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NOTE ON A VOLTERRA-FREDHOLM TYPE INTEGRODIFFERENTIAL EQUATION IN TWO VARIABLES

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Abstract. In this note, we prove the existence and the compactness of the set of solutions for a Volterra-Fredholm type integrodifferential equation in two variables. The main tool is the fixed point theorem of Krasnosel'skii together with a necessary and sufficient condition for subsets to be relatively compact in an appropriate Banach space. An illustrative example is given.

1. Introduction

In this paper, we consider the following Volterra-Fredholm type integrodifferential equation in two variables

$$\begin{aligned} u(x,y) = g(x,y) &+ \textstyle\int_0^x \int_0^y H(x,y,s,t,u(s,t),D_1u(s,t))dsdt \\ &+ \textstyle\int_0^1 \int_0^1 K(x,y,s,t,u(s,t),D_1u(s,t))dsdt, \end{aligned} \tag{1.1}$$

where $(x,y) \in \Omega = [0,1] \times [0,1]$ and $g : \Omega \to \mathbb{R}$; $H : \Delta \times \mathbb{R}^2 \to \mathbb{R}$; $K : \Omega \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are given functions with $\Delta = \{(x,y,s,t) \in \Omega \times \Omega : 0 \leq s \leq x \leq 1,$

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$0 \leq t \leq y \leq 1\}$. Denote by $D_1 u = \frac{\partial u}{\partial x}$, the partial derivative of a function $u(x,y)$ defined on Ω, with respect to the first variable.

Nonlinear integral and integrodifferential equations of various types appear in the mathematical description of the applications in other fields of science, such as economics, mechanics and physics, see Corduneanu ([3], Ch.6) and the references given therein. Solving such equations and studing the existence, uniqueness and some basic properties of solutions have been extensively interested by many authors. In general, the main results have been obtained via the fundamental methods in which the fixed point theorems are often applied.

In [8], Lungu and Rus established some results relative to existence, uniqueness, integral inequalities and data dependence for solutions of the following functional Volterra-Fredholm integral equation in two variables with deviating argument in a Banach space by Picard operators technique

$$\begin{aligned} u(x,y) &= g(x,y,h(u)(x,y)) \\ &\quad + \int_0^x \int_0^y K\left(x,y,s,t,u(s,t)\right) dsdt, \quad (x,y) \in \mathbb{R}_+^2. \end{aligned} \tag{1.2}$$

In [12], based on the applications of the well known Banach fixed point theorem coupled with Bielecki type norm and a certain integral inequality with explicit estimate, B. G. Pachpatte proved uniqueness and other properties of solutions of the following Fredholm type integrodifferential equation

$$x(t) = g(t) + \int_a^b f(t,s,x(s),x'(s),\cdots,x^{n-1}(s))ds, \quad t \in [a,b], \tag{1.3}$$

where x, g, f are real valued functions and $n \geq 2$ is an integer. With the same methods, B. G. Pachpatte studied the existence, uniqueness and some basic properties of solutions of the Fredholm type integral equation in two variables as follows, see [13],

$$u(x,y) = f(x,y) + \int_0^a \int_0^b g\left(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)\right) dtds \tag{1.4}$$

and those of certain Volterra integral and integrodifferential equations in two variables, see [14].

In [1], Aghajani *et al.* proved some results on the existence, uniqueness and estimation of solutions of the following Fredholm type integro-differential equations in two variables

$$u(x,y) = f(x,y) + \int_a^b \int_c^d g\left(x,y,s,t,u(s,t),D_1u(s,t),D_2u(s,t)\right) dtds, \tag{1.5}$$

by using Perov's fixed point theorem, where f, g are given real valued functions, u is the unknown function to be found and $D_i u(x_1,x_2) = \frac{\partial u}{\partial x_i}(x_1,x_2)$, for $i = 1, 2$.

In [7], Lauran established sufficient conditions for the existence of solutions of the integral equation of Volterra type by using the concepts of nonexpansive operators, contraction principles and the Schaefer's fixed point theorem.

In [9]–[11], using tools of functional analysis and a fixed point theorem of Krasnosel'skii type, solvability and asymptotically stable of nonlinear functional integral equations in one variable or two variables, or N variables have been investigated.

Recently in [4], the authors have proved the existence of a solution in the function space $X_1 = \{u \in C(\Omega;\mathbb{R}) : D_1u \in C(\Omega;\mathbb{R})\}$ for the following nonlinear integral equation of type Fredholm in two dimensional of the form

$$u(x,y) = g(x,y) + \int_0^1 \int_0^1 K(x,y,s,t,u(s,t),D_1u(s,t))dsdt, \tag{1.6}$$

where $(x,y) \in \Omega = [0,1] \times [0,1]$ and $g : \Omega \to \mathbb{R}$; $K : \Omega \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are given functions. On the other hand, the uniqueness of a solution or the compactness of set of solutions have been also deduced. The main tool is the Banach fixed point theorem or Schauder fixed point theorem together with the definitions of suitable Banach spaces and appropriate conditions for subsets to be relatively compact in these spaces.

Continuing the above mentioned works, because of mathematical context, we show that the fixed point theorem of Krasnosel'skii can be applied in order to obtain the existence result and the compactness of the set of solutions for (1.1). This is also a development based on the paper [4]. Our paper is organized as follows. Section 2 is devoted to the presentation of preliminaries, it consists of the definition of Banach space $(X_1, \|\cdot\|_{X_1})$

$$\begin{aligned} &X_1 = \{u \in C(\Omega;\mathbb{R}) : D_1u \in C(\Omega;\mathbb{R})\}, \\ &\|u\|_{X_1} = \sup_{(x,y)\in\Omega} \left(|u(x,y)| + |D_1u(x,y)|\right), \; u \in X_1 \end{aligned} \tag{1.7}$$

and a necessary and sufficient condition for relatively compactness of subsets in this space. The existence of solutions for (1.1) will be presented in section 3. On the other hand, the compactness of the solutions set is also proved. Finally, we give an illustrated example.

2. Preliminaries

Put $\Omega = [0,1] \times [0,1]$, we denote $\Delta = \{(x,y,s,t) \in \Omega \times \Omega : 0 \le s \le x \le 1, 0 \le t \le y \le 1\}$. Denote by $D_1u = \frac{\partial u}{\partial x}$, the partial derivative of a function $u(x,y)$ defined on Ω with respect to the first variable. By $X = C(\Omega;\mathbb{R})$, we denote the space of all continuous functions from Ω into $\mathbb{R}$ equipped with the standard norm:

$$\|u\|_X = \sup_{(x,y)\in\Omega} |u(x,y)|, \quad u \in X. \tag{2.1}$$

Put

$$X_1 = \{u \in X = C(\Omega;\mathbb{R}) : D_1 u \in X\}. \tag{2.2}$$

It is clear that $C^1(\Omega;\mathbb{R}) \subset X_1 \subset X$ and that they do not coincide. We have the following lemmas, see [4].

Lemma 2.1. *X_1 is a Banach space with the norm defined by*

$$\|u\|_{X_1} = \|u\|_X + \|D_1 u\|_X\,, \ \ u \in X_1. \tag{2.3}$$

Lemma 2.2. *Let $F \subset X_1$. Then F is relatively compact in X_1 if and only if the following conditions are satisfied*

(i) $\exists M > 0 : \|u\|_{X_1} \le M, \quad \forall\, u \in F$;

(ii)

$$\begin{aligned}&\forall \varepsilon > 0, \exists \delta > 0 : \forall (x,y), (\bar{x},\bar{y}) \in \Omega, |x-\bar{x}| + |y-\bar{y}| < \delta \\ &\Longrightarrow \sup_{u\in F} \left(|u(x,y) - u(\bar{x},\bar{y})| + |D_1 u(x,y) - D_1 u(\bar{x},\bar{y})|\right) < \varepsilon.\end{aligned} \tag{2.4}$$

For convenient, we recall the following theorem that will be used in Section 3. It is well known that, two main results of fixed point theory are Schauder's and Banach's theorems (also called contraction mapping principle), Krasnoselskii combined them into the following result.

Theorem 2.3. ([2], [5]) *Let M be a nonempty bounded closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that $U : M \to X$ is a contraction and $C : M \to X$ is a completely continuous operator such that $U(x) + C(y) \in M$, $\forall\, x, y \in M$. Then $U + C$ has a fixed point in M.*

3. The existence and compactness of the solutions set

In order to obtain the main result, in this section, we make the following assumptions.

(A1) $g \in X_1$;

(A2) $H \in C(\Delta \times \mathbb{R}^2;\mathbb{R})$, $\frac{\partial H}{\partial x} \in C(\Delta \times \mathbb{R}^2;\mathbb{R})$ such that there exist nonnegative functions h_0, $h_1 : \Delta \to \mathbb{R}$ with the following properties:

(i) $|H(x,y,s,t,u,v) - H(x,y,s,t,\bar{u},\bar{v})|$
$\le h_0(x,y,s,t)\,[|u-\bar{u}| + |v-\bar{v}|]\,,$

(ii) $\left|\frac{\partial H}{\partial x}(x,y,s,t,u,v) - \frac{\partial H}{\partial x}(x,y,s,t,\bar{u},\bar{v})\right|$
$\le h_1(x,y,s,t)\,[|u-\bar{u}| + |v-\bar{v}|]\,,$

for all $(x,y,s,t) \in \Delta$, $u, v, \bar{u}, \bar{v} \in \mathbb{R}$;

(A3) $K \in C(\Omega \times \Omega \times \mathbb{R}^2;\mathbb{R})$ such that $\frac{\partial K}{\partial x} \in C(\Omega \times \Omega \times \mathbb{R}^2;\mathbb{R})$ and there exist nonnegative functions k_0, $k_1 : \Omega \times \Omega \to \mathbb{R}$ with the following properties:

(i) $|K(x,y,s,t,u,v)| \leq k_0(x,y,s,t)\,(1+|u|+|v|)$,

(ii) $\left|\frac{\partial K}{\partial x}(x,y,s,t,u,v)\right| \leq k_1(x,y,s,t)\,(1+|u|+|v|)$,

for all $(x,y,s,t) \in \Omega\times\Omega,\ u,v\in\mathbb{R}$;

(A4) $\bar{\beta}_1+\bar{\beta}_2<1$, where

$$\bar{\beta}_1 = \sup_{(x,y)\in\Omega}\int_0^y h_0(x,y,x,t)dt + \sup_{(x,y)\in\Omega}\int_0^x\int_0^y \left(h_0(x,y,s,t)+h_1(x,y,s,t)\right)dsdt,$$

$$\bar{\beta}_2 = \sup_{(x,y)\in\Omega}\int_0^1\int_0^1 k_0(x,y,s,t)dsdt + \sup_{(x,y)\in\Omega}\int_0^1\int_0^1 k_1(x,y,s,t)dsdt.$$

Theorem 3.1. *Let the functions* g, H, K *in* (1.1) *satisfy the assumptions* (A1)-(A4). *Then the equation* (1.1) *has a solution in* X_1. *Furthermore, the set of solutions of this equation is compact in* X_1.

Proof. We rewrite (1.1) as follows

$$u(x,y) = (Au)(x,y), \quad (x,y)\in\Omega, \tag{3.1}$$

where

$$\begin{cases} (Au)(x,y) = (Uu)(x,y) + (Cu)(x,y), \\ (Uu)(x,y) = g(x,y) + \int_0^x\int_0^y H(x,y,s,t,u(s,t),D_1u(s,t))dsdt, \\ (Cu)(x,y) = \int_0^1\int_0^1 K(x,y,s,t,u(s,t),D_1u(s,t))dsdt,\ (x,y)\in\Omega,\ u\in X_1. \end{cases} \tag{3.2}$$

A simple verification shows that $Uu,\ Cu\in X_1,\ \forall\ u\in X_1$. For $M>0$, considering a closed ball in X_1 as follows

$$B_M = \{u\in X_1 : \|u\|_{X_1}\leq M\}. \tag{3.3}$$

We can show that there exists $M>0$ such that

(i) $Uu+Cv\in B_M$, for every $u,\ v\in B_M$,

and the operators U, C satisfies the conditions as below,

(ii) $U: B_M\to X_1$ is a contraction map,

(iii) $C: B_M\to X_1$ is continuous,

(iv) $F = C(B_M)$ is relatively compact in X_1.

We verify (i). Indeed, for every $u \in B_M$, for all $(x,y) \in \Omega$, we have

$$\begin{aligned}
&|(Uu)(x,y)| \\
&\leq \|g\|_X + \int_0^x \int_0^y |H(x,y,s,t,u(s,t),D_1u(s,t))|\, dsdt \\
&\leq \|g\|_X + \int_0^x \int_0^y |H(x,y,s,t,u(s,t),D_1u(s,t)) - H(x,y,s,t,0,0)|\, dsdt \\
&\quad + \int_0^x \int_0^y |H(x,y,s,t,0,0)|\, dsdt \\
&\leq \|g\|_X + \int_0^x \int_0^y h_0(x,y,s,t)\,(1+|u(s,t)|+|D_1u(s,t)|)\, dsdt \\
&\quad + \int_0^x \int_0^y |H(x,y,s,t,0,0)|\, dsdt \\
&\leq \|g\|_X + \int_0^x \int_0^y h_0(x,y,s,t)\left(1+\|u\|_{X_1}\right) dsdt \\
&\quad + \int_0^x \int_0^y |H(x,y,s,t,0,0)|\, dsdt,
\end{aligned}$$

so

$$\begin{aligned}
&|(Uu)(x,y)| \\
&\leq \|g\|_X + (1+M) \sup_{(x,y)\in\Omega} \int_0^x \int_0^y h_0(x,y,s,t) dsdt \\
&\quad + \sup_{(x,y)\in\Omega} \int_0^x \int_0^y |H(x,y,s,t,0,0)|\, dsdt,
\end{aligned} \tag{3.4}$$

it gives

$$\begin{aligned}
\|Uu\|_X \leq \|g\|_X &+ (1+M) \sup_{(x,y)\in\Omega} \int_0^x \int_0^y h_0(x,y,s,t) dsdt \\
&+ \sup_{(x,y)\in\Omega} \int_0^x \int_0^y |H(x,y,s,t,0,0)|\, dsdt.
\end{aligned} \tag{3.5}$$

On the other hand, we get

$$\begin{aligned}
&D_1(Uu)(x,y) \\
&= D_1g(x,y) + \int_0^y H(x,y,x,t,u(x,t),D_1u(x,t))dt \\
&\quad + \int_0^x \int_0^y \frac{\partial H}{\partial x}(x,y,s,t,u(s,t),D_1u(s,t))dsdt \\
&= D_1g(x,y) + \int_0^y \left[H(x,y,x,t,u(x,t),D_1u(x,t)) - H(x,y,x,t,0,0)\right] dt \\
&\quad + \int_0^y H(x,y,x,t,0,0)dt \\
&\quad + \int_0^x \int_0^y \left[\frac{\partial H}{\partial x}(x,y,s,t,u(s,t),D_1u(s,t)) - \frac{\partial H}{\partial x}(x,y,s,t,0,0)\right] dsdt \\
&\quad + \int_0^x \int_0^y \frac{\partial H}{\partial x}(x,y,s,t,0,0)dsdt.
\end{aligned} \tag{3.6}$$

Therefore

$$\begin{aligned}
&|D_1(Au)(x,y)| \\
&\leq \|D_1g\|_X + \int_0^y |H(x,y,x,t,u(x,t),D_1u(x,t)) - H(x,y,x,t,0,0)|\, dt \\
&\quad + \int_0^y |H(x,y,x,t,0,0)|\, dt \\
&\quad + \int_0^x \int_0^y \left|\frac{\partial H}{\partial x}(x,y,s,t,u(s,t),D_1u(s,t)) - \frac{\partial H}{\partial x}(x,y,s,t,0,0)\right| dsdt \\
&\quad + \int_0^x \int_0^y \left|\frac{\partial H}{\partial x}(x,y,s,t,0,0)\right| dsdt
\end{aligned}$$

$$
\begin{aligned}
&\le \|D_1 g\|_X + \int_0^y h_0(x,y,x,t)\,(1+|u(x,t)|+|D_1 u(x,t)|)\,dt \\
&\quad + \int_0^y |H(x,y,x,t,0,0)|\,dt \\
&\quad + \int_0^x \int_0^y h_1(x,y,s,t)\,(1+|u(s,t)|+|D_1 u(s,t)|)\,dsdt \\
&\quad + \int_0^x \int_0^y \left|\frac{\partial H}{\partial x}(x,y,s,t,0,0)\right| dsdt \\
&\le \|D_1 g\|_X + (1+M)\left[\int_0^y h_0(x,y,x,t)dt + \int_0^x \int_0^y h_1(x,y,s,t)dsdt\right] \qquad (3.7)\\
&\quad + \int_0^y |H(x,y,x,t,0,0)|\,dt + \int_0^x \int_0^y \left|\frac{\partial H}{\partial x}(x,y,s,t,0,0)\right| dsdt \\
&\le \|D_1 g\|_X + (1+M) \sup_{(x,y)\in\Omega}\left[\int_0^y h_0(x,y,x,t)dt + \int_0^x \int_0^y h_1(x,y,s,t)dsdt\right] \\
&\quad + \sup_{(x,y)\in\Omega}\left[\int_0^y |H(x,y,x,t,0,0)|\,dt + \int_0^x \int_0^y \left|\frac{\partial H}{\partial x}(x,y,s,t,0,0)\right| dsdt\right].
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\|D_1(Uu)\|_X \\
&\le \|D_1 g\|_X + (1+M) \sup_{(x,y)\in\Omega}\left[\int_0^y h_0(x,y,x,t)dt + \int_0^x \int_0^y h_1(x,y,s,t)dsdt\right] \\
&\quad + \sup_{(x,y)\in\Omega}\left[\int_0^y |H(x,y,x,t,0,0)|\,dt + \int_0^x \int_0^y \left|\frac{\partial H}{\partial x}(x,y,s,t,0,0)\right| dsdt\right].
\end{aligned} \tag{3.8}
$$

It follows that

$$
\|Uu\|_{X_1} \le \|g\|_{X_1} + \bar{\alpha}_1 + (1+M)\,\bar{\beta}_1, \tag{3.9}
$$

where

$$
\begin{aligned}
\bar{\alpha}_1 = &\sup_{(x,y)\in\Omega} \int_0^x \int_0^y |H(x,y,s,t,0,0)|\,dsdt \\
&+ \sup_{(x,y)\in\Omega}\left[\int_0^y |H(x,y,x,t,0,0)|\,dt + \int_0^x \int_0^y \left|\frac{\partial H}{\partial x}(x,y,s,t,0,0)\right| dsdt\right], \\
\bar{\beta}_1 = &\sup_{(x,y)\in\Omega} \int_0^x \int_0^y h_0(x,y,s,t)dsdt \\
&+ \sup_{(x,y)\in\Omega}\left[\int_0^y h_0(x,y,x,t)dt + \int_0^x \int_0^y h_1(x,y,s,t)dsdt\right].
\end{aligned} \tag{3.10}
$$

On the other hand, for every $v \in B_M$, for all $(x,y) \in \Omega$, we obtain

$$
\begin{aligned}
|(Cv)(x,y)| &\le \int_0^1 \int_0^1 |K(x,y,s,t,v(s,t),D_1 v(s,t))|\,dsdt \\
&\le \int_0^1 \int_0^1 k_0(x,y,s,t)\,(1+|v(s,t)|+|D_1 v(s,t)|)\,dsdt \\
&\le \int_0^1 \int_0^1 k_0(x,y,s,t)\left(1+\|v\|_{X_1}\right) dsdt \\
&\le (1+M) \sup_{(x,y)\in\Omega} \int_0^1 \int_0^1 k_0(x,y,s,t)dsdt.
\end{aligned} \tag{3.11}
$$

Therefore

$$
\|Cv\|_X \le (1+M) \sup_{(x,y)\in\Omega}\left[\int_0^1 \int_0^1 k_0(x,y,s,t)dsdt\right]. \tag{3.12}
$$

Similarly,

$$\begin{aligned} |D_1(Cv)(x,y)| &\leq \int_0^1 \int_0^1 \left| \frac{\partial K}{\partial x}(x,y,s,t,u(s,t),D_1u(s,t)) \right| dsdt \\ &\leq (1+M) \sup_{(x,y)\in\Omega} \int_0^1 \int_0^1 k_1(x,y,s,t)dsdt, \end{aligned} \tag{3.13}$$

hence

$$\|D_1(Cv)\|_X \leq (1+M) \sup_{(x,y)\in\Omega} \int_0^1 \int_0^1 k_1(x,y,s,t)dsdt. \tag{3.14}$$

This yields

$$\begin{aligned} &\|Cv\|_{X_1} \\ &\leq (1+M) \left[\sup_{(x,y)\in\Omega} \int_0^1 \int_0^1 k_0(x,y,s,t)dsdt + \sup_{(x,y)\in\Omega} \int_0^1 \int_0^1 k_1(x,y,s,t)dsdt \right] \\ &\leq (1+M)\,\bar{\beta}_2. \end{aligned} \tag{3.15}$$

It follows from (3.9) and (3.15) that

$$\|Uu+Cv\|_{X_1} \leq \|Uu\|_{X_1} + \|Cv\|_{X_1} \leq \|g\|_{X_1} + \bar{\alpha}_1 + (1+M)\left(\bar{\beta}_1 + \bar{\beta}_2\right). \tag{3.16}$$

Choosing $M \geq \|g\|_{X_1} + \bar{\alpha}_1 + (1+M)\left(\bar{\beta}_1 + \bar{\beta}_2\right)$, *i.e.*, $M \geq \frac{\|g\|_{X_1} + \bar{\alpha}_1 + \bar{\beta}_1 + \bar{\beta}_2}{1 - \bar{\beta}_1 - \bar{\beta}_2}$. Then $Uu + Cv \in B_M$, for all $u, v \in B_M$.

We verify (ii). It is obviously that $U : B_M \to X_1$ is a contraction map, if we show that

$$\|Uu - U\bar{u}\|_{X_1} \leq \bar{\beta}_1 \|u - \bar{u}\|_{X_1}, \quad \forall\, u,\, \bar{u} \in B_M. \tag{3.17}$$

For every u, $\bar{u} \in B_M$, for all $(x,y) \in \Omega$, using (A2,i), (3.2) gives

$$\begin{aligned} &|(Uu)(x,y) - (U\bar{u})(x,y)| \\ &\leq \int_0^x \int_0^y |H(x,y,s,t,u(s,t),D_1u(s,t)) - H(x,y,s,t,\bar{u}(s,t),D_1\bar{u}(s,t))| \, dsdt \\ &\leq \int_0^x \int_0^y h_0(x,y,s,t) \left[|u(s,t) - \bar{u}(s,t)| + |D_1u(s,t) - D_1\bar{u}(s,t)| \right] dsdt \\ &\leq \left(\sup_{(x,y)\in\Omega} \int_0^x \int_0^y h_0(x,y,s,t)dsdt \right) \|u - \bar{u}\|_{X_1}. \end{aligned}$$

Thus

$$\|Uu - U\bar{u}\|_X \leq \left(\sup_{(x,y)\in\Omega} \int_0^x \int_0^y h_0(x,y,s,t)dsdt \right) \|u - \bar{u}\|_{X_1}. \tag{3.18}$$

Similarly, by

$$\begin{aligned} &D_1(Uu)(x,y) - D_1(U\bar{u})(x,y) \\ &= \int_0^y \left[H(x,y,x,t,u(x,t),D_1u(x,t)) - H(x,y,x,t,\bar{u}(x,t),D_1\bar{u}(x,t)) \right] dt \\ &\quad + \int_0^x \int_0^y \left[\frac{\partial H}{\partial x}(x,y,s,t,u(s,t),D_1u(s,t)) \right. \\ &\quad \left. - \frac{\partial H}{\partial x}(x,y,s,t,\bar{u}(s,t),D_1\bar{u}(s,t)) \right] dsdt, \end{aligned} \tag{3.19}$$

using (A2,ii), we get

$$
\begin{aligned}
&|D_1(Uu)(x,y) - D_1(U\bar{u})(x,y)| \\
&\leq \int_0^y |H(x,y,x,t,u(x,t),D_1u(x,t)) - H(x,y,x,t,\bar{u}(x,t),D_1\bar{u}(x,t))|\, dt \\
&\quad + \int_0^x \int_0^y \left| \frac{\partial H}{\partial x}(x,y,s,t,u(s,t),D_1u(s,t)) \right. \\
&\quad \left. - \frac{\partial H}{\partial x}(x,y,s,t,\bar{u}(s,t),D_1\bar{u}(s,t)) \right| dsdt \\
&\leq \int_0^y h_0(x,y,x,t)\left[|u(x,t)-\bar{u}(x,t)| + |D_1u(x,t) - D_1\bar{u}(x,t)|\right] dt \\
&\quad + \int_0^x \int_0^y h_1(x,y,s,t)\left[|u(s,t)-\bar{u}(s,t)| + |D_1u(s,t) - D_1\bar{u}(s,t)|\right] dsdt \\
&\leq \left[\int_0^y h_0(x,y,x,t)dt + \int_0^x \int_0^y h_1(x,y,s,t)dsdt\right] \|u-\bar{u}\|_{X_1} \\
&\leq \sup_{(x,y)\in\Omega} \left[\int_0^y h_0(x,y,x,t)dt + \int_0^x \int_0^y h_1(x,y,s,t)dsdt\right] \|u-\bar{u}\|_{X_1}.
\end{aligned} \tag{3.20}
$$

Then

$$
\begin{aligned}
&\|D_1(Uu) - D_1(U\bar{u})\|_X \\
&\leq \sup_{(x,y)\in\Omega} \left[\int_0^y h_0(x,y,x,t)dt + \int_0^x \int_0^y h_1(x,y,s,t)dsdt\right] \|u-\bar{u}\|_{X_1}.
\end{aligned} \tag{3.21}
$$

This yields

$$
\|Uu - U\bar{u}\|_{X_1} \leq \bar{\beta}_1 \|u-\bar{u}\|_{X_1}, \tag{3.22}
$$

where

$$
\begin{aligned}
\bar{\beta}_1 = &\sup_{(x,y)\in\Omega} \int_0^x \int_0^y h_0(x,y,s,t)dsdt \\
&+ \sup_{(x,y)\in\Omega} \left[\int_0^y h_0(x,y,x,t)dt + \int_0^x \int_0^y h_1(x,y,s,t)dsdt\right].
\end{aligned} \tag{3.23}
$$

We verify (iii). Let $\{u_m\} \subset B_M$, $\|u_m - u\|_{X_1} \to 0$ as $m \to \infty$, we have to prove that

$$
\|Cu_m - Cu\|_X \to 0 \text{ and } \|D_1(Cu_m) - D_1(Cu)\|_X \to 0 \text{ as } m \to \infty. \tag{3.24}
$$

Note that

$$
\begin{aligned}
|(Cu_m)(x,y) - (Cu)(x,y)| \leq &\int_0^1 \int_0^1 |K(x,y,s,t,u_m(s,t),D_1u_m(s,t)) \\
&- K(x,y,s,t,u(s,t),D_1u(s,t))|\, dsdt.
\end{aligned} \tag{3.25}
$$

Let give $\varepsilon > 0$. Since the function K is uniformly continuous on $\Omega \times \Omega \times [-M,M] \times [-M,M]$, there exists $\delta > 0$ such that

$$
\begin{aligned}
&\forall\, (x,y,s,t) \in \Omega \times \Omega, \quad \forall\, u,v,\bar{u},\bar{v} \in [-M,M], \\
&|u-\bar{u}| + |v-\bar{v}| < \delta \Longrightarrow |K(x,y,s,t,u,v) - K(x,y,s,t,\bar{u},\bar{v})| < \varepsilon.
\end{aligned} \tag{3.26}
$$

By $\|u_m - u\|_X \to 0$ and $\|D_1u_m - D_1u\|_X \to 0$, there is $m_0 \in \mathbb{N}$ such that

$$
\forall\, m \in \mathbb{N},\ m \geq m_0 \implies \|u_m - u\|_X + \|D_1u_m - D_1u\|_X < \delta. \tag{3.27}
$$

It follows that for all $m \in \mathbb{N}$,

$$
\begin{aligned}
m \geq m_0 \implies &|K(x,y,s,t,u_m(s,t),D_1u_m(s,t)) \\
&- K(x,y,s,t,u(s,t),D_1u(s,t))| < \varepsilon,
\end{aligned} \tag{3.28}
$$

for all $(x, y, s, t) \in \Omega \times \Omega$, so

$$|(Cu_m)(x,y) - (Cu)(x,y)| < \varepsilon, \quad \forall\, (x,y) \in \Omega \times \Omega, \quad \forall\, m \geq m_0, \tag{3.29}$$

it means that

$$\|Cu_m - Cu\|_X < \varepsilon, \quad \forall\, m \geq m_0, \tag{3.30}$$

i.e., $\|Cu_m - Cu\|_X \to 0$ as $m \to \infty$. By the same way, we get

$$\|D_1(Cu_m) - D_1(Cu)\|_X \to 0$$

as $m \to \infty$.

We verify (iv). We use Lemma 2.2. The condition (2.2)(i) holds because of $F = C(B_M) \subset B_M$. It remains to show (2.2)(ii). We have

$$\begin{aligned} &(Cu)(x,y) - (Cu)(\bar{x},\bar{y}) \\ &= \int_0^1 \int_0^1 [K(x,y,s,t,u(s,t),D_1u(s,t)) \\ &\qquad -K(\bar{x},\bar{y},s,t,u(s,t),D_1u(s,t))]\, dsdt, \end{aligned} \tag{3.31}$$

for all (x,y), $(\bar{x},\bar{y}) \in \Omega$, $u \in B_M$.

Let $\varepsilon > 0$. By the fact that K is uniformly continuous on $\Omega \times \Omega \times [-M,M] \times [-M,M]$, there exists $\delta_1 > 0$ such that $\forall (x,y)$, $(\bar{x},\bar{y}) \in \Omega$,

$$\begin{aligned} &|x-\bar{x}| + |y-\bar{y}| < \delta_1 \\ &\Longrightarrow |K(x,y,s,t,\bar{u},\bar{v}) - K(\bar{x},\bar{y},s,t,\bar{u},\bar{v})| < \tfrac{\varepsilon}{2}, \end{aligned} \tag{3.32}$$

for all $(s,t,\bar{u},\bar{v}) \in \Omega \times [-M,M]$. Then, for all (x,y), $(\bar{x},\bar{y}) \in \Omega$,

$$\begin{aligned} |x-\bar{x}| + |y-\bar{y}| < \delta_1 \Longrightarrow &|K(x,y,s,t,u(s,t),D_1u(s,t)) \\ &-K(\bar{x},\bar{y},s,t,u(s,t),D_1u(s,t))| < \tfrac{\varepsilon}{2}, \end{aligned} \tag{3.33}$$

for all $(s,t,u) \in \Omega \times B_M$, so, $\forall (x,y)$, $(\bar{x},\bar{y}) \in \Omega$, if $|x-\bar{x}| + |y-\bar{y}| < \delta_1$ then

$$\begin{aligned} &|(Cu)(x,y) - (Cu)(\bar{x},\bar{y})| \\ &\leq \int_0^1 \int_0^1 |K(x,y,s,t,u(s,t),D_1u(s,t)) \\ &\quad -K(\bar{x},\bar{y},s,t,u(s,t),D_1u(s,t))|\, dsdt \\ &< \tfrac{\varepsilon}{2}, \quad \forall\, u \in B_M. \end{aligned} \tag{3.34}$$

It is similar to $\frac{\partial K}{\partial x}$, we also have, there exists $\delta_2 > 0$ such that for all (x,y), $(\bar{x},\bar{y}) \in \Omega$,

$$\begin{aligned} &|x-\bar{x}| + |y-\bar{y}| < \delta_2 \\ &\Longrightarrow |D_1(Cu)(x,y) - D_1(Cu)(\bar{x},\bar{y})| < \tfrac{\varepsilon}{2}, \quad \forall\, u \in B_M. \end{aligned} \tag{3.35}$$

Consequently, by choosing $\delta = \min\{\delta_1, \delta_2\}$, we obtain that $\forall (x,y)$, $(\bar{x},\bar{y}) \in \Omega$,

$$\begin{aligned} &|x-\bar{x}| + |y-\bar{y}| < \delta \\ &\Longrightarrow |(Cu)(x,y) - (Cu)(\bar{x},\bar{y})| + |D_1(Cu)(x,y) - D_1(Cu)(\bar{x},\bar{y})| < \varepsilon, \end{aligned} \tag{3.36}$$

for all $u \in B_M$. Lemma 2.2 implies that $F = C(B_M)$ is relatively compact in X_1. Applying the fixed point theorem of Krasnosel'skii (Theorem 2.3), the existence is proved. Furthermore, based on the structure of the set of all fixed points of the operator $U + C$, we will obtain the structure of the set of solutions. For the details of such structure, it can be found in, for example, Deimling ([5], p.212) or Krasnosel'skii and Zabreiko ([6], p.315). Here, we will consider the set Σ of solutions defined as follows

$$\Sigma = \{u \in B_M : u = Uu + Cu\} = \{u \in B_M : u = (I - U)^{-1} Cu\},$$

we show that it is compact in X_1.

From the compactness of the operator $C : B_M \to B_M$ and the continuity of $(I - U)^{-1} : B_M \to B_M$ and $\Sigma = (I - U)^{-1} C(\Sigma)$, we only prove that Σ is closed. Let $\{u_m\} \subset \Sigma$ and $\|u_m - u\|_{X_1} \to 0$. The continuity of $(I - U)^{-1}C$ leads to

$$\begin{aligned}
&\left\|u - (I - U)^{-1} Cu\right\|_{X_1} \\
&\leq \|u - u_m\|_{X_1} + \left\|u_m - (I - U)^{-1} Cu\right\|_{X_1} \\
&= \|u - u_m\|_{X_1} + \left\|(I - U)^{-1} Cu_m - (I - U)^{-1} Cu\right\|_{X_1} \to 0,
\end{aligned}$$

so $u = (I - U)^{-1} Cu \in \Sigma$. Theorem 3.1 is proved. □

Let us illustrate the result obtained here by means of an example.

Example 3.2. Consider (1.1), with the functions g, H, K as follows

$$\begin{cases}
H(x, y, s, t, u, v) = h(x, y)\left[4s^\sigma t^\sigma \sin\left(\frac{\pi u}{2u_0(s,t)}\right) + s^\gamma t^\gamma \cos\left(\frac{2\pi v}{D_1 u_0(s,t)}\right)\right], \\
K(x, y, s, t, u, v) = h(x, y) K_1(s, t, u, v), \\
g(x, y) = u_0(x, y) - \left(\frac{2+4x^{\sigma+1} y^{\sigma+1}}{(1+\sigma)^2} + \frac{2+x^{\gamma+1} y^{\gamma+1}}{(1+\gamma)^2}\right) h(x, y),
\end{cases} \tag{3.37}$$

where

$$\begin{cases}
K_1(s, t, u, v) = s^\sigma t^\sigma \left[\frac{|u|}{u_0(s,t)} + \left|\frac{u}{u_0(s,t)}\right|^{2/3}\right] \\
\qquad\qquad + s^\gamma t^\gamma \left[\frac{|v|}{D_1 u_0(s,t)} + \left|\frac{v}{D_1 u_0(s,t)}\right|^{3/5}\right], \\
u_0(x, y) = e^x + x^{\gamma_1} |y - \alpha|^{\gamma_2}, \;\; h(x, y) = x^{\tilde{\gamma}_1} |y - \tilde{\alpha}|^{\tilde{\gamma}_2},
\end{cases} \tag{3.38}$$

and σ, γ, α, γ_2, γ_1, $\tilde{\alpha}$, $\tilde{\gamma}_2$, $\tilde{\gamma}_1$ are positive constants satisfying

$$\begin{cases}
0 < \alpha < 1, \quad 0 < \gamma_2 \leq 1 < \gamma_1, \\
0 < \tilde{\alpha} < 1, \quad 0 < \tilde{\gamma}_2 \leq 1 < \tilde{\gamma}_1, \\
2\pi \left[\left(\frac{1}{1+\sigma} + \frac{1}{1+\gamma}\right) + \left(1 + \dfrac{1}{\pi}\right)\left(\frac{1}{(1+\sigma)^2} + \frac{1}{(1+\gamma)^2}\right)(1 + \tilde{\gamma}_1)\right] \\
\times \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1 - \tilde{\alpha})^{\tilde{\gamma}_2}\} < 1.
\end{cases} \tag{3.39}$$

Note that $u_0(x, y) = e^x + x^{\gamma_1} |y - \alpha|^{\gamma_2}$, $D_1 u_0(x, y) = e^x + \gamma_1 x^{\gamma_1 - 1} |y - \alpha|^{\gamma_2}$, so $u_0 \in X_1$ and $u_0(x, y) \geq 1$, $D_1 u_0(x, y) \geq 1$.

We can prove that $(A1)-(A4)$ hold. It is clear to see that $(A1)$ holds, since u_0, $h \in X_1$.

$(A2)$ holds, by the following. First, $H \in C(\Delta \times \mathbb{R}^2; \mathbb{R})$. On the other hand, $\frac{\partial h}{\partial x} \in X$ and $\frac{\partial H}{\partial x} = \frac{\partial h}{\partial x}(x,y)\left[4s^\sigma t^\sigma \sin\left(\frac{\pi u}{2u_0(s,t)}\right) + s^\gamma t^\gamma \cos\left(\frac{2\pi v}{D_1 u_0(s,t)}\right)\right]$, hence $\frac{\partial H}{\partial x} \in C(\Delta \times \mathbb{R}^2; \mathbb{R})$.

Next, it is obviously that

$$\begin{aligned} &|H(x,y,s,t,u,v) - H(x,y,s,t,\bar{u},\bar{v})| \\ &\leq h(x,y)\left[4s^\sigma t^\sigma \tfrac{\pi|u-\bar{u}|}{2u_0(s,t)} + s^\gamma t^\gamma \tfrac{2\pi|v-\bar{v}|}{D_1 u_0(s,t)}\right] \\ &\leq 2\pi h(x,y)\left(s^\sigma t^\sigma + s^\gamma t^\gamma\right)\left[|u-\bar{u}| + |v-\bar{v}|\right] \\ &\equiv h_0(x,y,s,t)\left[|u-\bar{u}| + |v-\bar{v}|\right], \end{aligned} \tag{3.40}$$

where

$$h_0(x,y,s,t) = 2\pi h(x,y)\left(s^\sigma t^\sigma + s^\gamma t^\gamma\right). \tag{3.41}$$

Similarly

$$\begin{aligned} &\left|\tfrac{\partial H}{\partial x}(x,y,s,t,u,v) - \tfrac{\partial H}{\partial x}(x,y,s,t,\bar{u},\bar{v})\right| \\ &\leq h_1(x,y,s,t)\left[|u-\bar{u}| + |v-\bar{v}|\right], \end{aligned} \tag{3.42}$$

in which

$$h_1(x,y,s,t) = 2\pi\left|\tfrac{\partial h}{\partial x}(x,y)\right|\left(s^\sigma t^\sigma + s^\gamma t^\gamma\right). \tag{3.43}$$

Assumption $(A3)$ also holds, by the following. First, $K \in C(\Omega \times \Omega \times \mathbb{R}^2; \mathbb{R})$.

On the other hand, $\frac{\partial h}{\partial x} \in X$, $\frac{\partial K}{\partial x} = \frac{\partial h}{\partial x}(x,y)K_1(s,t,u,v)$, so $\frac{\partial K}{\partial x} \in C(\Omega \times \Omega \times \mathbb{R}^2; \mathbb{R})$. Next, applying the inequality

$$x \leq 1 + x^q, \quad \forall\, x \geq 0, \quad \forall\, q \geq 1, \tag{3.44}$$

we have

$$\begin{aligned} |K_1(s,t,u,v)| &\leq s^\sigma t^\sigma \left[1 + \tfrac{2|u|}{u_0(s,t)}\right] + s^\gamma t^\gamma \left[1 + \tfrac{2|v|}{D_1 u_0(s,t)}\right] \\ &\leq 2\left(s^\sigma t^\sigma + s^\gamma t^\gamma\right)\left[1 + |u| + |v|\right]. \end{aligned} \tag{3.45}$$

It follows that

$$\begin{aligned} |K(x,y,s,t,u,v)| &= h(x,y)\,|K_1(s,t,u,v)| \\ &\leq k_0(x,y,s,t)\left[1 + |u| + |v|\right], \end{aligned} \tag{3.46}$$

with

$$k_0(x,y,s,t) = 2h(x,y)\left(s^\sigma t^\sigma + s^\gamma t^\gamma\right). \tag{3.47}$$

Similarly

$$\left|\tfrac{\partial K}{\partial x}(x,y,s,t,u,v)\right| \leq k_1(x,y,s,t)\left[1 + |u| + |v|\right], \tag{3.48}$$

in which

$$k_1(x,y,s,t) = 2\left|\tfrac{\partial h}{\partial x}(x,y)\right|\left(s^\sigma t^\sigma + s^\gamma t^\gamma\right). \tag{3.49}$$

And $(A4)$ holds, by the following. We have

$$\begin{aligned}
&\int_0^x \int_0^y \left(h_0(x,y,s,t) + h_1(x,y,s,t)\right) dsdt \\
&= 2\pi \left(h(x,y) + \left|\tfrac{\partial h}{\partial x}(x,y)\right|\right) \int_0^x \int_0^y \left(s^\sigma t^\sigma + s^\gamma t^\gamma\right) dsdt \\
&= 2\pi \left(\tfrac{x^{\sigma+1}y^{\sigma+1}}{(1+\sigma)^2} + \tfrac{x^{\gamma+1}y^{\gamma+1}}{(1+\gamma)^2}\right) \left(h(x,y) + \left|\tfrac{\partial h}{\partial x}(x,y)\right|\right), \\
&\int_0^y h_0(x,y,x,t)dt \\
&= 2\pi h(x,y) \int_0^y \left(x^\sigma t^\sigma + x^\gamma t^\gamma\right) dt = 2\pi \left(\tfrac{x^\sigma y^{\sigma+1}}{1+\sigma} + \tfrac{x^\gamma y^{\gamma+1}}{1+\gamma}\right) h(x,y), \\
&\int_0^1 \int_0^1 k_0(x,y,s,t)dsdt \\
&= 2h(x,y) \int_0^1 \int_0^1 \left(s^\sigma t^\sigma + s^\gamma t^\gamma\right) dsdt = 2\left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right) h(x,y), \\
&\int_0^1 \int_0^1 k_1(x,y,s,t)dsdt \\
&= 2\left|\tfrac{\partial h}{\partial x}(x,y)\right| \int_0^1 \int_0^1 \left(s^\sigma t^\sigma + s^\gamma t^\gamma\right) dsdt = 2\left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right)\left|\tfrac{\partial h}{\partial x}(x,y)\right|.
\end{aligned} \tag{3.50}$$

Now, we need the following lemma, it is not difficult to prove, so we omit.

Lemma 3.3. *Let positive constants* $\alpha, \gamma_1, \gamma_2$ *satisfy* $0 < \alpha < 1$, $0 < \gamma_2 \le 1 < \gamma_1$. *Then*

$$\begin{aligned}
&0 \le x^{\gamma_1} |y - \alpha|^{\gamma_2} \le \max\{\alpha^{\gamma_2}, (1-\alpha)^{\gamma_2}\}, \quad \forall\, (x,y) \in \Omega, \\
&0 \le x^{\gamma_1 - 1} |y - \alpha|^{\gamma_2} \le \max\{\alpha^{\gamma_2}, (1-\alpha)^{\gamma_2}\}, \quad \forall\, (x,y) \in \Omega.
\end{aligned} \tag{3.51}$$

Now, using Lemma 3.3, we get

$$\begin{cases}
0 \le h(x,y) = x^{\tilde{\gamma}_1} |y - \tilde{\alpha}|^{\tilde{\gamma}_2} \le \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\}; \\
0 \le \frac{\partial h}{\partial x}(x,y) = \tilde{\gamma}_1 x^{\tilde{\gamma}_1 - 1} |y - \tilde{\alpha}|^{\tilde{\gamma}_2} \le \tilde{\gamma}_1 \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\},
\end{cases} \tag{3.52}$$

for all $(x,y) \in \Omega$. Thus

$$\begin{aligned}
&\int_0^x \int_0^y \left(h_0(x,y,s,t) + h_1(x,y,s,t)\right) dsdt \\
&\le 2\pi \left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right) (1 + \tilde{\gamma}_1) \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\}, \\
&\int_0^y h_0(x,y,x,t)dt = 2\pi h(x,y) \int_0^y \left(x^\sigma t^\sigma + x^\gamma t^\gamma\right) dt \\
&\le 2\pi \left(\tfrac{1}{1+\sigma} + \tfrac{1}{1+\gamma}\right) \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\}, \\
&\int_0^1 \int_0^1 k_0(x,y,s,t)dsdt = 2\left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right) h(x,y) \\
&\le 2\left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right) \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\}, \\
&\int_0^1 \int_0^1 k_1(x,y,s,t)dsdt = 2\left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right) \left|\tfrac{\partial h}{\partial x}(x,y)\right| \\
&\le 2\left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right) \tilde{\gamma}_1 \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\}.
\end{aligned} \tag{3.53}$$

It follows that

$$\begin{aligned}\bar{\beta}_1 &= \sup_{(x,y)\in\Omega} \int_0^y h_0(x,y,x,t)dt + \sup_{(x,y)\in\Omega} \int_0^x \int_0^y (h_0(x,y,s,t)+h_1(x,y,s,t))\,dsdt \\ &\le 2\pi \left[\left(\tfrac{1}{1+\sigma} + \tfrac{1}{1+\gamma}\right) + \left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right)(1+\tilde{\gamma}_1)\right] \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\}, \\ \bar{\beta}_2 &= \sup_{(x,y)\in\Omega} \int_0^1 \int_0^1 k_0(x,y,s,t)dsdt + \sup_{(x,y)\in\Omega} \int_0^1 \int_0^1 k_1(x,y,s,t)dsdt \\ &\le 2\left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right)(1+\tilde{\gamma}_1)\max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\bar{\beta}_1 + \bar{\beta}_2 \\ &\le 2\pi \left[\left(\tfrac{1}{1+\sigma} + \tfrac{1}{1+\gamma}\right) + (1+\frac{1}{\pi})\left(\tfrac{1}{(1+\sigma)^2} + \tfrac{1}{(1+\gamma)^2}\right)(1+\tilde{\gamma}_1)\right] \\ &\quad \times \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} < 1.\end{aligned}$$

We conclude Theorem 3.1 holds in this case. Furthermore, it is not difficult to verify that $u_0 \in X_1$ is exactly a solution of (1.1).

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