\sum_{s=n}^{\infty} p_s \quad \leq \quad \frac{M_2 - \alpha}{M_2} \quad \text{for all} \;\; n \geq N \tag{2.3}$$

and

$$\sum_{s=n}^{\infty} q_s \quad \leq \quad \frac{\alpha - (a+b)M_2 - M_1}{M_2} \quad \text{for all} \;\; n \geq N, \tag{2.4}$$

where M_1 and M_2 are positive constants such that

$$(a+b)M_2 + M_1 < M_2 \quad \text{and} \quad \alpha \in ((a+b)M_2 + M_1, M_2).$$

Let B be the set of all bounded real sequences $\{x_n\}$ defined for all $n \geq n_0$ with supremum norm $\|x\| = \sup_{n \geq n_0} |x_n|$. Then clearly B is a Banach space. Set

$$S = \{x \in B : M_1 \leq x_n \leq M_2, n \geq n_0\}.$$

It is clear that S is a bounded, closed, and convex subset of B. Define an operator $T : S \to B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} + \sum_{s=n}^{\infty} [p_s x_{s-k} - q_s x_{s+r}], & n \geq N, \\ (Tx)_N, \quad n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, we have

$$(Tx)_n \leq \alpha + \sum_{s=n}^{\infty} p_s x_{s-k} \leq \alpha + M_2 \sum_{s=n}^{\infty} p_s \leq M_2$$

and

$$\begin{aligned} (Tx)_n &\geq \alpha - a_n x_{n-l} - b_n x_{n+m} - \sum_{s=n}^{\infty} q_s x_{s+r} \\ &\geq \alpha - aM_2 - bM_2 - M_2 \sum_{s=n}^{\infty} q_s \geq M_1, \end{aligned}$$

where we have used (2.3) and (2.4). Thus $TS \subset S$.

Next, we show that T is a contraction mapping on S. Let $x, y \in S$ and $n \geq N$. Then

$$|(Tx)_n - (Ty)_n| \leq a_n|x_{n-l} - y_{n-l}| + b_n|x_{n+m} - y_{n+m}| + \sum_{s=n}^{\infty}(p_s|x_{s-k} - y_{s-k}| + q_s|x_{s+r} - y_{s+r}|)$$

or

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq \|x - y\|(a + b + \sum_{s=n}^{\infty}(p_s + q_s)) \\ &\leq \left(a + b + \frac{M_2 - \alpha}{M_2} + \frac{\alpha - (a+b)M_2 - M_1}{M_2}\right)\|x - y\| \\ &= \lambda_1\|x - y\|, \end{aligned}$$

where $\lambda_1 = (1 - \frac{M_1}{M_2})$. This implies that $\|Tx - Ty\| \leq \lambda_1\|x - y\|$. Since $\lambda_1 < 1$, T is a contraction mapping on S. By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed. □

Theorem 2.3. *With respect to the difference equation* (1.1)*, assume that* $0 \leq a_n \leq a < 1$*, and* $a - 1 < b \leq b_n \leq 0$ *for all* $n \geq n_0$*. If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), we can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty} p_s \leq \frac{(1+a)M_4 - \alpha}{M_4} \quad \text{for all } n \geq N \tag{2.5}$$

and

$$\sum_{s=n}^{\infty} q_s \leq \frac{\alpha - aM_4 - M_3}{M_4} \quad \text{for all } n \geq N, \tag{2.6}$$

where M_3 and M_4 are positive constants such that

$$M_3 + aM_4 < (1+b)M_4 \quad \text{and} \quad \alpha \in (M_3 + aM_4, (1+b)M_4).$$

Let B be the space as defined in Theorem 2.2. Set

$$S = \{x \in B : M_3 \leq x_n \leq M_4, \ \ n \geq n_0\}.$$

It is clear that S is a bounded, closed, and convex subset of S. Define an operator $T : S \to B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} + \sum_{s=n}^{\infty} [p_s x_{s-k} - q_s x_{s+r}], & n \geq N, \\ (Tx)_N, \quad n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, we have from (2.5) and (2.6)

$$(Tx)_n \leq \alpha - bM_4 + M_4 \sum_{s=n}^{\infty} p_s \leq M_4,$$

and

$$(Tx)_n \geq \alpha - a_n M_4 - M_4 \sum_{s=n}^{\infty} q_s \geq M_3.$$

This proves that $TS \subset S$. Next we show that T is a contraction mapping. Let $x, y \in S$ and $n \geq N$. Then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq \|x - y\| \left(a - b + \sum_{s=n}^{\infty} (p_s + q_s) \right) \\ &\leq \lambda_2 \|x - y\|, \end{aligned}$$

where $\lambda_2 = (1 - \frac{M_3}{M_4})$. This implies that $\|Tx - Ty\| \leq \lambda_2 \|x - y\|$. Since $\lambda_2 < 1$, T is a contraction mapping on S. Hence by Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. □

Theorem 2.4. *With respect to the difference equation* (1.1)*, assume that* $1 < a \leq a_n \leq d < \infty$ *and* $0 \leq b_n \leq b < a - 1$ *for all* $n \geq n_0$*. If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ so that

$$N + l \geq n_0 + k, \tag{2.7}$$

sufficiently large such that

$$\sum_{s=n}^{\infty} p_s \leq aM_6 - \alpha, \quad n \geq N, \tag{2.8}$$

and

$$\sum_{s=n}^{\infty} q_s \leq \frac{\alpha - dM_5 - (1 + b)M_6}{M_6}, \quad n \geq N, \tag{2.9}$$

where M_5 and M_6 are positive constants such that

$$dM_5 + (1+b)M_6 < aM_6 \quad \text{and} \quad \alpha \in (dM_5 + (1+b)M_6, aM_6).$$

Let B be the space as defined in Theorem 2.2. Set

$$S = \{x \in B : M_5 \le x_n \le M_6, \ \ n \ge n_0\}.$$

Obviously S is a bounded, closed, and convex subset of B. Define a mapping $T : S \to B$ as follows:

$$(Tx)_n = \begin{cases} \dfrac{1}{a_{n+l}} \left\{ \alpha - x_{n+l} - b_{n+l}x_{n+l+m} + \displaystyle\sum_{s=n+l}^{\infty} [p_s x_{s-k} - q_s x_{s+r}] \right\}, & n \ge N, \\ (Tx)_N, \quad n_0 \le n \le N. \end{cases}$$

Clearly, Tx is continuous. For $n \ge N$ and $x \in S$, we have from (2.8) and (2.9), respectively, that

$$(Tx)_n \le \frac{1}{a_{n+l}} \left(\alpha + M_6 \sum_{s=n}^{\infty} p_s \right) \le \frac{1}{a} \left(\alpha + M_6 \sum_{s=n}^{\infty} p_s \right) \le M_6$$

and

$$\begin{aligned} (Tx)_n &\ge \frac{1}{a_{n+l}} \left(\alpha - (1+b)M_6 - M_6 \sum_{s=n}^{\infty} q_s \right) \\ &\ge \frac{1}{d} \left(\alpha - (1+b)M_6 - M_6 \sum_{s=n}^{\infty} q_s \right) \ge M_5. \end{aligned}$$

Thus $TS \subset S$. Next we show that T is a contraction mapping on S. If $x, y \in S$, and $n \ge N$, then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\le \frac{1}{a} \|x - y\| \left(1 + b + \sum_{s=n}^{\infty} [p_s + q_s] \right) \\ &\le \lambda_3 \|x - y\|, \end{aligned}$$

where $\lambda_3 = (1 - \frac{dM_5}{M_6})$. This implies that $\|Tx - Ty\| \le \lambda_3 \|x - y\|$. Since $\lambda_3 < 1$, T is a contraction mapping on S. Therefore, by Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed. □

Theorem 2.5. *With respect to the difference equation* (1.1)*, assume that* $1 < a \le a_n \le d < \infty$ *and* $1 - a < b \le b_n \le 0$ *for all* $n \ge n_0$*. If condition* (2.1) *holds then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty} p_s \leq \frac{(a+b)M_8 - \alpha}{M_8}, \quad n \geq N \tag{2.10}$$

and

$$\sum_{s=n}^{\infty} q_s \leq \frac{\alpha - dM_7 - M_8}{M_8}, \quad n \geq N, \tag{2.11}$$

where M_7 and M_8 are positive constants such that

$$dM_7 + M_8 < (a+b)M_8, \quad \text{and} \quad \alpha \in (dM_7 + M_8, (a+b)M_8).$$

Let B be the space as defined in Theorem 2.2. Set

$$S = \{x \in B : M_7 \leq x_n \leq M_8, \ \ n \geq n_0\}.$$

Clearly S is a bounded, closed and convex subset of B. Define a mapping $T : S \to B$ as follows:

$$(Tx)_n = \begin{cases} \dfrac{1}{a_{n+l}} \left\{ \alpha - x_{n+l} - b_{n+l} x_{n+l+m} + \displaystyle\sum_{s=n+l}^{\infty} (p_s x_{s-k} - q_s x_{s+r}) \right\}, & n \geq N, \\ (Tx)_N, \quad n_0 \leq n \leq N. \end{cases}$$

It is clearly that Tx is continuous. For $n \geq N$ and $x \in S$, we have from (2.10), and (2.11), respectively that

$$(Tx)_n \leq \frac{1}{a} \left(\alpha - bM_8 + M_8 \sum_{s=n}^{\infty} p_s \right) \leq M_8$$

and

$$(Tx)_n \geq \frac{1}{d} \left(\alpha - M_8 - M_8 \sum_{s=n}^{\infty} q_s \right) \geq M_7.$$

This implies that $TS \subset S$. Further if $x, y \in S$ and $n \geq N$, then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq \frac{1}{a} \|x - y\| \left(1 - b + \sum_{s=n}^{\infty} (p_s + q_s) \right) \\ &= \lambda_4 \|x - y\|, \end{aligned}$$

where $\lambda_4 = (1 - \frac{dM_7}{M_8})$. This implies that $\|Tx - Ty\| \leq \lambda_4 \|x - y\|$. Since $\lambda_4 < 1$, T is a contraction mapping on S. By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. □

Theorem 2.6. *With respect to the difference equation* (1.1)*, assume that* $-1 < a \le a_n \le 0$ *and* $0 \le b_n \le b \le 1 + a$ *for all* $n \ge n_0$*. If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty} p_s \le \frac{(1+a)M_{10} - \alpha}{M_{10}}, \quad n \ge N \tag{2.12}$$

and

$$\sum_{s=n}^{\infty} q_s \le \frac{\alpha - bM_{10} - M_9}{M_{10}}, \quad n \ge N, \tag{2.13}$$

where M_9 and M_{10} are positive constants such that

$$M_9 + bM_{10} < (a+b)M_{10} \quad \text{and} \quad \alpha \in (M_9 + bM_{10}, (1+a)M_{10}).$$

Let B be the space as defined in Theorem 2.1. Set

$$S = \{x \in B : M_9 \le x_n \le M_{10}, \ \ n \ge n_0\}.$$

Clearly S is a bounded, closed and convex subset of B. Define a mapping $T : S \to B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} + \displaystyle\sum_{s=n}^{\infty} \left(p_s x_{s-k} - q_s x_{s+r}\right), & n \ge N, \\ (Tx)_N, \quad n_0 \le n \le N. \end{cases}$$

Obviously, Tx is continuous. For $n \ge N$ and $x \in S$, from (2.12) and (2.13), it follows that

$$(Tx)_n \le \alpha - aM_{10} + M_{10} \sum_{s=n}^{\infty} p_s \le M_{10}$$

and

$$(Tx)_n \ge \alpha - bM_{10} - M_{10} \sum_{s=n}^{\infty} q_s \ge M_9.$$

Thus, $TS \subset S$. Next we show that T is a contraction mapping on S. If $x, y \in S$, and $n \ge N$, then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\le \|x-y\| \left(-a + b + \sum_{s=n}^{\infty} (p_s + q_s) \right) \\ &= \lambda_5 \|x - y\| \end{aligned}$$

where $\lambda_5 = (1 - \frac{M_9}{M_{10}})$. This implies that $\|Tx - Ty\| \le \lambda_5 \|x-y\|$. Since $\lambda_5 < 1$, T is a contraction mapping on S. By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed. □

Theorem 2.7. *With respect to the difference equation* (1.1)*, assume that* $-1 < a \le a_n \le 0$ *and* $-1 - a < b \le b_n \le 0$ *for all* $n \ge n_0$*. If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty} p_s \le \frac{(1+a+b)M_{12} - \alpha}{M_{12}}, \quad n \ge N \tag{2.14}$$

and

$$\sum_{s=n}^{\infty} q_s \le \frac{\alpha - M_{11}}{M_{12}}, \quad n \ge N, \tag{2.15}$$

where M_{11} and M_{12} are positive constants such that

$$M_{11} < (1+a+b)M_{12} \quad \text{and} \quad \alpha \in (M_{11}, (1+a+b)M_{12}).$$

Let B be the space as defined in Theorem 2.2. Set

$$S = \{x \in B : M_{11} \le x_n \le M_{12}, \ \ n \ge n_0\}.$$

It is clear that S is a bounded, closed and convex subset of B. Define an operator $T : S \to B$ as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} + \displaystyle\sum_{s=n}^{\infty}(p_s x_{s-k} - q_s x_{s+r}), & n \ge N, \\ (Tx)_N, \quad n_0 \le n \le N. \end{cases}$$

Clearly Tx is continuous. For $n \ge N$ and $x \in S$, from (2.14) and (2.15), it follows that

$$(Tx)_n \le \alpha - aM_{12} - bM_{12} + M_{12} \sum_{s=n}^{\infty} p_s \le M_{12}$$

and

$$(Tx)_n \ge \alpha - M_{12} \sum_{s=n}^{\infty} q_s \ge M_{11}.$$

This implies that $TS \subset S$. If $x, y \in S$ and $n \ge N$, then we have

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\le \|x - y\| \left(-a - b + \sum_{s=n}^{\infty}(p_s + q_s) \right) \\ &= \lambda_6 \|x - y\|, \end{aligned}$$

where $\lambda_6 = (1 - \frac{M_{11}}{M_{12}})$. This implies that $\|Tx - Ty\| \leq \lambda_6 \|x - y\|$. Since $\lambda_6 < 1$, T is a contraction mapping on S. By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. □

Theorem 2.8. *With respect to the difference equation* (1.1)*, assume that* $-\infty < d \leq a_n \leq a < -1$ *and* $0 \leq b_n \leq b < -a - 1$ *for all* $n \geq n_0$*. If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty} p_s \leq \frac{dM_{13} + \alpha}{M_{14}}, \quad n \geq N \tag{2.16}$$

and

$$\sum_{s=n}^{\infty} q_s \leq \frac{(-a - 1 - b)M_{14} - \alpha}{M_{14}}, \quad n \geq N, \tag{2.17}$$

where M_{13} and M_{14} are positive constants such that

$$-dM_{13} < (-a - 1 - b)M_{14}, \quad \text{and} \quad \alpha \in (-dM_{13}, (-a - 1 - b)M_{14}).$$

Let B be the space as defined in Theorem 2.2. Set

$$S = \{x \in B : M_{13} \leq x_n \leq M_{14}, \ \ n \geq n_0\}.$$

Clearly S is a bounded, closed and convex subset of B. Define a mapping $T : S \to B$ as follows:

$$(Tx)_n = \begin{cases} -\dfrac{1}{a_{n+l}}\{\alpha + x_{n+l} + b_{n+l}x_{n+l+m} - \displaystyle\sum_{s=n+l}^{\infty}(p_s x_{n-k} - q_s x_{s+r})\}, & n \geq N, \\ (Tx)_N, \quad n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, from (2.16) and (2.17), that

$$(Tx)_n \leq -\frac{1}{a}\left(\alpha + M_{14} + bM_{14} + M_{14}\sum_{s=n}^{\infty} p_s\right) \leq M_{14}$$

and

$$(Tx)_n \geq -\frac{1}{d}\left(\alpha - M_{14}\sum_{s=n}^{\infty} q_s\right) \geq M_{13}.$$

Thus $TS \subset S$. If $x, y \in S$ and $n \geq N$, then we have

$$|(Tx)_n - (Ty)_n| \leq -\frac{1}{a}\|x-y\| \left(1 + b + \sum_{s=n}^{\infty}(p_s + q_s)\right)$$
$$= \lambda_7 \|x-y\|,$$

where $\lambda_7 = (1 - \frac{M_{13}}{M_{14}})$. This implies that $\|Tx - Ty\| \leq \lambda_7\|x-y\|$. Since $\lambda_7 < 1$, T is a contraction mapping on S. By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. □

Theorem 2.9. *With respect to the difference equation* (1.1)*, assume that* $-\infty < d \leq a_n \leq a < -1$ *and* $a + 1 < b \leq b_n \leq 0$ *for all* $n \geq n_0$*. If condition* (2.1) *holds, then equation* (1.1) *has a bounded nonoscillatory solution.*

Proof. In view of condition (2.1), one can choose an integer $N > n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty} p_s \quad \leq \quad \frac{dM_{15} + bM_{16} + \alpha}{M_{16}}, \quad n \geq N \tag{2.18}$$

and

$$\sum_{s=n}^{\infty} q_s \quad \leq \quad \frac{(-a-1)M_{16} - \alpha}{M_{16}}, \quad n \geq N, \tag{2.19}$$

where M_{15} and M_{16} are positive constants such that

$$-dM_{15} - bM_{16} < (-a-1)M_{16} \quad \text{and} \quad \alpha \in (-dM_{15} - bM_{16}, (-a-1)M_{16}).$$

Let B be the space as defined in Theorem 2.2. Set

$$S = \{x \in B : M_{15} \leq x_n \leq M_{16}, \ \ n \geq n_0\}.$$

It is clear that S is a bounded, closed and convex subset of B. Define a mapping $T : S \to B$ as follows:

$$(Tx)_n = \begin{cases} -\dfrac{1}{a_{n+l}}\left\{\alpha + x_{n+l} + b_{n+l}x_{n+l+m} - \displaystyle\sum_{s=n+l}^{\infty}(p_s x_{n-k} - q_s x_{s+r})\right\}, & n \geq N, \\ (Tx)_N, \quad n_0 \leq n \leq N. \end{cases}$$

Clearly Tx is continuous. For $n \geq N$ and $x \in S$, we have from (2.18) and (2.19) that

$$(Tx)_n \leq -\frac{1}{a}\left(\alpha + M_{16} + M_{16}\sum_{s=n}^{\infty} p_s\right) \leq M_{16}$$

and

$$(Tx)_n \geq -\frac{1}{d}\left(\alpha + bM_{16} - M_{16}\sum_{s=n}^{\infty} q_s\right) \geq M_{15}.$$

This implies that $TS \subset S$. If $x, y \in S$ and $n \geq N$, then

$$\begin{aligned}|(Tx)_n - (Ty)_n| &\leq -\frac{1}{a}\|x-y\|\left(1 - b + \sum_{s=n}^{\infty}(p_s + q_s)\right)\\ &= \lambda_8\|x-y\|,\end{aligned}$$

where $\lambda_8 = (1 - \frac{dM_{15}}{M_{16}})$. This implies that $\|Tx - Ty\| \leq \lambda_8\|x-y\|$. Since $\lambda_8 < 1$, T is a contraction mapping on S. By Theorem 2.1, T has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof. □

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the neutral difference equation of the form

$$\begin{aligned}&\Delta\left(x_n + \frac{1}{2}x_{n-1} + \frac{1}{3}x_{n+2}\right) + \left(\frac{4(n-1)}{3n(n+2)(n+3)} + \frac{3n-1}{2n^2(n+1)}\right)x_{n-1}\\ &\quad + \frac{1}{(n+3)^2}x_{n+2} = 0, \quad n \geq 1. \qquad (3.1)\end{aligned}$$

Here $a_n = \frac{1}{2}$, $b_n = \frac{1}{3}$, $p_n = \frac{4(n-1)}{3n(n+2)(n+3)} + \frac{3n-1}{2n^2(n+1)}$, $q_n = \frac{1}{(n+3)^2}$. One can easily verify that all conditions of Theorem 2.2 are satisfied, and hence equation (3.1) has a bounded nonoscillatory solution. In fact $\{x_n\} = \{\frac{n+1}{n}\}$ is one such solution of equation (3.1).

Example 3.2. Consider a neutral difference equation of the form

$$\Delta\left(x_n + \frac{1}{4}x_{n-3} - \left(\frac{3}{4} - \frac{1}{4^n}\right)x_{n+1}\right) + \frac{1}{4^n}x_{n-1} - \frac{1}{4^{n+1}}x_{n+2} = 0, \quad n \geq 1. \qquad (3.2)$$

Here $a_n = \frac{1}{4}$, $b_n = -(\frac{3}{4} - \frac{1}{4^n})$, $p_n = \frac{1}{4^n}$, $q_n = \frac{1}{4^{n+1}}$. A straight forward verification shows that all conditions of Theorem 2.3 are satisfied, and hence equation (3.2) has a bounded nonoscillatory solution. In fact $\{x_n\} = \{2 + (-1)^n\}$ is one such solution of equation (3.2).

Example 3.3. Consider the neutral difference equation of the form

$$\Delta\left(x_n - \frac{1}{2}x_{n-1} + \frac{1}{4}x_{n+1}\right) + \frac{1}{2(n-1)(n+1)}x_{n-1} - \frac{(5n+13)}{4(n+1)^2(n+3)}x_{n+1} = 0, \quad n \geq 2. \tag{3.3}$$

Here $a_n = \frac{-1}{2}$, $b_n = \frac{1}{4}$, $p_n = \frac{1}{2(n-1)(n+1)}$, $q_n = \frac{(5n+13)}{4(n+1)^2(n+3)}$. It is easy to verify that all conditions of Theorem 2.6 are valid. We note that $\{x_n\} = \{\frac{n}{n+1}\}$ is a bounded nonoscillatory solution of equation (3.3).

Example 3.4. Consider a neutral difference equation of the form

$$\Delta\left(x_n - \frac{1}{2}\left(\frac{3}{4} - \frac{1}{2^n}\right)x_{n-2} - \frac{1}{4}x_{n+2}\right) + \frac{1}{2^n}x_{n-1} - \frac{1}{2^n}x_{n+1} = 0, \quad n \geq 1. \tag{3.4}$$

Here $a_n = -\frac{1}{2}(\frac{3}{4} - \frac{1}{2^n})$, $b_n = \frac{-1}{4}$, $p_n = q_n = \frac{1}{2^n}$. It is easy to verify that all conditions of Theorem 2.7 are satisfied. In fact $\{x_n\} = \{1 + \frac{1}{2^n}\}$ is a bounded nonoscillatory solution of equation (3.4).

Example 3.5. Consider a neutral difference equation of the form

$$\Delta(x_n - 4x_{n-1} - 2x_{n+1}) + \frac{1}{2^n(2+2^n)}x_{n-1} - \frac{4}{2^n}x_{n+2} = 0, \quad n \geq 1. \tag{3.5}$$

Here $a_n = -4$, $b_n = -2$, $p_n = \frac{1}{2^n(2+2^n)}$ and $q_n = \frac{4}{2^n}$. One can easily verify that all conditions of Theorem 2.9 are valid. Hence equation (3.5) has a bounded nonoscillatory solution. In fact $\{x_n\} = \{1 + \frac{1}{2^n}\}$ is one such solution of equation (3.5).

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