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FURTHER INVESTIGATION ON BEST PROXIMITY POINT OF GENERALIZED CYCLIC WEAK φ-CONTRACTION IN ORDERED METRIC SPACES

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Abstract. In this manuscript, a generalized cyclic weak φ-contraction is introduced (two examples are also given) and a new best proximity point theorem for this cyclic mapping are stated under certain conditions. The results obtained in this paper improve and extend previous results that have been proved for cyclic mappings.

1. Introduction

Let (X, d) be a metric space and A be nonempty subsets of X. f is a given self-mapping defined on A, if there exists x such that $fx = x$, we say that x is a fixed point of f. Banach fixed point theorem states that when (X, d) be a complete metric space and $f : X \to X$ is a contraction, then f has a unique fixed point in X. We known that Banach fixed point theorem is a classical tool in fixed point theory and it has wide applications. In particular, in micro economics, fixed point theorems are used to study the existence of Nash equilibria(see [3, 13]).

With the fixed point problem, another obvious problem arises. If f is a non-self-mapping from A to B, where A and B are nonempty subsets of X.

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Solution of equation $fx = x$ may not exist, Particularly when $A \cap B = \emptyset$, then we want to find a solution x^* such that $d(fx^*, x^*) = \min d(fx, x)$, where $x \in A$. This is the problem to be solved by best proximity problem. Therefore, the best proximity point problem becomes a hot topic recently. Moreover, research on cyclic contraction have received considerable interest. In 2003, Kirk-Srinavasan-Veeramani [12] stated the first result in this area. Later, other authors also obtained many important results in this area (see [1], [4], [7], [8], [10], [12]).

First, we give the definition of cyclic map.

Definition 1.1. Let A and B be nonempty subsets of a metric space(X, d). A mapping $f : A \bigcup B \to A \bigcup B$ is called a cyclic map provided that $f(A) \subset B$ and $f(B) \subset A$.

Next, we recall the definitions of several cyclic maps and best proximity point.

Definition 1.2. ([4, 9]) Let A and B be non-empty subsets of a metric space (X, d). A mapping f is a cyclic map, we say that:

(i) f is a *cyclic contraction*: for any $x \in A, y \in B$ and some $\alpha \in (0, 1)$

$$d(fx, fy) \leq \alpha d(x, y);$$

(ii) f is a *Kannan Type cyclic contraction*: for any $x \in A, y \in B$ and some $\alpha \in (0, \frac{1}{2})$

$$d(fx, fy) \leq \alpha[d(fx, x) + d(fy, y)];$$

(iii) f is a *Chatterjee Type cyclic contraction*: for any $x \in A, y \in B$ and some $\alpha \in (0, \frac{1}{2})$

$$d(fx, fy) \leq \alpha[d(fx, y) + d(fy, x)];$$

(iv) f is a *Reich type cyclic contraction*: for any $x \in A, y \in B$ and some $\alpha \in (0, \frac{1}{3})$

$$d(fx, fy) \leq \alpha M(x, y),$$

where $M(x, y) = \max\{d(x, y), d(fx, x), d(fy, y)\}$.

In 2003, the first fixed point theorem for the cyclic contraction was stated by Kirk-Srinavasan-Veeramani [12]. In 2011, Karapinar and Erhan [9] proved fixed point theorems for the above cyclic maps.

Recently, several authors presented many results for cyclic mappings satisfying various (nonlinear) contractive conditions based on altering distance function φ which were introduced by Khan *et al.* [11].

Definition 1.3. ([2]) Let A and B be nonempty subsets of a metric space(X, d). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A cyclic map $f : A \bigcup B \to A \bigcup B$ is called a cyclic weak φ-contraction if for all $x \in A$ and $y \in B$

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)). \tag{1.1}$$

Let X be a nonempty set, we know that $(X, d, \preceq)$ is an ordered metric space if and only if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set. Two elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$.

Recently, Rezapour-Derafshpour-Shahzad (see [7, 14]) stated the following theorem about cyclic weak φ-contraction:

Theorem 1.4. *Let $(X, d, \preceq)$ be an ordered metric space, A and B be nonempty subsets of X and $f : A \bigcup B \to A \bigcup B$ be a decreasing, cyclic weak φ-contraction. Suppose there exists $x_0 \in A$ such that $x_0 \preceq f^2x_0 \preceq fx_0$. Define $x_{n+1} = fx_n$ and $d_n := d(x_{n+1}, x_n)$ for all $n \in N$. Then $d_n \to d(A, B)$.*

Definition 1.5. Let A and B be nonempty subsets of a metric space(X, d). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A cyclic map $f : A \bigcup B \to A \bigcup B$ is called a Kannan type cyclic weak φ-contraction if for all $x \in A$ and $y \in B$

$$d(fx, fy) \leq h(x, y) - \varphi(h(x, y)) + \varphi(d(A, B)), \tag{1.2}$$

where $h(x, y) = \frac{1}{2}[d(x, fx) + d(y, fy)]$.

In 2012, Karapinar [8] proved the best proximity point theorem of Kannan type cyclic weak φ-contractions in ordered metric spaces.

Motivated by the above mentioned results and the on-going research, the purpose of this paper is to introduce a generalization of cyclic mappings known as a generalized cyclic weak φ-contraction and obtain a corresponding best proximity point theorem for this cyclic mapping under certain condition.

2. Main results

In this section, we first introduce a generalized cyclic weak φ-contraction and give an example of the cyclic map which is different from other cyclic maps, then we prove a new best proximity point theorem for this cyclic mapping satisfing certain condition.

Definition 2.1. Let A and B be nonempty subsets of a metric space(X, d). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A cyclic map $f : A \bigcup B \to A \bigcup B$ is called a generalized cyclic weak φ-contraction if for any

$x \in A$ and $y \in B$

$$d(fx, fy) \leq p(x, y) - \varphi(p(x, y)) + \varphi(d(A, B)), \tag{2.1}$$

where $p(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}$.

Example 2.2. Let $X := \mathbb{R}$ with the metric $|\cdot|$. Suppose $A = B = [0, 1]$ and define $f : A \bigcup B \to A \bigcup B$ by

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } \ x = 1, \\ \frac{1}{2}x + \frac{1}{4}, & \text{if } \ x \in [0, 1) \end{cases}$$

and $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{3}{10}t$. Obviously, f is a cyclic map. Then, we prove that f is a generalized cyclic weak φ-contraction but neither a Kannan type cyclic weak φ-contraction nor a cyclic weak φ-contraction. Indeed, for $x = 1$ and $y = \frac{2}{3}$, we have

$$\begin{aligned} d(fx, fy) &\leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)), \\ \left|f1 - f\frac{2}{3}\right| &\leq \left|1 - \frac{2}{3}\right| - \frac{3}{10}\left|1 - \frac{2}{3}\right|, \\ \frac{1}{3} &\leq \frac{1}{3} - \frac{1}{10}, \end{aligned}$$

which is a contradiction. Therefore f is not a cyclic weak φ-contraction.

$$d(fx, fy) \leq h(x, y) - \varphi(h(x, y)) + \varphi(d(A, B)), \tag{2.2}$$

$$\begin{aligned} h(x, y) &= \frac{1}{2}[d(x, fx) + d(y, fy)] \\ &= \frac{1}{2}\left(\left|1 - \frac{1}{4}\right| + \left|\frac{2}{3} - \frac{1}{2} \times \frac{2}{3} - \frac{1}{4}\right|\right) \qquad (2.3) \\ &= \frac{5}{12}. \end{aligned}$$

Substituting (2.3) into (2.2), we obtain

$$\begin{aligned} \left|f1 - f\frac{2}{3}\right| &\leq \frac{5}{12} - \frac{3}{10} \times \frac{5}{12}, \\ \frac{1}{3} &\leq \frac{7}{24}, \end{aligned}$$

which shows that f is not yet a Kannan type cyclic weak φ-contraction.

Next, we show that f is a generalized cyclic weak φ-contraction.

Case 1. $x = y = 1$, f obviously satisfies (2.1).

Case 2. $x = 1$ and $y \in [0, 1)$.

$$d(fx, fy) = \left|\frac{1}{4} - \frac{1}{2}y - \frac{1}{4}\right| = \frac{1}{2}y,$$

$$d(x,y)=1-y,\ \ d(x,fx)=\frac{3}{4},\ \ d(y,fy)=\left|\frac{1}{2}y-\frac{1}{4}\right|,$$
$$d(x,fy)=\frac{3}{4}-\frac{1}{2}y,\ \ d(y,fx)=\frac{1}{2}y.$$

It is obvious that $p(x,y)=\max\{1-y,\frac{3}{4}\}$. If $y\in[0,\frac{1}{4}]$, then $p(x,y)=1-y$, we have

$$\begin{aligned}&p(x,y)-\varphi(p(x,y))+\varphi(d(A,B))-d(fx,fy)\\&=1-y-\frac{3}{10}(1-y)-\frac{1}{2}y>0.\end{aligned}$$

That is, f obviously satisfies (2.1). If $y\in[\frac{1}{4},1)$, then $p(x,y)=\frac{3}{4}$, we have

$$\begin{aligned}&p(x,y)-\varphi(p(x,y))+\varphi(d(A,B))-d(fx,fy)\\&=\frac{3}{4}-\frac{3}{10}\times\frac{3}{4}-\frac{1}{2}y>0,\end{aligned}$$

which holds for every $y\in[\frac{1}{4},1)$. Therefore, f obviously satisfies (2.1).
Case 3. $x\in[0,1)$ and $y\in[0,1)$.

$$\begin{aligned}d(fx,fy)&=\frac{1}{2}|x-y|,\quad d(x,y)=|x-y|,\\d(x,fx)&=|\frac{1}{2}x-\frac{1}{4}|,\quad d(y,fy)=\left|\frac{1}{2}y-\frac{1}{4}\right|,\\d(x,fy)&=\left|x-\frac{1}{2}y-\frac{1}{4}\right|,\quad d(y,fx)=\left|y-\frac{1}{2}x-\frac{1}{4}\right|.\end{aligned}$$

If $p(x,y)=d(x,y)$, f obviously satisfies (2.1). If $d(x,y)<p(x,y)\neq d(x,y)$, we have

$$\begin{aligned}&p(x,y)-\varphi(p(x,y))+\varphi(d(A,B))-d(fx,fy)\\&=p(x,y)-\frac{3}{10}p(x,y)-\frac{1}{2}|x-y|>0,\end{aligned}$$

which holds for for all $x\in[0,1)$ and $y\in[0,1)$. So, f obviously satisfies (2.1). Therefore, f is a generalized cyclic weak φ-contraction.

Example 2.3. Consider the Euclidean ordered space $X=\mathbb{R}$ with the usual metric. Suppose $A=[-2,-1], B=[1,2]$ and define $T:A\bigcup B\to A\bigcup B$ by

$$T(x)=\begin{cases}-\frac{1}{2}x+\frac{1}{2}, & \text{if } x\in A,\\ \\ -\frac{1}{2}x-\frac{1}{2}, & \text{if } x\in B.\end{cases}$$

If $\varphi:[0,\infty)\to[0,\infty)$ is defined by $\varphi(t)=\frac{1}{2}t$, where

$$d(A,B)=2,\ \ d(Tx,Ty)=\frac{1}{2}|y-x|+1,$$

$$\begin{aligned}&p(x,y)-\varphi(p(x,y))+\varphi(d(A,B))\\&\geq \frac{1}{2}d(x,y)+1=d(Tx,Ty).\end{aligned}$$

Therefore, T is a generalized cyclic weak φ-contraction.

Remark 2.4. By the above examples, we can see that it is significant to study the generalized cyclic weak φ-contraction which is different from Kannan type cyclic weak φ-contraction and cyclic weak φ-contraction.

Theorem 2.5. *Let A and B be nonempty subsets of a metric space (X,d). Suppose that $f : A\bigcup B \to A\bigcup B$ is a generalized cyclic weak φ-contraction and there exists $y_0 \in A$. Define $y_{n+1} = fy_n$ for any $n \in \mathbb{N}$. Then $d(y_n, y_{n+1}) \to d(A,B)$ as $n \to \infty$.*

Proof. Let $d_n = d(y_n, y_{n+1})$. First we claim that the sequence $\{d_n\}$ is non-increasing. By the assumption, we have

$$\begin{aligned}d_{n+1}&=d(y_{n+1},y_{n+2})\\&=d(fy_n,fy_{n+1})\\&\leq p(y_n,y_{n+1})-\varphi(p(y_n,y_{n+1}))+\varphi(d(A,B)),\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}&p(y_n,y_{n+1})=\max\{d(y_n,y_{n+1}),d(y_n,fy_n),d(y_{n+1},fy_{n+1}),\\&\frac{1}{2}[d(y_n,fy_{n+1})+d(y_{n+1},fy_n)]\}=\max\{d(y_n,y_{n+1}),d(y_{n+1},y_{n+2})\}.\end{aligned}$$

Assume that there exists $n_0 \in \mathbb{N}$ such that $p(y_{n_0}, y_{n_0+1}) = d(y_{n_0+1}, y_{n_0+2})$, from $d(y_{n_0+1}, y_{n_0+2}) > d(y_{n_0}, y_{n_0+1})$. We have

$$d(y_{n_0+1},y_{n_0+2})\leq d(y_{n_0+1},y_{n_0+2})-\varphi(d(y_{n_0+1},y_{n_0+2}))+\varphi(d(A,B)),$$

thus,

$$\varphi(d(y_{n_0+1},y_{n_0+2}))\leq\varphi(d(A,B)).$$

Again since φ is a strictly increasing map, we have

$$d(y_{n_0+1},y_{n_0+2})\leq d(A,B)\leq d(y_{n_0+1},y_{n_0+2}).$$

Obviously, $d(y_{n_0+1}, y_{n_0+2}) = d(A,B) \leq d(y_{n_0}, y_{n_0+1})$, which is a contradiction. Hence, for all $n \in \mathbb{N}$,

$$p(y_n,y_{n+1})=d(y_n,y_{n+1}).$$

Then the expression (2.4) turns into

$$d(y_{n+1},y_{n+2})\leq d(y_n,y_{n+1})-\varphi(d(y_n,y_{n+1}))+\varphi(d(A,B)),\tag{2.5}$$

therefore,

$$d(y_{n+1},y_{n+2})\leq d(y_n,y_{n+1}).$$

That is, the sequence $\{d_n\}$ is non-increasing and bounded below, it is obvious that $\lim_{n\to\infty} d_n$ exists.

If $d_{n_0} = 0$ for some $n_0 \in N$, obviously, $d_n \to 0$ and $d(A, B) = 0$, that is, $d_n \to d(A, B)$.

If $d_n \neq 0$ for all $n \in \mathbb{N}$. Put $d_n \to \gamma$, thus $\gamma \geq d(A, B)$. Since φ is a strictly increasing map, we have $\varphi(\gamma) \geq \varphi(d(A, B))$. From the expression (2.5), we get that

$$\varphi(d(y_n, y_{n+1})) \leq d(y_n, y_{n+1}) - d(y_{n+1}, y_{n+2}) + \varphi(d(A, B)).$$

From which it follows that

$$\varphi(\gamma) \leq \varphi(d(A, B)).$$

Therefore, $\gamma = d(A, B)$, that is, $d_n \to d(A, B)$. The proof is complete. □

Corollary 2.6. *Let (X, d) be a complete metric space. Suppose that $f : X \to X$ is a generalized cyclic weak φ-contraction and there exists $x_0 \in A$. Define $y_{n+1} = f y_n$ for any $n \in \mathbb{N}$. Then there exists a unique fixed point $y \in X$ such that $fy = y$.*

Proof. By the assumption and Theorem 2.5, we have $d(y_{n+1}, y_n) \to 0$.

Next, we show that $\{y_n\}$ is a Cauchy sequence in the metric space (X, d). Suppose $\{y_n\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ for which we can find two subsequences $\{y_{m(i)}\}$ and $\{y_{n(i)}\}$ of $\{y_n\}$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \quad d(y_{m(i)}, y_{n(i)}) \geq \epsilon. \tag{2.6}$$

from which it follows that

$$d(y_{m(i)}, y_{n(i)-1}) < \epsilon. \tag{2.7}$$

From (2.6), (2.7) and the triangular inequality, we get that

$$\begin{aligned} \epsilon &\leq d(y_{m(i)}, y_{n(i)}) \\ &\leq d(y_{m(i)}, y_{n(i)-1}) + d(y_{n(i)-1}, y_{n(i)}) \\ &\leq \epsilon + d(y_{n(i)-1}, y_{n(i)}). \end{aligned}$$

Letting $i \to \infty$ in the above inequalities and using $d(y_{n+1}, y_n) \to 0$, we get that

$$\lim_{i\to\infty} d(y_{m(i)}, y_{n(i)}) = \epsilon. \tag{2.8}$$

Again, regarding (2.6) and the triangular inequality, we have

$$\begin{aligned}\epsilon &\le d(y_{m(i)}, y_{n(i)})\\ &\le d(y_{n(i)}, y_{n(i)-1}) + d(y_{n(i)-1}, y_{m(i)})\\ &\le d(y_{n(i)}, y_{n(i)-1}) + d(y_{n(i)-1}, y_{m(i)+1}) + d(y_{m(i)+1}, y_{m(i)})\\ &\le d(y_{n(i)}, y_{n(i)-1}) + d(y_{n(i)-1}, y_{m(i)}) + 2d(y_{m(i)+1}, y_{m(i)})\\ &\le 2d(y_{n(i)}, y_{n(i)-1}) + d(y_{n(i)}, y_{m(i)}) + 2d(y_{m(i)+1}, y_{m(i)}).\end{aligned}$$

Letting $i \to \infty$ in the above inequalities, using $d(y_{n+1}, y_n) \to 0$ and (2.8), we get that

$$\begin{aligned}\lim_{i\to\infty} d(y_{m(i)}, y_{n(i)}) &= \lim_{i\to\infty} d(y_{m(i)}, y_{n(i)-1})\\ &= \lim_{i\to\infty} d(y_{m(i)+1}, y_{n(i)-1}) = \epsilon.\end{aligned}$$

Since

$$\begin{aligned}d(y_{m(i)+1}, y_{n(i)}) &= d(fy_{m(i)}, fy_{n(i)-1})\\ &\le p(y_{m(i)}, y_{n(i)-1}) - \varphi(p(y_{m(i)}, y_{n(i)-1})),\end{aligned}$$

where

$$\begin{aligned}&p(y_{m(i)}, y_{n(i)-1})\\ &= \max\Big\{ d(y_{m(i)}, y_{n(i)-1}), d(y_{m(i)}, y_{m(i)+1}), d(y_{n(i)-1}, y_{n(i)}),\\ &\qquad \frac{1}{2}[d(y_{m(i)}, y_{n(i)}) + d(y_{m(i)+1}, y_{n(i)-1})]\Big\}.\end{aligned}$$

Letting $i \to \infty$ and considering the continuity of φ, we have

$$\epsilon \le \epsilon - \varphi(\epsilon) < \epsilon,$$

hence, $\epsilon = 0$. Which is a contradiction. Thus $\{y_n\}$ is a Cauchy sequence in (X, d) and since X is complete, there exists y such that $y_n \to y$. We have

$$\begin{aligned}d(y, fy) &\le d(y, y_{n+1}) + d(y_{n+1}, fy)\\ &= d(y, y_{n+1}) + d(fy_n, fy)\end{aligned} \tag{2.9}$$

and

$$d(fy_n, fy) \le p(y_n, y) - \varphi(p(y_n, y)), \tag{2.10}$$

$$\begin{aligned}&p(y_n, y)\\ &= \max\Big\{ d(y_n, x), d(y_n, fy_n), d(y, fy), \frac{1}{2}[d(y_n, fy) + d(y, fy_n)]\Big\}\\ &= \max\Big\{ d(y_n, y), d(y_n, y_{n+1}), d(y, fy), \frac{1}{2}[d(y_n, fy) + d(y, y_{n+1})]\Big\}.\end{aligned} \tag{2.11}$$

Substituting (2.11) and (2.10) into (2.9), taking the limit as $n \to \infty$, we get that

$$d(y, fy) \leq d(y, fy) - \varphi(d(y, fy)).$$

Since φ is a strictly increasing, we have $\varphi(d(y, fy)) > 0$. If $d(y, fy) \neq 0$, then $d(y, fy) < d(y, fy)$, which is a contradiction. So, $d(y, fy) = 0$, that is, $fy = y$.

For the uniqueness proof of y, we can suppose that there exists $y^* \in X$ such that $fy^* = y^*$ but $y^* \neq y$. Since $f : X \to X$ is a generalized cyclic weak φ-contraction, then

$$d(y, y^*) = d(fy, fy^*) \leq p(y, y^*) - \varphi(p(y, y^*)),$$

where

$$\begin{aligned} p(y, y^*) &= \max\left\{d(y, y^*), d(y, y), d(y^*, y^*), \frac{1}{2}[d(y, y^*) + d(y, y^*)]\right\} \\ &= d(y, y^*). \end{aligned}$$

Thus, $d(y, y^*) \leq d(y, y^*) - \varphi(d(y, y^*))$ implying $d(y, y^*) = 0$, that is, $y^* = y$, which completes the uniqueness proof. □

Definition 2.7. ([5, 6]) Let (X, d) be a metric space with a mapping $f : X \to X$, if $\lim_{n\to\infty} f^{n_i}(y) = z \Rightarrow \lim_{n\to\infty} f(f^{n_i}(y)) = fz$, we call mapping f to be orbitally continuous.

Remark 2.8. Obviously, if f is orbitally continuous, then f^m is also orbitally continuous for any $m \in \mathbb{N}$.

Theorem 2.9. *Let A and B be nonempty subsets of a metric space (X, d). Suppose that $f : A \bigcup B \to A \bigcup B$ is a generalized cyclic weak φ-contraction and f is also orbitally continuous. Assume that A is closed and there exists $y_0 \in A$. Define $y_{n+1} = fy_n$ for any $n \in \mathbb{N}$. If $\{y_{2n}\}$ has a convergent subsequence in A, then there exists $y \in A$ such that $d(y, fy) = d(A, B)$.*

Proof. By the assumption, we know that the subsequence $\{y_{2n_k}\}$ of sequence $\{y_{2n}\}$ converges to a point $y \in A$. Regarding Theorem 2.5, we have

$$d(y_{2n_k}, y_{2n_k+1}) = d(y_{2n_k}, fy_{2n_k}) \to d(A, B).$$

From orbital continuity of f, we have $d(y, fy) = d(A, B)$. □

Definition 2.10. A metric space (X, d) is called regular if every bounded monotone sequence of X is convergent.

Corollary 2.11. *Let A and B be nonempty subsets of a regular ordered metric space $(X, d, \preceq)$. Suppose that $f : A \bigcup B \to A \bigcup B$ is a decreasing generalized cyclic weak φ-contraction and f is also orbitally continuous. Assume that A is closed and there exists $y_0 \in A$ such that $y_0 \preceq f^2 y_0 \preceq f y_0$. Define $y_{n+1} = f y_n$ for any $n \in \mathbb{N}$. Then there exists $y \in A$ such that $d(y, fy) = d(A, B)$.*

Proof. By the assumption, we can obtain

$$y_0 \preceq y_2 \preceq \cdots \preceq y_{2n} \preceq y_1.$$

Since X is regular and A is closed, the sequence $\{y_{2n}\}$ converges to a point $y \in A$. From the result of Theorem 2.9, we have $d(y, fy) = d(A, B)$. □

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