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AN N-ORDER ITERATIVE SCHEME FOR A NONLINEAR CARRIER WAVE EQUATION IN THE ANNULAR WITH ROBIN-DIRICHLET CONDITIONS

Le Thi Phuong Ngoc[1], Le Huu Ky Son[2,4], Tran Minh Thuyet[3] and Nguyen Thanh Long[4]

[1]University of Khanh Hoa
01 Nguyen Chanh Str., Nha Trang City, Vietnam
e-mail: ngoc1966@gmail.com

[2]Department of Fundamental Sciences, Ho Chi Minh City University of Food Industry
140 Le Trong Tan Str., Tan Phu Dist., Ho Chi Minh City, Vietnam
e-mail: kyson85@gmail.com

[3]Department of Mathematics, University of Economics of HoChiMinh City
59C Nguyen Dinh Chieu Str., Dist. 3, HoChiMinh City, Vietnam
e-mail: tmthuyet@ueh.edu.vn

[4]Department of Mathematics and Computer Science, University of Natural Sciences
Vietnam National University Ho Chi Minh City
227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam
e-mail: longnt2@gmail.com

Abstract. This paper is devoted to the study of a nonlinear Carrier wave equation in the annular associated with Robin-Dirichlet conditions. Using a high order iterative scheme, the existence of a local unique weak solution is proved. Moreover, the sequence established here converges to a unique weak solution at a rate of order N with $N \geq 2$.

1. Introduction

In this paper, we consider the following nonlinear Carrier wave equation in the annular

$$u_{tt} - \mu(\|u(t)\|_0^2)(u_{xx} + \tfrac{1}{x}u_x) = f(x,t,u), \quad \rho < x < 1, \quad 0 < t < T, \tag{1.1}$$

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associated with Robin-Dirichlet conditions

$$u(\rho,t)=0,\ \ u_x(1,t)+\zeta u(1,t)=0 \tag{1.2}$$

and initial conditions

$$u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x), \tag{1.3}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and ρ, ζ are given constants with $0<\rho<1$. In (1.1), the nonlinear term $\mu(\|u(t)\|_0^2)$ depends on the integral $\|u(t)\|_0^2=\int_\rho^1 xu^2\,(x,t)\,dx$.

(1.1) herein is the bidimensional nonlinear wave equation describing nonlinear vibrations of the annular membrane $\Omega_1=\{(x,y):\rho^2<x^2+y^2<1\}$. In the vibration processing, the area of the annular membrane and the tension at various points change in time. The condition on the boundary $\Gamma_1=\{(x,y):x^2+y^2=1\}$, that is $u_x(1,t)+\zeta u(1,t)=0$, describes elastic constraints where ζ the constant has a mechanical signification. And with the boundary conditions on $\Gamma_\rho=\{(x,y):x^2+y^2=\rho^2\}$ requiring $u(\rho,t)=0$, the annular membrane is fixed.

In [1], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt}-\left(1+\tfrac{EA}{LT_0}\int_0^L u^2(y,t)dy\right)u_{xx}=0, \tag{1.5}$$

where $u(x,t)$ is the x-derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross-section of a string, L is the length of a string and ρ is the density of a material. Clearly, if properties of a material vary with x and t, then there is a hyperbolic equation of the type (Larkin [5])

$$u_{tt}-B\left(x,t,\int_0^1 u^2\,(y,t)\,dy\right)u_{xx}=0. \tag{1.6}$$

The Kirchhoff-Carrier equations of the form (1.1) received much attention. We refer the reader to, *e.g.*, Cavalcanti *et al.* [2], Ebihara, Medeiros and Miranda [4], Larkin [5], Medeiros [10], Miranda *et al.* [11], for many interesting results and further references.

Motivated by results for nonlinear wave equations in [8], [9], where recurrent sequences converge at a rate of order 1 or 2, we will construct a high order iterative scheme to obtain a convergent sequence at a rate of order N to a local weak solution of (1.1)–(1.3). This scheme is established based on a high order method for solving operator equation $F(x)=0$, it also has been applied in [12], [13], [17] and some other works. It is well known that, Newton's method and its variants are used to solve nonlinear operator equations, see [14] and references therein. In case $\lim\limits_{n\to\infty} u_n=u$, one speaks of *convergence of order*

N if $|u_{n+1} - u| \le C|u_n - u|^N$ for some $C > 0$ and all large N. In the special cases $N = 1$ with $C < 1$ and $N = 2$ one also speaks of linear and quadratic convergence, respectively, see [3]. Here we shall associate with (1.1) a recurrent sequence $\{u_m\}$ defined by

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial t^2} - \mu(\|u_m(t)\|_0^2)\left(\frac{\partial^2 u_m}{\partial x^2} + \frac{1}{x}\frac{\partial u_m}{\partial x}\right) \\ &= \sum\nolimits_{i=0}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i, \end{aligned} \tag{1.7}$$

$\rho < x < 1$, $0 < t < T$ with u_m satisfying (1.2), (1.3) and $u_0 \equiv 0$. If $f \in C^N([\rho,1] \times \mathbb{R}_+ \times \mathbb{R})$, we prove that the sequence $\{u_m\}$ converges at a rate of order N to a local weak solution of (1.1)–(1.3). We note more that, the result obtained here is local (in time T small enough), because T is chosen corresponding to the size of the initial data, see (3.40) in Section 3. In our proofs, the Faedo-Galerkin approximation method associated to a priori estimates, weak convergence, compactness techniques and a known fixed point theorem are used. Our results can be regarded as an extension and improvement of the corresponding results of [8], [9], [16].

2. Preliminaries

Put $\Omega = (\rho, 1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $(\cdot,\cdot)$ the scalar product in L^2. We denoted by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T ||u(t)||_X^p dt\right)^{1/p}, & \text{if } \ 1 \le p < \infty, \\ ess\sup\limits_{0<t<T} ||u(t)||_X, & \text{if } \ p = \infty. \end{cases}$$

With $f \in C^k([\rho,1] \times \mathbb{R}_+ \times \mathbb{R})$, $f = f(x,t,y)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial y}$ and $D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} f$, $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{Z}_+^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = k$, $D^{(0,0,0)} f = f$.

On H^1, H^2, we shall use the following norms

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{\frac{1}{2}} \tag{2.1}$$

and

$$\|v\|_{H^2} = \left(\|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2\right)^{\frac{1}{2}}, \tag{2.2}$$

respectively.

Note that L^2, H^1, H^2 are also the Hilbert spaces with respect to the corresponding scalar products

$$\begin{aligned} &\langle u, v\rangle = \int_\rho^1 xu(x)\, v(x)\, dx, \\ &\langle u, v\rangle + \langle u_x, v_x\rangle,\ \langle u, v\rangle + \langle u_x, v_x\rangle + \langle u_{xx}, v_{xx}\rangle, \end{aligned} \tag{2.3}$$

respectively. The norms in L^2, H^1 and H^2 induced by the corresponding scalar products (2.3) are denoted by $\|\cdot\|_0$, $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively.

We put

$$V = \left\{ v \in H^1 : v(\rho) = 0 \right\}. \tag{2.4}$$

Then V is a closed subspace of H^1 and on V two norms $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent norms. V_1 is continuously and densely embedded in L^2. Identifying L^2 with $\left(L^2\right)'$ (the dual of L^2), we have $V \hookrightarrow L^2 \hookrightarrow V'$. We remark that the notation $\langle \cdot, \cdot \rangle$ is also used for the pairing between V and V'.

We have the following lemmas.

Lemma 2.1. *We have the following inequalities*

(i) $\sqrt{\rho}\, \|v\| \le \|v\|_0 \le \|v\|, \quad \forall\, v \in L^2,$

(ii) $\sqrt{\rho}\, \|v\|_{H^1} \le \|v\|_1 \le \|v\|_{H^1}, \quad \forall\, v \in H^1.$

Proof. From the following inequalities

$$\begin{aligned} &\rho \textstyle\int_\rho^1 v^2(x)\, dx \le \int_\rho^1 x v^2(x)\, dx \le \int_\rho^1 v^2(x)\, dx, \quad \text{for all } v \in L^2, \\ &\rho \textstyle\int_\rho^1 v_x^2(x)\, dx \le \int_\rho^1 x v_x^2(x)\, dx \le \int_\rho^1 v_x^2(x)\, dx, \quad \text{for all } v \in H^1, \end{aligned}$$

the Lemma 2.1 is proved. □

Lemma 2.2. *The embedding $V \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact and for all $v \in V$, we have*

(i) $\|v\|_{C^0(\overline{\Omega})} \le \sqrt{1-\rho}\, \|v_x\|,$

(ii) $\|v\| \le \frac{1-\rho}{\sqrt{2}} \|v_x\|,$

(iii) $\|v\|_0 \le \frac{1-\rho}{\sqrt{2\rho}} \|v_x\|_0,$

(iv) $\|v_x\|_0^2 + v^2(1) \ge \|v\|_0^2,$

(v) $|v(1)| \le \sqrt{3}\, \|v\|_1.$

Proof. The embedding $V \hookrightarrow H^1$ is continuous and the embedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact, so the embedding $V \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact.

(i) For all $v \in V$ and $x \in [\rho, 1]$,

$$|v(x)| = \left| \int_\rho^x v_x(y)\, dy \right| \le \int_\rho^1 |v_x(y)|\, dy \le \sqrt{1-\rho}\, \|v_x\|. \tag{2.5}$$

(ii) For all $v \in V$ and $x \in [\rho, 1]$,

$$v^2(x) = \left| \int_\rho^x v_x(y)\, dy \right|^2 \leq (x-\rho) \int_\rho^x v_x^2(y)\, dy \leq (x-\rho) \|v_x\|^2. \tag{2.6}$$

Integrating over x from ρ to 1, we obtain

$$\|v\|^2 = \int_\rho^1 v^2(x)\, dx \leq \int_\rho^1 (x-\rho) \|v_x\|^2 dx = \frac{(1-\rho)^2}{2} \|v_x\|^2. \tag{2.7}$$

(iii) For all $v \in V$,

$$\|v\|_0 \leq \|v\| \leq \frac{1-\rho}{\sqrt{2}} \|v_x\| \leq \frac{1-\rho}{\sqrt{2\rho}} \|v_x\|_0. \tag{2.8}$$

(iv) By using integration by part we have, for any $v \in V$,

$$\begin{aligned} &\|v\|_0^2 = \int_\rho^1 xv^2(x)\, dx = \tfrac{1}{2}\left[x^2 v^2(x)\right]_\rho^1 - \int_\rho^1 x^2 v(x)\, v_x(x)\, dx, \\ &\tfrac{1}{2} v^2(1) - \int_\rho^1 x^2 v(x)\, v_x(x)\, dx \\ &\leq \tfrac{1}{2} v^2(1) + \|v\|_0 \|v_x\|_0 \leq \tfrac{1}{2} v^2(1) + \tfrac{1}{2}\left(\|v\|_0^2 + \|v_x\|_0^2\right), \end{aligned} \tag{2.9}$$

which implies (iv).

(v) By $\|v\|_0^2 = \frac{1}{2} v^2(1) - \int_\rho^1 x^2 v(x)\, v_x(x)\, dx$, we have,

$$\begin{aligned} v^2(1) &= 2\|v\|_0^2 + 2\int_\rho^1 x^2 v(x)\, v_x(x)\, dx \\ &\leq 2\|v\|_0^2 + 2\|v\|_0 \|v_x\|_0 \leq 2\|v\|_0^2 + \|v\|_0^2 + \|v_x\|_0^2 \leq 3\|v\|_1^2, \end{aligned} \tag{2.10}$$

it gives (v). The Lemma 2.2 is proved. □

Remark 2.3. On L^2, two norms $v \mapsto \|v\|$ and $v \mapsto \|v\|_0$ are equivalent. So are two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v\|_1$ on H^1, and five norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_1$, $v \mapsto \|v_x\|$, $v \mapsto \|v_x\|_0$ and $v \mapsto \sqrt{\|v_x\|_0^2 + v^2(1)}$ on V.

Now, we define the bilinear form

$$a(u, v) = \zeta u(1)\, v(1) + \int_\rho^1 x u_x(x)\, v_x(x)\, dx \text{ for all } u, v \in V_1, \tag{2.11}$$

where $\zeta \geq 0$ is a constant. We then have the following lemma.

Lemma 2.4. *The symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.11) *is continuous on $V \times V$ and coercive on V, i.e.,*

(i) $|a(u, v)| \leq C_1 \|u\|_1 \|v\|_1$,

(ii) $a(v, v) \geq C_0 \|v\|_1^2$,

for all $u,\ v \in V$, *where* $C_0 = \frac{1}{2}\min\{1, \frac{2\rho}{(1-\rho)^2}\}$ *and* $C_1 = 1 + 3\zeta$.

Proof. (i) By $\sqrt{1-\rho}\,\|v_x\| \geq \|v\|_{C^0(\overline{\Omega})} \geq |v(1)|$ and $\sqrt{\rho}\,\|v_x\| \leq \|v_x\|_0$ for all $v \in V$, we have

$$\begin{aligned} |a(u,v)| &\leq \zeta\,|u(1)|\,|v(1)| + \int_\rho^1 |xu_x(x)\,v_x(x)|\,dx \\ &\leq 3\zeta\,\|u\|_1\,\|v\|_1 + \|u_x\|_0\,\|v_x\|_0 \leq (3\zeta+1)\,\|u\|_1\,\|v\|_1 . \end{aligned}$$

(ii) By the inequality

$$\|v_x\|_0^2 \geq \frac{2\rho}{(1-\rho)^2}\,\|v\|_0^2 ,$$

we have

$$\begin{aligned} a(v,v) &= \zeta v^2(1) + \int_\rho^1 xv_x^2(x)\,dx = \zeta v^2(1) + \|v_x\|_0^2 \\ &\geq \|v_x\|_0^2 = \frac{1}{2}\|v_x\|_0^2 + \frac{1}{2}\|v_x\|_0^2 \\ &\geq \frac{1}{2}\|v_x\|_0^2 + \frac{1}{2}\frac{2\rho}{(1-\rho)^2}\|v\|_0^2 \geq \frac{1}{2}\min\left\{1, \frac{2\rho}{(1-\rho)^2}\right\}\|v\|_1^2 . \end{aligned}$$

The Lemma 2.4 is proved. □

Lemma 2.5. *There exists the Hilbert orthonormal base* $\{w_j\}$ *of the space* L^2 *consisting of eigenfunctions* w_j *corresponding to eigenvalues* λ_j *such that*

$$\begin{aligned} &\text{(i)}\ \ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots,\ \ \lim_{j\to+\infty}\lambda_j = +\infty, \\ &\text{(ii)}\ \ a(w_j, v) = \lambda_j\,\langle w_j, v\rangle \quad \textit{for all}\ v \in V, \quad j = 1, 2, \cdots . \end{aligned} \tag{2.12}$$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ *is also the Hilbert orthonormal base of* V *with respect to the scalar product* $a(\cdot,\cdot)$.

On the other hand, we also have w_j *satisfying the following boundary value problem*

$$\begin{cases} Aw_j \equiv -\left(w_{jxx} + \frac{1}{x}w_{jx}\right) = -\frac{1}{x}\frac{\partial}{\partial x}(xw_{jx}) = \lambda_j w_j,\ \ \textit{in}\ \Omega, \\ w_j(\rho) = w_{jx}(1) + \zeta w_j(1) = 0,\ \ w_j \in C^\infty([\rho,1]) . \end{cases} \tag{2.13}$$

Proof. The proof of Lemma 2.5 can be found in [[15], p.87, Theorem 7.7], with $H = L^2$ and $a(\cdot,\cdot)$ as defined by (2.11). □

We also note that the operator $A : V \longrightarrow V'$ in (2.13) is uniquely defined by the Lax-Milgram's lemma, *i.e.*,

$$a(u,v) = \langle Au, v\rangle \quad \text{for all} \ \ u, v \in V. \tag{2.14}$$

Lemma 2.6. *On $V \cap H^2$, three norms*

$$v \mapsto \|v\|_{H^2}, \quad v \mapsto \|v\|_2 = \sqrt{\|v\|_0^2 + \|v_x\|_0^2 + \|v_{xx}\|_0^2}$$

and

$$v \mapsto \|v\|_{2*} = \sqrt{\|v_x\|_0^2 + \|Av\|_0^2}$$

are equivalent.

Proof. (i) It is easy to see that two norms

$$v \mapsto \|v\|_{H^2}, \quad v \mapsto \|v\|_2 = \sqrt{\|v\|_0^2 + \|v_x\|_0^2 + \|v_{xx}\|_0^2}$$

are equivalent on $V \cap H^2$, because of

$$\sqrt{\rho}\, \|v\|_{H^2} \leq \|v\|_2 \leq \|v\|_{H^2} \quad \text{for all } \ v \in H^2. \tag{2.15}$$

(ii) For all $x \in [\rho, 1]$, and $v \in V \cap H^2$, we have

$$x\,|Au\,(x)|^2 = x\tfrac{1}{x^2}\left[\tfrac{\partial}{\partial x}\,(xu_x)\right]^2 = xu_{xx}^2 + 2u_x u_{xx} + \tfrac{1}{x}u_x^2. \tag{2.16}$$

(ii)-(a). We verify $\|u\|_2 \leq const\ \|u\|_{2*}$.
It follows from (2.16) that

$$xu_{xx}^2 \leq x\,|Au\,(x)|^2 + 2\,|u_x u_{xx}| + \tfrac{1}{x}u_x^2. \tag{2.17}$$

Hence

$$\begin{aligned} \|u_{xx}\|_0^2 &\leq \|Au\|_0^2 + \tfrac{2}{\rho}\,\|u_x\|_0\,\|u_{xx}\|_0 + \tfrac{1}{\rho^2}\,\|u_x\|_0^2 \\ &\leq \|Au\|_0^2 + \tfrac{1}{\rho}\left(\tfrac{2}{\rho}\,\|u_x\|_0^2 + \frac{\rho}{2}\,\|u_{xx}\|_0^2\right) + \tfrac{1}{\rho^2}\,\|u_x\|_0^2 \\ &= \|Au\|_0^2 + \tfrac{2}{\rho^2}\,\|u_x\|_0^2 + \tfrac{1}{2}\,\|u_{xx}\|_0^2 + \tfrac{1}{\rho^2}\,\|u_x\|_0^2. \end{aligned} \tag{2.18}$$

This implies that

$$\begin{aligned} \|u_{xx}\|_0^2 &\leq 2\,\|Au\|_0^2 + \tfrac{6}{\rho^2}\,\|u_x\|_0^2 \leq 2\left(1 + \tfrac{3}{\rho^2}\right)\left(\|Au\|_0^2 + \|u_x\|_0^2\right) \\ &\leq 2\left(1 + \tfrac{3}{\rho^2}\right)\|u\|_{2*}^2. \end{aligned} \tag{2.19}$$

By $\|v\|_0 \leq \frac{1-\rho}{\sqrt{2\rho}}\,\|v_x\|_0$, for all $v \in V$, we get

$$\begin{aligned} \|u\|_2^2 &= \|u\|_0^2 + \|u_x\|_0^2 + \|u_{xx}\|_0^2 \\ &\leq \tfrac{(1-\rho)^2}{2\rho}\,\|u_x\|_0^2 + \|u_x\|_0^2 + \|u_{xx}\|_0^2 \\ &= \left(1 + \tfrac{(1-\rho)^2}{2\rho}\right)\|u_x\|_0^2 + \|u_{xx}\|_0^2 \\ &\leq \left(1 + \tfrac{(1-\rho)^2}{2\rho}\right)\|u\|_{2*}^2 + 2\left(1 + \tfrac{3}{\rho^2}\right)\|u\|_{2*}^2 \\ &= \left(\tfrac{(1-\rho)^2}{2\rho} + 3 + \tfrac{6}{\rho^2}\right)\|u\|_{2*}^2. \end{aligned} \tag{2.20}$$

(ii)-(b). We verify $\|u\|_{2*} \leq const\ \|u\|_2$.

It follows from (2.16) that

$$x\left|Au\left(x\right)\right|^2 = x\tfrac{1}{x^2}\left[\tfrac{\partial}{\partial x}\left(xu_x\right)\right]^2 = xu_{xx}^2 + 2u_xu_{xx} + \tfrac{1}{x}u_x^2. \tag{2.21}$$

Hence

$$x\left|Au\left(x\right)\right|^2 \le xu_{xx}^2 + 2\left|u_xu_{xx}\right| + \tfrac{1}{x}u_x^2. \tag{2.22}$$

Thus

$$\begin{aligned} \left\|Au\right\|_0^2 &\le \left\|u_{xx}\right\|_0^2 + \tfrac{2}{\rho}\left\|u_x\right\|_0\left\|u_{xx}\right\|_0 + \tfrac{1}{\rho^2}\left\|u_x\right\|_0^2 \\ &\le \left\|u_{xx}\right\|_0^2 + \tfrac{1}{\rho}\left(\left\|u_x\right\|_0^2 + \left\|u_{xx}\right\|_0^2\right) + \tfrac{1}{\rho^2}\left\|u_x\right\|_0^2 \\ &= \left(1+\tfrac{1}{\rho}\right)\left[\left\|u_{xx}\right\|_0^2 + \tfrac{1}{\rho}\left\|u_x\right\|_0^2\right] \\ &\le \left(1+\tfrac{1}{\rho}\right)\tfrac{1}{\rho}\left[\left\|u_{xx}\right\|_0^2 + \left\|u_x\right\|_0^2\right] \le \left(1+\tfrac{1}{\rho}\right)\tfrac{1}{\rho}\left\|u\right\|_2^2. \end{aligned} \tag{2.23}$$

This implies

$$\begin{aligned} \left\|u\right\|_{2*}^2 &= \left\|u_x\right\|_0^2 + \left\|Au\right\|_0^2 \\ &\le \left\|u\right\|_2^2 + \left(1+\tfrac{1}{\rho}\right)\tfrac{1}{\rho}\left\|u\right\|_2^2 = \left(1+\tfrac{1}{\rho}+\tfrac{1}{\rho^2}\right)\left\|u\right\|_2^2. \end{aligned} \tag{2.24}$$

The Lemma 2.6 is proved. □

3. A high order iterative scheme

First, we say that u is a weak solution of (1.1)–(1.3) if

$$u \in L^\infty(0,T;V\cap H^2),\ u_t \in L^\infty(0,T;V),\ u_{tt} \in L^\infty(0,T;L^2)\} \tag{3.1}$$

and u satisfies the following variational equation

$$\langle u_{tt}(t), v\rangle + \mu\left(\left\|u(t)\right\|_0^2\right)a(u(t),v) = \left\langle f\left(x,t,u\right), v\right\rangle, \tag{3.2}$$

for all $v \in V$ and a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ \ u_t(0) = \tilde{u}_1, \tag{3.3}$$

where $a(\cdot,\cdot)$ is the symmetric bilinear form on V defined by (2.11).

Now, we make the following assumptions.

(H_1) $\tilde{u}_0 \in V\cap H^2$, $\tilde{u}_1 \in V$;

(H_2) $\mu \in C^1\left(\mathbb{R}_+\right)$, and there exist constants $p>1$, $\mu_*>0$, $\mu_1>0$, $\mu_2>0$ such that
- (i) $0<\mu_* \le \mu(z) \le \mu_1(1+z^p)$, for all $z \ge 0$,
- (ii) $|\mu'(z)| \le \mu_2(1+z^{p-1})$, for all $z \ge 0$;

(H_3) $f \in C^0([\rho,1]\times\mathbb{R}_+\times\mathbb{R})$ such that $f(\rho,t,0)=0$, $\forall\, t\ge 0$ and
- (i) $D_3^i f \in C^0([\rho,1]\times\mathbb{R}_+\times\mathbb{R})$, $1\le i\le N$,
- (ii) $D_1D_3^i f \in C^0([\rho,1]\times\mathbb{R}_+\times\mathbb{R})$, $0\le i\le N-1$.

Fix $T^* > 0$. For each $M > 0$ given, we set the constants $\tilde{K}_M(\mu)$, $\bar{K}_M(f)$ as follows

$$\begin{cases} \tilde{K}_M(\mu) = \sup\limits_{0 \le z \le M^2} (\mu(z) + |\mu'(z)|), \\ \bar{K}_M(f) = \sum_{i=0}^{N} \left\| D_3^i f \right\|_{C^0(A_*(M))} + \sum_{i=1}^{N-1} \left\| D_1 D_3^i f \right\|_{C^0(A_*(M))}, \\ \|f\|_{C^0(A_*(M))} = \sup\{|f(x,t,y)| : (x,t,y) \in A_*(M)\}, \end{cases}$$

where $A_*(M) = \left\{ (x,t,y) \in [\rho,1] \times [0,T^*] \times \mathbb{R} : |y| \le \sqrt{\frac{1-\rho}{\rho}} M \right\}$. For each $M > 0$ and $T \in (0,T^*]$, we put

$$\begin{aligned} W(M,T) = \{ & u \in L^\infty\left(0,T; V \cap H^2\right) : u_t \in L^\infty(0,T;V),\ u_{tt} \in L^2(Q_T), \\ & \|u\|_{L^\infty(0,T;V\cap H^2)} \le M, \|u_t\|_{L^\infty(0,T;V)} \le M, \|u_{tt}\|_{L^2(Q_T)} \le M\}, \\ W_1(M,T) = \{ & u \in W(M,T) : u_{tt} \in L^\infty\left(0,T;L^2\right)\}. \end{aligned}$$

Now, we establish the following recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.4}$$

we associate (3.2) with the following problem.

Find $u_m \in W_1(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle \ddot{u}_m(t), v\rangle + \bar{\mu}_m(t)\, a(u_m(t), v) = \left\langle \bar{F}_m(t), v\right\rangle, \ \forall\, v \in V, \\ u_m(0) = \tilde{u}_0, \ \dot{u}_m(0) = \tilde{u}_1, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} \bar{\mu}_m(t) &= \mu\left(\|u_m(t)\|_0^2\right), \\ \bar{F}_m(x,t) &= \sum_{i=0}^{N-1} \tfrac{1}{i!} D_3^i f(x,t,u_{m-1})(u_m - u_{m-1})^i. \end{aligned} \tag{3.6}$$

Then we have the following theorem.

Theorem 3.1. *Let (H_1)-(H_3) hold. Then there exist a constant $M > 0$ depending on $\tilde{u}_0$, $\tilde{u}_1$, μ, ζ, ρ and $T > 0$ depending on $\tilde{u}_0$, $\tilde{u}_1$, μ, f, ζ, ρ such that, for $u_0 \equiv 0$, there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (3.5) *and* (3.6)*.*

Proof. Step 1. *Approximating solutions.* Consider the basis $\{w_j\}$ for V as in Lemma 2.5. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.7}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of nonlinear differential equations

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + \mu_m^{(k)}(t)\, a(u_m^{(k)}(t), w_j) = \left\langle F_m^{(k)}(t), w_j \right\rangle, \ j = 1, \cdots, k, \\ u_m^{(k)}(0) = u_{0k}, \ \dot{u}_m^{(k)}(0) = u_{1k}, \end{cases} \tag{3.8}$$

in which

$$\begin{cases} u_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \longrightarrow \tilde{u}_0 \quad \text{strongly} \quad V \cap H^2, \\ u_{1k} = \sum_{j=1}^k \beta_j^{(k)} w_j \longrightarrow \tilde{u}_1 \quad \text{strongly} \quad V, \end{cases} \tag{3.9}$$

and

$$\begin{cases} \mu_m^{(k)}(t) = \mu\left(\left\| u_m^{(k)}(t) \right\|_0^2 \right), \\ F_m^{(k)}(x,t) = \sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1})(u_m^{(k)} - u_{m-1})^i \\ \qquad\qquad = \sum_{j=0}^{N-1} A_j(x,t,u_{m-1})(u_m^{(k)})^j, \end{cases} \tag{3.10}$$

with

$$A_j(x,t,u_{m-1}) = \sum_{i=j}^{N-1} \frac{(-1)^{i-j}}{j!(i-j)!} D_3^i f(x,t,u_{m-1}) u_{m-1}^{i-j}. \tag{3.11}$$

The system (3.8), (3.9) can be written in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda_j \mu_m^{(k)}(t) c_{mj}^{(k)}(t) = F_{mj}^{(k)}(t), \quad 1 \le j \le k, \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \quad \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \end{cases} \tag{3.12}$$

where

$$F_{mj}^{(k)}(t) = \left\langle F_m^{(k)}(t), w_j \right\rangle, \ 1 \le j \le k. \tag{3.13}$$

It is obviously that the system (3.13) is equivalent to the system of intergal equations

$$\begin{aligned} c_{mj}^{(k)}(t) = \alpha_j^{(k)} + \beta_j^{(k)} t - \lambda_j \int_0^t d\tau \int_0^\tau \mu_m^{(k)}(s) c_{mj}^{(k)}(s) ds \\ + \int_0^t d\tau \int_0^\tau F_{mj}^{(k)}(s) ds, \quad 1 \le j \le k. \end{aligned} \tag{3.14}$$

Note that by (3.4), it is not difficult to prove that the system (3.14) has a unique solution $c_{mj}^{(k)}(t)$, $1 \le j \le k$ on interval $[0, T_m^{(k)}] \subset [0, T]$, so let us omit the details.

The following estimates allow one to take $T_m^{(k)} = T$ independent of m and k.

Step 2. *A priori estimates.* We put

$$S_m^{(k)}(t) = X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|_0^2 ds, \tag{3.15}$$

where

$$\begin{cases} X_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|_0^2 + \mu_m^{(k)}(t)\, a(u_m^{(k)}(t), u_m^{(k)}(t)), \\ Y_m^{(k)}(t) = a\left(\dot{u}_m^{(k)}(t), \dot{u}_m^{(k)}(t)\right) + \mu_m^{(k)}(t) \left\|Au_m^{(k)}(t)\right\|_0^2 . \end{cases} \tag{3.16}$$

Then, it follows from (3.8), (3.15), (3.16), that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + \int_0^t \dot{\mu}_m^{(k)}(s) \left[a\left(u_m^{(k)}(s), u_m^{(k)}(s)\right) + \left\|Au_m^{(k)}(s)\right\|_0^2 \right] ds \\ &\quad + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds + 2\int_0^t \left\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\right\rangle ds \\ &\quad +2\int_0^t a\left(F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\right) ds \\ &\equiv S_m^{(k)}(0) + \sum_{j=1}^4 I_j. \end{aligned} \tag{3.17}$$

We shall estimate the terms I_j on the right–hand side of (3.17) as follows.

First term I_1. By the following inequalities

$$\begin{aligned} &\|v\|_0 \le \tfrac{1-\rho}{\sqrt{2\rho}} \|v_x\|_0 \le \tfrac{1-\rho}{\sqrt{2\rho}} \sqrt{a(v,v)} \quad \text{for all } v \in V, \\ &S_m^{(k)}(t) \ge \mu_m(t) \left[a(u_m^{(k)}(t), u_m^{(k)}(t)) + \left\|Au_m^{(k)}(t)\right\|_0^2 \right] \\ &\qquad \ge \mu_* \left[a(u_m^{(k)}(t), u_m^{(k)}(t)) + \left\|Au_m^{(k)}(t)\right\|_0^2 \right], \\ &\left\|u_m^{(k)}(t)\right\|_0 \le \tfrac{1-\rho}{\sqrt{2\rho}} \left\|u_{mx}^{(k)}(t)\right\|_0 \le \tfrac{1-\rho}{\sqrt{2\rho}} \sqrt{a(u_m^{(k)}(t), u_m^{(k)}(t)} \\ &\qquad \le \tfrac{1-\rho}{\sqrt{2\rho\mu_*}} \sqrt{S_m^{(k)}(t)} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \left|\dot{\mu}_m^{(k)}(t)\right| &= 2\left|\mu'\left(\left\|u_m^{(k)}(t)\right\|_0^2\right) \langle u_m^{(k)}(t), \dot{u}_m^{(k)}(t)\rangle\right| \\ &\le 2\mu_2 \left(1 + \left\|u_m^{(k)}(t)\right\|_0^{2p-2}\right) \left\|u_m^{(k)}(t)\right\|_0 \left\|\dot{u}_m^{(k)}(t)\right\|_0 \\ &\le 2\mu_2 \left[1 + \left(\tfrac{1-\rho}{\sqrt{2\rho\mu_*}} \sqrt{S_m^{(k)}(t)}\right)^{2p-2}\right] \tfrac{1-\rho}{\sqrt{2\rho\mu_*}} \sqrt{S_m^{(k)}(t)} \sqrt{S_m^{(k)}(t)} \\ &= 2\mu_2 \tfrac{1-\rho}{\sqrt{2\rho\mu_*}} \left[1 + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*}\right)^{p-1} \left(S_m^{(k)}(t)\right)^{p-1}\right] S_m^{(k)}(t), \end{aligned} \tag{3.19}$$

we have

$$\begin{aligned}
I_1 &= \int_0^t \dot{\mu}_m^{(k)}(s)\left[a\left(u_m^{(k)}(s), u_m^{(k)}(s)\right) + \left\|Au_m^{(k)}(s)\right\|_0^2\right] ds \\
&\le 2\mu_2 \tfrac{1-\rho}{\sqrt{2\rho\mu_*^3}} \int_0^t \left[1 + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*}\right)^{p-1}\left(S_m^{(k)}(s)\right)^{p-1}\right]\left(S_m^{(k)}(s)\right)^2 ds \\
&\le 2\mu_2 \tfrac{1-\rho}{\sqrt{2\rho\mu_*^3}} \int_0^t \left[\left(S_m^{(k)}(s)\right)^2 + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*}\right)^{p-1}\left(S_m^{(k)}(s)\right)^{p+1}\right] ds \\
&\le 2\mu_2 \tfrac{1-\rho}{\sqrt{2\rho\mu_*^3}} \left[1 + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*}\right)^{p-1}\right] \int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N_1}\right] ds \\
&= \tilde{\beta}_1 \int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N_1}\right] ds,
\end{aligned} \tag{3.20}$$

where

$$N_1 = \max\{p+1, N-1\}, \quad \tilde{\beta}_1 = 2\mu_2 \tfrac{1-\rho}{\sqrt{2\rho\mu_*^3}}\left[1 + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*}\right)^{p-1}\right]. \tag{3.21}$$

Second term I_2. $(3.8)_1$ can be rewritten as follows

$$\left\langle \ddot{u}_m^{(k)}(t), w_j\right\rangle + \mu_m^{(k)}(t)\left\langle Au_m^{(k)}(t), w_j\right\rangle = \left\langle F_m^{(k)}(t), w_j\right\rangle, \ j = 1, \cdots, k. \tag{3.22}$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$, we obtain that

$$\begin{aligned}
&\left\|\ddot{u}_m^{(k)}(t)\right\|_0^2 \\
&= -\mu_m^{(k)}(t)\left\langle Au_m^{(k)}(t), \ddot{u}_m^{(k)}(t)\right\rangle + \left\langle F_m^{(k)}(t), \ddot{u}_m^{(k)}(t)\right\rangle \\
&\le \left[\mu_m^{(k)}(t)\left\|Au_m^{(k)}(t)\right\|_0 + \left\|F_m^{(k)}(t)\right\|_0\right]\left\|\ddot{u}_m^{(k)}(s)\right\|_0 \\
&\le \left[\mu_m^{(k)}(t)\left\|Au_m^{(k)}(t)\right\|_0 + \left\|F_m^{(k)}(t)\right\|_0\right]^2 \\
&\le 2\left(\mu_m^{(k)}(t)\right)^2\left\|Au_m^{(k)}(t)\right\|_0^2 + 2\left\|F_m^{(k)}(t)\right\|_0^2 \\
&\le 2\mu_1\left(1 + \left\|u_m^{(k)}(t)\right\|_0^{2p}\right) S_m^{(k)}(t) + 2\left\|F_m^{(k)}(t)\right\|_0^2 \\
&\le 2\mu_1\left[1 + \left(\tfrac{1-\rho}{\sqrt{2\rho\mu_*}}\sqrt{S_m^{(k)}(t)}\right)^{2p}\right] S_m^{(k)}(t) + 2\tfrac{(1-\rho)^2}{2\rho}\left\|F_{mx}^{(k)}(t)\right\|_0^2 \\
&\le 2\mu_1\left[S_m^{(k)}(t) + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*}\right)^p\left(S_m^{(k)}(t)\right)^{p+1}\right] + \tfrac{(1-\rho)^2}{\rho}\left\|F_{mx}^{(k)}(t)\right\|_0^2 \\
&\le 2\mu_1\left[1 + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*}\right)^p\right]\left[1 + \left(S_m^{(k)}(t)\right)^{N_1}\right] + \tfrac{(1-\rho)^2}{\rho}\left\|F_{mx}^{(k)}(t)\right\|_0^2.
\end{aligned} \tag{3.23}$$

Integrating in t to get

$$
\begin{aligned}
I_2 &= \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|_0^2 ds \\
&\leq 2\mu_1 \left[1 + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*} \right)^p \right] \int_0^t \left[1 + \left(S_m^{(k)}(s) \right)^{N_1} \right] ds \\
&\quad + \tfrac{(1-\rho)^2}{\rho} \int_0^t \left\| F_{mx}^{(k)}(s) \right\|_0^2 ds \\
&= \tilde{\beta}_2 \int_0^t \left[1 + \left(S_m^{(k)}(s) \right)^{N_1} \right] ds + \tfrac{(1-\rho)^2}{\rho} \int_0^t \left\| F_{mx}^{(k)}(s) \right\|_0^2 ds,
\end{aligned} \tag{3.24}
$$

where

$$
\tilde{\beta}_2 = 2\mu_1 \left[1 + \left(\tfrac{(1-\rho)^2}{2\rho\mu_*} \right)^p \right]. \tag{3.25}
$$

Third integral I_3.

$$
\begin{aligned}
I_3 &= 2 \int_0^t \left\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s) \right\rangle ds \\
&\leq \int_0^t S_m^{(k)}(s) ds + \tfrac{(1-\rho)^2}{2\rho} \int_0^t \left\| F_{mx}^{(k)}(s) \right\|_0^2 ds.
\end{aligned} \tag{3.26}
$$

Fourth term I_4.

$$
\begin{aligned}
I_4 &= 2 \int_0^t a\left(F_m^{(k)}(s), \dot{u}_m^{(k)}(s) \right) ds \\
&\leq 2 \int_0^t \sqrt{a\left(\dot{u}_m^{(k)}(s), \dot{u}_m^{(k)}(s) \right)} \sqrt{a\left(F_m^{(k)}(s), F_m^{(k)}(s) \right)} ds \\
&\leq \int_0^t a\left(\dot{u}_m^{(k)}(s), \dot{u}_m^{(k)}(s) \right) ds + \int_0^t a\left(F_m^{(k)}(s), F_m^{(k)}(s) \right) ds \\
&\leq \int_0^t S_m^{(k)}(s) ds + C_1 \int_0^t \left\| F_m^{(k)}(s) \right\|_1^2 ds \\
&\leq \int_0^t S_m^{(k)}(s) ds + C_1 \int_0^t \left[\left\| F_m^{(k)}(s) \right\|_0^2 + \left\| F_{mx}^{(k)}(s) \right\|_0^2 \right] ds \\
&\leq \int_0^t S_m^{(k)}(s) ds + C_1 \left(1 + \tfrac{(1-\rho)^2}{2\rho} \right) \int_0^t \left\| F_{mx}^{(k)}(s) \right\|_0^2 ds.
\end{aligned} \tag{3.27}
$$

Therefore, we deduce from (3.17), (3.20), (3.24), (2.24), (3.27) that

$$
\begin{aligned}
S_m^{(k)}(t) \leq S_m^{(k)}(0) &+ \left(\tilde{\beta}_1 + \tilde{\beta}_2 \right) \int_0^t \left[1 + \left(S_m^{(k)}(s) \right)^{N_1} \right] ds + 2 \int_0^t S_m^{(k)}(s) ds \\
&+ \left[\tfrac{3(1-\rho)^2}{2\rho} + C_1 \left(1 + \tfrac{(1-\rho)^2}{2\rho} \right) \right] \int_0^t \left\| F_{mx}^{(k)}(s) \right\|_0^2 ds.
\end{aligned} \tag{3.28}
$$

The following property of $F_{mx}^{(k)}(t)$ is useful to continue estimates

$$
\left\| F_{mx}^{(k)}(t) \right\|_0 \leq \bar{c}_M \left[1 + \left(\sqrt{S_m^{(k)}(t)} \right)^{N-1} \right], \tag{3.29}
$$

where $\bar{c}_M = \sum_{i=0}^{N-1} \tilde{c}_i$,

$$\tilde{c}_i = \begin{cases} \bar{K}_M(f)\left[\sqrt{\frac{1-\rho^2}{2}} + M + \sum_{i=1}^{N-1} \frac{2^{i-1}}{i!}\gamma_i^*(M,\rho)M^i\right], & i = 0, \\ \bar{K}_M(f)\frac{2^{i-1}}{i!}\frac{\gamma_i^*(M,\rho)}{\sqrt{\mu_*^i}}, & i = 1,2,\cdots,N-1, \end{cases} \tag{3.30}$$

$$\gamma_i^*(M,\rho) = \left[\left(\sqrt{\frac{1-\rho^2}{2}} + M\right)\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i + i\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i-1}\right],$$
$$1 \le i \le N-1.$$

Indeed, by

$$\begin{aligned} &F_{mx}^{(k)}(x,t) \\ &= D_1 f(x,t,u_{m-1}) + D_3 f(x,t,u_{m-1})\nabla u_{m-1} \\ &\quad + \sum_{i=1}^{N-1}\left[\tfrac{1}{i!}D_1 D_3^i f(x,t,u_{m-1}) \right. \\ &\quad \left. + \tfrac{1}{i!}D_3^{i+1} f(x,t,u_{m-1})\nabla u_{m-1}\right](u_m^{(k)} - u_{m-1})^i \\ &\quad + \sum_{i=1}^{N-1}\tfrac{1}{i!}D_3^i f(x,t,u_{m-1}) i (u_m^{(k)} - u_{m-1})^{i-1}(\nabla u_m^{(k)} - \nabla u_{m-1}), \end{aligned} \tag{3.31}$$

using inequalities

$$(a+b)^p \le 2^{p-1}(a^p + b^p), \ \ \forall\, a,b > 0, \ \ p \ge 1,$$

and

$$s^i \le 1 + s^q, \ \ \forall\, s \ge 0, \ \ \forall\, i,q, \ 0 \le i \le q, \tag{3.32}$$

we get

$$\begin{aligned} &\left|F_{mx}^{(k)}(x,t)\right| \\ &\le |D_1 f(x,t,u_{m-1}) + D_3 f(x,t,u_{m-1})\nabla u_{m-1}| \\ &\quad + \sum_{i=1}^{N-1}\left|\left[\tfrac{1}{i!}D_1 D_3^i f(x,t,u_{m-1}) + \tfrac{1}{i!}D_3^{i+1} f(x,t,u_{m-1})\nabla u_{m-1}\right](u_m^{(k)} - u_{m-1})^i\right| \\ &\quad + \sum_{i=1}^{N-1}\left|\tfrac{1}{i!}D_3^i f(x,t,u_{m-1}) i (u_m^{(k)} - u_{m-1})^{i-1}(\nabla u_m^{(k)} - \nabla u_{m-1})\right| \\ &\le \bar{K}_M(f)\left(1 + |\nabla u_{m-1}|\right) + \bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(1 + |\nabla u_{m-1}|\right)\left|u_m^{(k)} - u_{m-1}\right|^i \\ &\quad + \bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{i}{i!}\left|(u_m^{(k)} - u_{m-1})^{i-1}(\nabla u_m^{(k)} - \nabla u_{m-1})\right| \\ &\le \bar{K}_M(f)\left(1 + |\nabla u_{m-1}|\right) \\ &\quad + \bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(1 + |\nabla u_{m-1}|\right)\left(\sqrt{\frac{1-\rho}{\rho}}\left(\left\|u_{mx}^{(k)}(t)\right\|_0 + \|\nabla u_{m-1}(t)\|_0\right)\right)^i \\ &\quad + \bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{i}{i!}\left|\nabla u_m^{(k)} - \nabla u_{m-1}\right|\left(\sqrt{\frac{1-\rho}{\rho}}\left(\left\|u_{mx}^{(k)}(t)\right\|_0 + \|\nabla u_{m-1}(t)\|_0\right)\right)^{i-1} \\ &\le \bar{K}_M(f)\left(1 + |\nabla u_{m-1}|\right) \end{aligned}$$

$$
\begin{aligned}
&+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(1+|\nabla u_{m-1}|\right)\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i}\left(\left\|u_{mx}^{(k)}(t)\right\|_0+\|\nabla u_{m-1}(t)\|_0\right)^{i}\\
&+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{i}{i!}\left|\nabla u_m^{(k)}-\nabla u_{m-1}\right|\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i-1}\\
&\times\left(\left\|u_{mx}^{(k)}(t)\right\|_0+\|\nabla u_{m-1}(t)\|_0\right)^{i-1}.
\end{aligned}
\tag{3.33}
$$

Hence

$$
\begin{aligned}
&\left\|F_{mx}^{(k)}(t)\right\|_0\\
&\le \bar{K}_M(f)\left(\sqrt{\frac{1-\rho^2}{2}}+\|\nabla u_{m-1}\|_0\right)\\
&\quad+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(\sqrt{\frac{1-\rho^2}{2}}+\|\nabla u_{m-1}\|_0\right)\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i}\left(\left\|u_{mx}^{(k)}(t)\right\|_0+\|\nabla u_{m-1}(t)\|_0\right)^{i}\\
&\quad+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{i}{i!}\left\|u_{mx}^{(k)}(t)-\nabla u_{m-1}(t)\right\|_0\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i-1}\left(\left\|u_{mx}^{(k)}(t)\right\|_0+\|\nabla u_{m-1}(t)\|_0\right)^{i-1}\\
&\le \bar{K}_M(f)\left(\sqrt{\frac{1-\rho^2}{2}}+M\right)\\
&\quad+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{1}{i!}\left(\sqrt{\frac{1-\rho^2}{2}}+M\right)\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i}\left(\left\|u_{mx}^{(k)}(t)\right\|_0+M\right)^{i}\\
&\quad+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{i}{i!}\left(\sqrt{\frac{1-\rho}{\rho}}\right)^{i-1}\left(\left\|u_{mx}^{(k)}(t)\right\|_0+M\right)^{i}\\
&= \bar{K}_M(f)\left(\sqrt{\frac{1-\rho^2}{2}}+M\right)+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{1}{i!}\gamma_i^*(M,\rho)\left(\left\|u_{mx}^{(k)}(t)\right\|_0+M\right)^{i}\\
&\le \bar{K}_M(f)\left(\sqrt{\frac{1-\rho^2}{2}}+M\right)+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{1}{i!}\gamma_i^*(M,\rho)2^{i-1}\left(\left\|u_{mx}^{(k)}(t)\right\|_0^{i}+M^i\right)\\
&\le \bar{K}_M(f)\left(\sqrt{\frac{1-\rho^2}{2}}+M\right)+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\gamma_i^*(M,\rho)\left[\left(\sqrt{\tfrac{S_m^{(k)}(t)}{\mu_*}}\right)^{i}+M^i\right]\\
&= \bar{K}_M(f)\left(\sqrt{\frac{1-\rho^2}{2}}+M\right)+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\gamma_i^*(M,\rho)\left[\left(\sqrt{\tfrac{S_m^{(k)}(t)}{\mu_*}}\right)^{i}+M^i\right]\\
&= \bar{K}_M(f)\left[\sqrt{\frac{1-\rho^2}{2}}+M+\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\gamma_i^*(M,\rho)M^i\right]\\
&\quad+\bar{K}_M(f)\sum_{i=1}^{N-1}\tfrac{2^{i-1}}{i!}\frac{\gamma_i^*(M,\rho)}{\sqrt{\mu_*^i}}\left(\sqrt{S_m^{(k)}(t)}\right)^{i}
\end{aligned}
$$

$$
\begin{aligned}
&= \textstyle\sum_{i=0}^{N-1} \tilde{c}_i \left(\sqrt{S_m^{(k)}(t)}\right)^i \leq \sum_{i=0}^{N-1} \tilde{c}_i \left[1 + \left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] \\
&= \bar{c}_M \left[1 + \left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right],
\end{aligned} \tag{3.34}
$$

where $\gamma_i^*(M,\rho)$, $1 \leq i \leq N-1$, $\tilde{c}_j$, $0 \leq j \leq N-1$, $\bar{c}_M$ are defined by (3.30).

Now, we can estimate the intergal $\int_0^t \left\| F_{mx}^{(k)}(s) \right\|_0^2 ds$. Using the property of $F_{mx}^{(k)}(t)$ as in (3.29), we obtain

$$
\begin{aligned}
\textstyle\int_0^t \left\| F_{mx}^{(k)}(s) \right\|_0^2 ds &\leq \bar{c}_M^2 \textstyle\int_0^t \left[1 + \left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]^2 ds \\
&\leq 2\bar{c}_M^2 \textstyle\int_0^t \left[1 + \left(S_m^{(k)}(s)\right)^{N-1}\right] ds \\
&\leq 4\bar{c}_M^2 \left[T + \textstyle\int_0^t \left(S_m^{(k)}(s)\right)^{N_1} ds\right],
\end{aligned} \tag{3.35}
$$

since $N_1 = \max\{2, p+1, N-1\} \geq N-1$. Combining (3.28) and (3.35), it gives

$$
S_m^{(k)}(t) \leq S_m^{(k)}(0) + TC_1(M) + C_1(M) \textstyle\int_0^t \left(S_m^{(k)}(s)\right)^{N_1} ds, \tag{3.36}
$$

in which

$$
C_1(M) = 2 + \tilde{\beta}_1 + \tilde{\beta}_2 + 4\bar{c}_M^2 \left[\tfrac{3(1-\rho)^2}{2\rho} + C_1 \left(1 + \tfrac{(1-\rho)^2}{2\rho}\right)\right]. \tag{3.37}
$$

By means of the convergences (2.5), there exists a constant $M > 0$ independent of k and m such that, for all $m, k \in \mathbb{N}$,

$$
\begin{aligned}
&S_m^{(k)}(0) \\
&= \|u_{1k}\|_0^2 + a\left(u_{1k}, u_{1k}\right) + \mu\left(\|u_{0k}\|_0^2\right)\left[a(u_{0k}, u_{0k}) + \|Au_{0k}\|_0^2\right] \\
&\leq \tfrac{M^2}{4}.
\end{aligned} \tag{3.38}
$$

Finally, it follows from (3.36), (3.38) that

$$
S_m^{(k)}(t) \leq \tfrac{M^2}{4} + TC_1(M) + C_1(M) \textstyle\int_0^t \left(S_m^{(k)}(s)\right)^{N_1} ds, \tag{3.39}
$$

for $0 \leq t \leq T_m^{(k)} \leq T$.

Then by solving a nonlinear Volterra integral inequality (3.39) (based on the methods in [7]), the following lemma is proved.

Lemma 3.2. *There exists a constant $T > 0$ independent of k and m such that*

$$
S_m^{(k)}(t) \leq M^2, \ \forall\, t \in [0,T], \quad \textit{for all } k \textit{ and } m \in \mathbb{N}. \tag{3.40}
$$

By Lemma 3.2, we can take constant $T_m^{(k)} = T$ for all m and k. Therefore, we have

$$u_m^{(k)} \in W(M,T), \quad \text{for all } m \quad \text{and} \quad k \in \mathbb{N}. \tag{3.41}$$

Step 3. *Convergence.* Thanks to (3.41), there exists a subsequence $\{u_m^{(k_j)}\}$ of $\{u_m^{(k)}\}$ such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;V) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(Q_T) \text{ weakly}, \\ u_m \in W(M,T). \end{cases} \tag{3.42}$$

By the compactness lemma of Lions ([6], p.57) and applying the theorem's Fischer-Riesz, from (3.42), one has a subsequence of $\{u_m^{(k)}\}$, denoted by the same symbol satisfying

$$\begin{cases} u_m^{(k)} \to u_m & \text{strongly in} \quad L^2(0,T;V) & \text{and a.e. in } Q_T, \\ \dot{u}_m^{(k)} \to u_m' & \text{strongly in} \quad L^2(Q_T) & \text{and a.e. in } Q_T. \end{cases} \tag{3.43}$$

On the other hand, using the inequality

$$\left|a^j - b^j\right| \le jM_1^{j-1}\left|a-b\right|, \ \forall\, a,b \in [-M_1, M_1], \ \forall\, M_1 > 0, \ \forall\, j \in \mathbb{N}, \tag{3.44}$$

we deduce from (3.41) and $(3.42)_4$, that

$$\left|(u_m^{(k)})^j - (u_m)^j\right| \le j\left(\sqrt{\frac{1-\rho}{\rho}}M\right)^{j-1}\left|u_m^{(k)} - u_m\right|, \ j = \overline{1, N-1}. \tag{3.45}$$

Thus

$$\begin{aligned} &\left\|(u_m^{(k)})^j - (u_m)^j\right\|_{L^2(Q_T)} \\ &\le j\left(\sqrt{\frac{1-\rho}{\rho}}M\right)^{j-1}\left\|u_m^{(k)} - u_m\right\|_{L^2(Q_T)}, \quad j = \overline{1, N-1}. \end{aligned} \tag{3.46}$$

Therefore, (3.43) and (3.46) give

$$(u_m^{(k)})^j \to (u_m)^j \quad \text{strongly in} \quad L^2(Q_T). \tag{3.47}$$

We note that

$$\begin{aligned} &\left\|F_m^{(k)} - \bar{F}_m\right\|_{L^2(Q_T)} \\ &\le \textstyle\sum_{j=0}^{N-1}\left\|A_j(\cdot,\cdot,u_{m-1})\right\|_{L^\infty(Q_T)}\left\|(u_m^{(k)})^j - (u_m)^j\right\|_{L^2(Q_T)}, \end{aligned} \tag{3.48}$$

so (3.43) leads to

$$F_m^{(k)} \to \bar{F}_m \quad \text{strongly in} \quad L^2(Q_T). \tag{3.49}$$

On the other hand, we have

$$\begin{aligned}\left|\mu_m^{(k)}(t) - \bar{\mu}_m(t)\right| &= \left|\mu\left(\left\|u_m^{(k)}(t)\right\|_0^2\right) - \mu\left(\|u_m(t)\|_0^2\right)\right| \\ &\leq 2M\tilde{K}_M(\mu)\left\|u_m^{(k)}(t) - u_m(t)\right\|_0 .\end{aligned} \tag{3.50}$$

Hence, from (3.43) and (3.50), we obtain

$$\mu_m^{(k)} \to \bar{\mu}_m \text{ strongly in } L^2(0,T). \tag{3.51}$$

Passing to limit in (3.8), (3.9), we have u_m satisfying (3.5), (3.6) in $L^2(0,T)$.

On the other hand, it follows from $(3.5)_1$ and $(3.42)_4$ that

$$u_m'' = -\bar{\mu}_m(t)\, Au_m + \bar{F}_m \in L^\infty(0,T;L^2). \tag{3.52}$$

Therefore, $u_m \in W_1(M,T)$ and Theorem 3.1 is proved. □

Next, in order to obtain the main result in this section, we put

$$W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\},$$

then $W_1(T)$ is a Banach space with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;L^2)} .$$

Theorem 3.3. *Let* (H_1)-(H_3) *hold. Then, there exist constants* $M > 0$ *and* $T > 0$ *such that*

(i) (1.1)-(1.3) *has a unique weak solution* $u \in W_1(M,T)$.

(ii) *The recurrent sequence* $\{u_m\}$, *defined by* (3.5) *and* (3.6), *converges at a rate of order* N *to the solution* u *strongly in the space* $W_1(T)$ *in the sense*

$$\|u_m - u\|_{W_1(T)} \leq C\,\|u_{m-1} - u\|_{W_1(T)}^N , \tag{3.53}$$

for all $m \geq 1$, *where* C *is a suitable constant. On the other hand, the estimate is fulfilled*

$$\|u_m - u\|_{W_1(T)} \leq C_T\beta_T^{N^m}, \quad \text{for all } m \in \mathbb{N}, \tag{3.54}$$

in which C_T *and* $0 < \beta_T < 1$ *are the constants depending only on* T.

Proof. Existence. We can prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$.

Indeed, let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), v\rangle + \bar{\mu}_{m+1}(t)\, a(w_m(t), v) + [\bar{\mu}_{m+1}(t) - \bar{\mu}_m(t)]\, \langle Au_m(t), v\rangle \\ = \left\langle \bar{F}_{m+1}(t) - \bar{F}_m(t), v\right\rangle, \quad \forall\, v \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.55}$$

Taking $v = w_m'$ in $(3.55)_1$, after integrating in t, we get

$$\begin{aligned} Z_m(t) = &\int_0^t \bar{\mu}_{m+1}'(s)\, a(w_m(s), w_m(s))ds \\ &-2\int_0^t [\bar{\mu}_{m+1}(s) - \bar{\mu}_m(s)] \langle Au_m(s), w_m'(s)\rangle ds \\ &+2\int_0^t \left\langle \bar{F}_{m+1}(s) - \bar{F}_m(s), w_m'(s)\right\rangle ds \\ \equiv\, &J_1 + J_2 + J_3, \end{aligned} \tag{3.56}$$

where

$$\begin{aligned} Z_m(t) &= \|w_m'(t)\|_0^2 + \bar{\mu}_{m+1}(t)\, a(w_m(t), w_m(t)) \\ &\geq \|w_m'(t)\|_0^2 + \mu_* a(w_m(t), w_m(t)) \\ &\geq \|w_m'(t)\|_0^2 + \mu_* C_0 \|w_m(t)\|_1^2 \\ &\geq 2\sqrt{\mu_* C_0}\, \|w_m'(t)\|_0 \|w_m(t)\|_1, \end{aligned} \tag{3.57}$$

and all integrals on the right – hand side of (3.56) are estimated as follows.

Estimating J_1. It follows from $(3.42)_4$ that

$$\begin{aligned} |\bar{\mu}_m'(t)| &= 2\left|\mu'\left(\|u_m(t)\|_0^2\right)\right| |\langle u_m(t), u_m'(t)\rangle| \\ &\leq 2\tilde{K}_M(\mu) \|u_m(t)\|_0 \|u_m'(t)\|_0 \\ &\leq 2\tilde{K}_M(\mu) \|u_m(t)\|_1 \|u_m'(t)\|_0 \leq 2M^2\tilde{K}_M(\mu), \end{aligned} \tag{3.58}$$

this implies that

$$J_1 = \int_0^t \bar{\mu}_{m+1}'(s)\, a(w_m(s), w_m(s))ds \leq \frac{2}{\mu_*} M^2 \tilde{K}_M(\mu) \int_0^t Z_m(s)ds. \tag{3.59}$$

Estimating J_2.

$$\begin{aligned} |\bar{\mu}_{m+1}(t) - \bar{\mu}_m(t)| &= \left|\mu\left(\|u_{m+1}(t)\|_0^2\right) - \mu\left(\|u_m(t)\|_0^2\right)\right| \\ &\leq \tilde{K}_M(\mu) \left|\|u_{m+1}(t)\|_0^2 - \|u_m(t)\|_0^2\right| \\ &\leq 2M\tilde{K}_M(\mu) \|w_m(t)\|_0. \end{aligned} \tag{3.60}$$

Thus

$$\begin{aligned} J_2 &= -2\int_0^t [\bar{\mu}_{m+1}(s) - \bar{\mu}_m(s)] \langle Au_m(s), w_m'(s)\rangle ds \\ &\leq 4M\tilde{K}_M(\mu) \int_0^t \|w_m(s)\|_0 \|Au_m(s)\|_0 \|w_m'(s)\|_0\, ds \\ &\leq \frac{4}{\mu_*} M^2\tilde{K}_M(\mu) \int_0^t \|w_m(s)\|_1 \|w_m'(s)\|_0\, ds \\ &\leq \frac{4}{\mu_*} M^2\tilde{K}_M(\mu) \int_0^t \frac{Z_m(s)}{2\sqrt{\mu_* C_0}} ds = \frac{2}{\sqrt{\mu_*^3 C_0}} M^2 \tilde{K}_M(\mu) \int_0^t Z_m(s)ds. \end{aligned} \tag{3.61}$$

Estimating J_3. Using Taylor's expansion of the function

$$f(x, t, u_m) = f(x, t, u_{m-1} + w_{m-1})$$

around the point u_{m-1} up to order N, we obtain

$$\begin{aligned} & f(x,t,u_m) - f(x,t,u_{m-1}) \\ & = \sum_{i=1}^{N-1} \tfrac{1}{i!} D_3^i f(x,t,u_{m-1}) w_{m-1}^i + \tfrac{1}{N!} D_3^N f(x,t,\tilde{\lambda}_m) w_{m-1}^N, \end{aligned} \tag{3.62}$$

where $\tilde{\lambda}_m = \tilde{\lambda}_m(x,t) = u_{m-1} + \theta_1 (u_m - u_{m-1})$, $0 < \theta_1 < 1$. Hence, it follows from (3.6) and (3.62) that

$$\begin{aligned} & \bar{F}_{m+1}(x,t) - \bar{F}_m(x,t) \\ & = \sum_{i=1}^{N-1} \tfrac{1}{i!} D_3^i f(x,t,u_m) w_m^i + \tfrac{1}{N!} D_3^N f(x,t,\tilde{\lambda}_m) w_{m-1}^N. \end{aligned} \tag{3.63}$$

It implies that

$$\begin{aligned} & \left|\bar{F}_{m+1}(x,t) - \bar{F}_m(x,t)\right| \\ & \leq \bar{K}_M(f) \sum_{i=1}^{N-1} \tfrac{1}{i!} \left|w_m^i\right| + \tfrac{1}{N!} \bar{K}_M(f) \left|w_{m-1}^N\right| \\ & \leq \bar{K}_M(f) \sum_{i=1}^{N-1} \tfrac{1}{i!} \left(\sqrt{\frac{1-\rho}{\rho}} \left\|w_{mx}(t)\right\|_0\right)^i \\ & \quad + \tfrac{1}{N!} \bar{K}_M(f) \left(\sqrt{\frac{1-\rho}{\rho}} \left\|w_{m-1,\, x}(t)\right\|_0\right)^N \\ & = \bar{K}_M(f) \sum_{i=1}^{N-1} \tfrac{1}{i!} \left(\sqrt{\frac{1-\rho}{\rho}}\right)^i \left\|w_{mx}(t)\right\|_0^{i-1} \left\|w_{mx}(t)\right\|_0 \\ & \quad + \tfrac{1}{N!} \bar{K}_M(f) \left(\sqrt{\frac{1-\rho}{\rho}}\right)^N ||w_{m-1}||_{W_1(T)}^N \\ & \leq \bar{K}_M(f) \sum_{i=1}^{N-1} \tfrac{1}{i!} \left(\sqrt{\frac{1-\rho}{\rho}}\right)^i M^{i-1} \frac{1}{\sqrt{\mu_*}} \sqrt{Z_m(t)} \\ & \quad + \tfrac{1}{N!} \bar{K}_M(f) \left(\sqrt{\frac{1-\rho}{\rho}}\right)^N ||w_{m-1}||_{W_1(T)}^N. \end{aligned} \tag{3.64}$$

Hence

$$\begin{aligned} & \left\|\bar{F}_{m+1}(t) - \bar{F}_m(t)\right\|_0 \\ & \leq \sqrt{\frac{1-\rho^2}{2}} \bar{K}_M(f) \sum_{i=1}^{N-1} \tfrac{1}{i!} \left(\sqrt{\frac{1-\rho}{\rho}}\right)^i M^{i-1} \frac{1}{\sqrt{\mu_*}} \sqrt{Z_m(t)} \\ & \quad + \sqrt{\frac{1-\rho^2}{2}} \tfrac{1}{N!} \bar{K}_M(f) \left(\sqrt{\frac{1-\rho}{\rho}}\right)^N \left\|w_{m-1}\right\|_{W_1(T)}^N \\ & = \zeta_T^{(1)} \sqrt{Z_m(t)} + \zeta_T^{(2)} \left\|w_{m-1}\right\|_{W_1(T)}^N, \end{aligned} \tag{3.65}$$

where

$$\begin{aligned} \zeta_T^{(1)} &= \frac{1}{\sqrt{\mu_*}} \bar{K}_M(f) \sqrt{\frac{1-\rho^2}{2}} \textstyle\sum_{i=1}^{N-1} \frac{1}{i!} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^i M^{i-1}, \\ \zeta_T^{(2)} &= \tfrac{1}{N!} \bar{K}_M(f) \sqrt{\frac{1-\rho^2}{2}} \left(\sqrt{\frac{1-\rho}{\rho}} \right)^N . \end{aligned} \tag{3.66}$$

It leads to

$$\begin{aligned} J_3 &= 2 \textstyle\int_0^t \left\langle \bar{F}_{m+1}(s) - \bar{F}_m(s), w_m'(s) \right\rangle ds \\ &\leq 2 \textstyle\int_0^t \left\| \bar{F}_{m+1}(s) - \bar{F}_m(s) \right\|_0 \left\| w_m'(s) \right\|_0 ds \\ &\leq 2 \textstyle\int_0^t \left(\zeta_T^{(1)} \sqrt{Z_m(s)} + \zeta_T^{(2)} \left\| w_{m-1} \right\|_{W_1(T)}^N \right) \sqrt{Z_m(s)} ds \\ &\leq 2 \left(\zeta_T^{(1)} + \zeta_T^{(2)} \right) \textstyle\int_0^t Z_m(s) ds + \frac{1}{2} T \zeta_T^{(2)} \left\| w_{m-1} \right\|_{W_1(T)}^{2N} . \end{aligned} \tag{3.67}$$

Then we deduce from (3.56), (3.59), (3.61) and (3.67) that

$$\begin{aligned} Z_m(t) \leq & \frac{1}{2} T \zeta_T^{(2)} \left\| w_{m-1} \right\|_{W_1(T)}^{2N} \\ & + 2 \left[\frac{1}{\mu_*} \left(1 + \frac{1}{\sqrt{\mu_* C_0}} \right) M^2 \tilde{K}_M(\mu) + \zeta_T^{(1)} + \zeta_T^{(2)} \right] \textstyle\int_0^t Z_m(s) ds. \end{aligned} \tag{3.68}$$

By using Gronwall's lemma, (3.68) yields

$$\left\| w_m \right\|_{W_1(T)} \leq \mu_T \left\| w_{m-1} \right\|_{W_1(T)}^N , \tag{3.69}$$

where

$$\mu_T = \left(1 + \frac{1}{\sqrt{\mu_* C_0}} \right) \sqrt{\frac{1}{2} T \zeta_T^{(2)}} \exp \left[T \left(\frac{1}{\mu_*} \left(1 + \frac{1}{\sqrt{\mu_* C_0}} \right) M^2 \tilde{K}_M(\mu) + \zeta_T^{(1)} + \zeta_T^{(2)} \right) \right] .$$

Then, it follows from (3.69) that, for all m and p,

$$||u_m - u_{m+p}||_{W_1(T)} \leq (1 - \beta_T)^{-1} (\mu_T)^{\frac{-1}{N-1}} \beta_T^{N^m} . \tag{3.70}$$

Choosing T small enough such that $\beta_T = M \mu_T^{\frac{1}{N-1}} < 1$. It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \quad \text{strongly in} \quad W_1(T). \tag{3.71}$$

Note that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T; V \cap H^2) \quad \text{weakly*}, \\ u_{m_j}' \to u' & \text{in} \quad L^\infty(0,T; V) \quad \text{weakly*}, \\ u_{m_j}'' \to u'' & \text{in} \quad L^2(Q_T) \quad \text{weakly}, \\ u \in W(M,T). & \end{cases} \tag{3.72}$$

We have

$$\begin{aligned} &\left\|\bar{F}_m(\cdot,t) - f(\cdot,t,u(t))\right\|_0 \\ &\leq \|f(\cdot,t,u_{m-1}) - f(\cdot,t,u(t))\|_0 \\ &\quad + \left\|\sum_{i=1}^{N-1} \tfrac{1}{i!}\tfrac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i\right\|_0 \\ &\leq \bar{K}_M(f)\,\|u_{m-1} - u\|_{W_1(T)} \\ &\quad + \bar{K}_M(f)\sum_{i=1}^{N-1} \tfrac{1}{i!}\sqrt{\frac{1-\rho^2}{2}}\left(\sqrt{\frac{1-\rho}{\rho}}\right)^i \|u_m - u_{m-1}\|^i_{W_1(T)}. \end{aligned} \tag{3.73}$$

Hence, it implies from (3.71) and (3.73) that

$$\bar{F}_m \to f(\cdot,t,u(t)) \quad \text{strongly in} \quad L^\infty(0,T;L^2). \tag{3.74}$$

Furthermore, we have

$$\begin{aligned} \left|\bar{\mu}_m(t) - \mu\left(\|u(t)\|_0^2\right)\right| &\leq 2M\tilde{K}_M(\mu)\,\|u_m(t) - u(t)\|_0 \\ &\leq 2M\tilde{K}_M(\mu)\,\|u_m - u\|_{W_1(T)}. \end{aligned} \tag{3.75}$$

Hence, from (3.71) and (3.75), we obtain

$$\bar{\mu}_m(t) \to \mu\left(\|u(t)\|_0^2\right) \quad \text{strongly in} \quad L^\infty(0,T). \tag{3.76}$$

Finally, passing to limit in (3.5), (3.6) as $m = m_j \to \infty$, there exists $u \in W(M,T)$ satisfying the equation

$$\langle u''(t), v\rangle + \mu\left(\|u(t)\|_0^2\right) a(u(t),v) = \langle f(\cdot,t,u(t)), v\rangle, \tag{3.77}$$

for all $v \in V$ and the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \tag{3.78}$$

Uniqueness. Applying a similar argument used in the proof of Theorem 3.1, $u \in W_1(M,T)$ is a unique local weak solution of (1.1)–(1.3).

Passing to the limit in (3.70) as $p \to +\infty$ for fixed m, we get (3.54). Also with a similar argument, (3.53) follows. Theorem 3.3 is proved completely. □

Remark 3.4. In order to construct a N-order iterative scheme, we need the condition (H_3). Then, we get a convergent sequence at a rate of order N to a local unique weak solution of the problem and the existence follows. This condition of f can be relaxed if we only consider the existence of solutions, see [8], [16].

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