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EXISTENCE AND APPROXIMATION OF SOLUTIONS FOR GENERALIZED QUADRATIC FRACTIONAL INTEGRAL EQUATIONS

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Abstract. In this paper we prove the existence as well as approximation of the solutions for a nonlinear generalized quadratic fractional integral equation of mixed type. An algorithm for the solutions is developed and it is shown that the sequence of successive approximations starting with a lower or an upper solution converges monotonically to the solution of the related quadratic fractional integral equation under some suitable mixed hybrid conditions. The existence of minimal and maximal solutions and the related integral inequalities are also proved under certain monotonic conditions. We base our main results on the Dhage iteration principle embodied in a recent hybrid fixed point theorem of Dhage (2014) in partially ordered normed linear spaces. A couple of examples are also provided to illustrate the hypotheses and abstract theory developed in the paper.

1. Introduction

The quadratic integral equations have been a topic of interest since long time because of their occurrence in the problems of some natural and physical processes of the universe. See Argyros [1], Deimling [4], Chandrasekher [2] and the references therein. The study gained momentum after the formulation of the hybrid fixed point principles in Banach algebras due to Dhage [5]–[8]. The existence results for such quadratic operators equations are generally proved under the mixed Lipschitz and compactness type conditions together with a

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certain growth condition on the nonlinearities involved in the quadratic operator or functional equations. The hybrid fixed point theorems in Banach algebras find numerous applications in the theory of nonlinear quadratic differential and integral equations. See Dhage [6, 7, 8, 14, 15], Dhage and Dhage [18, 19], Dhage *et.al.* [20] and the references therein. The Lipschitz and compactness hypotheses are considered to be very strong conditions in the theory of nonlinear differential and integral equations but nevertheless do not yield any algorithm to determine the numerical solutions. Therefore, it is of interest to relax or weaken these conditions in the existence and approximation theory of quadratic integral equations. This is the main motivation of the present paper. In this paper we prove the existence as well as approximations of the solutions of a certain generalized quadratic integral equation via an algorithm based on successive approximations under weak partial Lipschitz and compactness type conditions.

Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ for some $T > 0$, we consider the quadratic functional integral equation (in short QFIE)

$$\begin{aligned} x(t) &= k(t, x(t), x(\alpha(t))) \\ &\quad + \big[f(t, x(t), x(\beta(t)))\big]\left(\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s, x(s), x(\eta(s)))\, ds\right) \end{aligned} \tag{1.1}$$

for all $t \in J$, where the functions $k, f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\alpha, \beta, \eta : J \to J$ and $v : J \times J \to \mathbb{R}$ are continuous functions, $1 \leq q \leq 2$ and Γ is the Euler's Gamma function.

By a *solution* of the QFIE (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equation (1.1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The QFIE (1.1) is well-known in the literature and studied earlier in the works of Dhage [5], Dhage and Ntouyas [21], El-Sayed and Hashem [22, 23]. In particular, If $f(t, x, y) = 0$ for all $t \in J$ and $x, y \in \mathbb{R}$ the QFIE (1.1) reduces to the nonlinear functional equation

$$x(t) = k(t, x(t), x(\alpha(t))), \quad t \in J, \tag{1.2}$$

and if $k(t, x, y) = q(t)$ and $f(t, x, y) = 1$ for all $t \in J$ and $x, y \in \mathbb{R}$, it is reduced to nonlinear usual Volterra integral equation

$$x(t) = q(t) + \int_0^t v(t,s) g(s, x(s), x(\eta(s)))\, ds, \quad t \in J. \tag{1.3}$$

Again, if $f(t,x,y)=1$ for all $t\in J$ and $x,y\in\mathbb{R}$, then (1.1) reduces to the following well-known nonlinear functional integral equation

$$\begin{aligned} x(t) &= k(t,x(t),x(\alpha(t))) \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds, \quad t\in J. \end{aligned} \tag{1.4}$$

Next, if $k(t,x,y)=0$, $f(t,x,y)=f(t,x)$, $v(t,s)=1$ and $q=1$, then the QFIE (1.1) reduces to the following quadratic integral equation

$$x(t) = \big[f(t,x(t))\big]\left(\int_0^t g(s,x(s),x(\eta(s)))\,ds\right), \tag{1.5}$$

which is discussed in Dhage [5] via classical fixed point theory.

Finally, if $k(t,x,y)=k(t,x)+f(t,x)\,h(t)$, $f(t,x,y)=f(t,x)$, $g(t,x,y)=g(t,x)$ and $q=1$, then the QFIE (1.1) reduces to the following quadratic integral equation

$$x(t) = k(t,x(t)) + \big[f(t,x(t))\big]\left(q(t)+\int_0^t v(t,s)g(s,x(s))\,ds\right),\ t\in J. \tag{1.6}$$

The QFIE (1.6) has been discussed in Dhage [14] for the existence and approximation of the solutions under hybrid conditions via Dhage iteration method. Therefore, the QFIE (1.1) is more general and the existence theorem for which is proved in Dhage and Ntouyas [21] for the general case $q>0$ under usual classic Lipschitz and compactness type conditions via a hybrid fixed point theorem of Dhage [5]. In this paper we prove the existence as well as approximations of the solutions, integral inequalities, maximal and minimal solutions and comparison principle etc. for the QFIE (1.1) under weaker conditions which include the existence and approximation results for all the above nonlinear functional and functional Volterra integral equations as special cases under weak partial Lipschitz and partial compactness type conditions.

The rest of the paper is organized as follows: In the following Section 2 we give the preliminaries and auxiliary results needed in the subsequent part of the paper. The main existence and approximations results are given in Section 3. In Section 4, a result concerning the maximal and minimal solutions are proved for the considered nonlinear quadratic integral equation.

2. Auxiliary results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of a partially ordered normed linear space appear in Dhage [8], Heikkilä and Lakshmikantham [24] and the references therein.

Two elements x and y in E are said to be *comparable* if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a *chain* or *totally ordered* if all the elements of C are comparable. It is known that E is *regular* if $\{x_n\}$ is a nondecreasing (resp., nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp., $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [24] and the references therein.

We need the following definitions (see Dhage [6]–[10] and the references therein) in what follows.

Definition 2.1. A mapping $\mathcal{T} : E \to E$ is called *isotone* or *monotone nondecreasing* if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called *monotone nonincreasing* if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called *monotonic* or simply *monotone* if it is either monotone nondecreasing or monotone nonincreasing on E.

Definition 2.2. A mapping $\mathcal{T} : E \to E$ is called *partially continuous* at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3. A non-empty subset S of the partially ordered Banach space E is called *partially bounded* if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called *partially bounded* if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called *uniformly partially bounded* if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant.

Definition 2.4. A non-empty subset S of the partially ordered Banach space E is called *partially compact* if every chain C in S is a relatively compact subset of E. A mapping $\mathcal{T} : E \to E$ is called *partially compact* if $\mathcal{T}(E)$ is a partially relatively compact subset of E. $\mathcal{T}$ is called *uniformly partially compact* if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called *partially totally bounded* if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on E.

Remark 2.5. Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

Definition 2.6. The order relation $\preceq$ and the metric d on a non-empty set E are said to be *compatible* if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called *Janhavi* if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a *Janhavi metric* or *Janhavi Banach space*.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property. In general every finite dimensional Banach space with a standard norm and an order relation is a *Janhavi Banach space*.

Definition 2.7. ([7]) A upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called *partially nonlinear $\mathcal{D}$-Lipschitz* if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{2.2}$$

The elements of $\mathcal{K}$ are called the positive vectors of the normed linear algebra E. The following lemma follows immediately from the definition of the set $\mathcal{K}$ and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.8. ([8]) *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.9. An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.

The *Dhage iteration principle* may be described as "*the monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation*" and it is a powerful tool in the existence theory of nonlinear analysis. The procedure involved in the application of Dhage iteration principle to nonlinear equation is called the "*Dhage iteration method*." It is clear that Dhage iteration method is different for different nonlinear problems and also different from the usual Picard's successive iteration method. The Dhage iteration method embodied in the following applicable hybrid fixed point theorems of Dhage [10] is used as the key tool for our work contained in this paper. A few other hybrid fixed point theorems involving the Dhage iteration method may be found in Dhage [9]–[12].

Theorem 2.10. ([11]) *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ and $\mathcal{C} : E \to E$ be three nondecreasing operators such that*

- (a) *$\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ respectively,*
- (b) *$\mathcal{B}$ is partially continuous and uniformly partially compact, and*
- (c) *$M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$,*
  where $M = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and
- (d) *there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0 + \mathcal{C}x_0$.*

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x \tag{2.3}$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n + \mathcal{C}x_n$, $n = 0, 1, \cdots$, converges monotonically to x^*.

Remark 2.11. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact has been utilized to prove the main results of this paper.

Remark 2.12. The hypothesis (a) of Theorem 2.10 implies that the operators $\mathcal{A}$ and $\mathcal{C}$ are partially continuous and consequently all the three operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in the theorem are partially continuous on E. The regularity of E in above Theorem 2.10 may be replaced with a stronger continuity condition of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on E.

3. Existence and approximation result

The QFIE (1.1) is considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{3.1}$$

and

$$x \leq y \iff x(t) \leq y(t), \quad \forall\, t \in J, \tag{3.2}$$

respectively. Clearly, $C(J,\mathbb{R})$ is a Banach algebra with respect to the above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach algebra $C(J,\mathbb{R})$ has some nice properties concerning the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in certain subsets of it. The following lemma in this connection follows by an application of Arzelá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by* (3.1) *and* (3.2) *respectively. Then every partially compact subset S of $C(J,\mathbb{R})$ is Janhavi, i.e., $\|\cdot\|$ and $\leq$ are compatible in every compact chain C in S.*

Proof. The lemma mentioned in Dhage [9, 10], but the proof appears in Dhage and Dhage [16, 18, 19]. Since the proof is not well-known, we give the details of the proof. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots, \tag{3.3}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x point-wise on J. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.2. A function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the QFIE (1.1) if it satisfies

$$\begin{aligned} u(t) &\leq k(t,u(t),u(\alpha(t))) \\ &\quad + \big[f(t,u(t),u(\beta(t)))\big] \left(\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s,u(s),u(\eta(s)))\,ds\right) \end{aligned}$$

for all $t \in J$. Similarly, a function $v \in C(J,\mathbb{R})$ is said to be an upper solution of the QFIE (1.1) if it satisfies the above inequalities with reverse sign.

Definition 3.3. A function $g(t,x,y)$ is called Carathéodory if

(i) the map $t \mapsto g(t,x,y)$ is measurable for each $x,y \in \mathbb{R}$ and
(ii) the map $(x,y) \mapsto g(t,x,y)$ is jointly continuous for each $t \in J$.

A Caratheódory function g is called L^2-Carathéodory if

(iii) there exists a function $h \in L^2(J,\mathbb{R})$ such that

$$|g(t,x,y)| \leq h(t) \quad \text{a.e.} \quad t \in J$$

for all $x,y \in \mathbb{R}$.

We consider the following set of assumptions in what follows:

(A_1) The function f is nonnegative on $J \times \mathbb{R} \times \mathbb{R}$.
(A_2) There exists a $\mathcal{D}$-function ψ_f such that

$$0 \leq f(t,x_1,x_2) - f(t,y_1,y_2) \leq \psi_f(\max\{x_1-y_1, x_1-y_1\})$$

for all $t \in J$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$, $x_1 \geq y_1$, $x_2 \geq y_2$.
(A_3) There exists a constant $M_f > 0$ such that $0 \leq f(t,x,y) \leq M_f$ for all $t \in J$ and $x,y \in \mathbb{R}$.
(B_1) The function v is nonnegative on $J \times J$.
(B_2) g defines a L^2-Carathéodory function $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$.
(B_3) $g(t,x,y)$ is nondecreasing in x and y for all $t \in J$.
(C_1) There exists a $\mathcal{D}$-function ψ_k such that

$$0 \leq k(t,x_1,x_2) - k(t,y_1,y_2) \leq \psi_k(\max\{x_1-y_1, x_1-y_1\})$$

for all $t \in J$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$.
(C_2) There exists a constant $M_k > 0$ such that $|k(t,x,y)| \leq M_k$ for all $t \in J$ and $x \in \mathbb{R}$.
(C_3) The QFIE (1.1) has a lower solution $u \in C(J,\mathbb{R})$.

Theorem 3.4. *Assume that hypotheses* (A_1)-(A_3), (B_1)-(B_3) and (C_1)-(C_3) hold. Furthermore, assume that

$$\left(\frac{V\,T^{q-1}\,\|h\|_{L^2}}{\Gamma(q)}\right)\psi_f(r)+\psi_k(r)<r, \quad r>0, \tag{3.4}$$

then the QFIE (1.1) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of successive approximations defined by

$$\begin{aligned} &x_{n+1}(t) \\ &= k(t,x_n(t),x_n(\alpha(t))) \\ &\quad + \left[f(t,x_n(t),x_n(\beta(t)))\right]\left(\frac{1}{\Gamma(q)}\int_{t_0}^{t}\frac{v(t,s)}{(t-s)^{1-q}}\; g(s,x_n(s),x_n(\eta(s)))\,ds\right) \end{aligned} \tag{3.5}$$

for all $t\in J$, where $x_0=u$, converges monotonically to x^*.

Proof. Set $E=C(J,\mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define three operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on E by

$$\mathcal{A}x(t)=f(t,x(t),x(\alpha(t))), \quad t\in J, \tag{3.6}$$

$$\mathcal{B}x(t)=\frac{1}{\Gamma(q)}\int_0^t\frac{v(t,s)}{(t-s)^{1-q}}\; g(s,x(s),x(\eta(s)))\,ds, \quad t\in J, \tag{3.7}$$

and

$$\mathcal{C}x(t)=k(t,x(t),x(\alpha(t))), \quad t\in J. \tag{3.8}$$

From the continuity of the integral and the hypotheses (A_0)-(A_1) and (B_1), it follows that $\mathcal{A}$ and $\mathcal{B}$ define the maps $\mathcal{A},\mathcal{B}:E\to\mathcal{K}$. Now by definitions of the operators $\mathcal{A}$ and $\mathcal{B}$, the QFIE (1.1) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t)+\mathcal{C}x(t)=x(t), \;\; t\in J. \tag{3.9}$$

We shall show that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.10. This is achieved in the series of following steps.

Step I: *$\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are nondecreasing on E.* Let $x,y\in E$ be such that $x\geq y$. Then by hypothesis (A_2) and (C_2), we obtain

$$\mathcal{A}x(t)=f(t,x(t),x(\alpha(t)))\geq f(t,y(t),y(\alpha(t)))=\mathcal{A}y(t)$$

and

$$\mathcal{C}x(t)=k(t,x(t),x(\alpha(t)))\geq k(t,y(t),y(\alpha(t)))=\mathcal{C}y(t),$$

for all $t \in J$. This shows that $\mathcal{A}$ and $\mathcal{C}$ are nondecreasing operators on E into E. Similarly, using hypothesis (B_3),

$$\begin{aligned} \mathcal{B}x(t) &= \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} \; g(s, x(s), x(\eta(s)))\, ds \\ &\leq \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} \; g(s, y(s), y(\eta(s)))\, ds \\ &= \mathcal{B}y(t) \end{aligned}$$

for all $t \in J$. Hence, it is follows that the operator $\mathcal{B}$ is also a nondecreasing operator on E into itself. Thus, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are nondecreasing positive operators on E into itself.

Step II: *$\mathcal{A}$ and $\mathcal{C}$ are partially bounded and partially $\mathcal{D}$-Lipschitz on E.* Let $x \in E$ be arbitrary. Then by (A_2),

$$|\mathcal{A}x(t)| \leq \left| f(t, x(t), x(\alpha(t))) \right| \leq M_f$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E. Similarly, using hypothesis (C_1), it is shown that $\|\mathcal{C}x\| \leq M_k$ and consequently $\mathcal{C}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then, by hypothesis (A_3),

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= \left| f(t, x(t), x(\alpha(t))) - f(t, y(t), y(\alpha(t))) \right| \\ &\leq \psi_f(\max\{|x(t) - y(t)|\,,\, |x(\alpha(t)) - y(\alpha(t))|\}) \\ &\leq \psi_f(\|x - y\|) \end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \psi_f(\|x - y\|)$$

for all $x, y \in E$ with $x \geq y$. Similarly, by hypothesis (C_2),

$$\|\mathcal{C}x - \mathcal{C}y\| \leq \psi_k(\|x - y\|)$$

for all $x, y \in E$ with $x \geq y$. Hence $\mathcal{A}$ and $\mathcal{C}$ are partially nonlinear $\mathcal{D}$-Lipschitz operators on E which further implies that they are also a partially continuous on E into itself.

Step III: *$\mathcal{B}$ is a partially continuous operator on E.* Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\mathcal{B}x_n(t) &= \lim_{n\to\infty}\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\; g(s,x_n(s),x_n(\eta(s)))\,ds\\ &= \frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\;\Big[\lim_{n\to\infty} g(s,x_n(s),x_n(\eta(s))\Big]\,ds\\ &= \frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\; g(s,x(s),x(\eta(s))\,ds\\ &= \mathcal{B}x(t),\end{aligned}$$

for all $t\in J$. This shows that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1,t_2\in J$ be arbitrary with $t_1<t_2$. Then

$$\begin{aligned}&\Big|Bx_n(t_2)-Bx_n(t_1)\Big|\\ &= \Big|\frac{1}{\Gamma(q)}\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}\,g(s,x_n(s),x_n(\eta(s)))\,ds\\ &\qquad -\frac{1}{\Gamma(q)}\int_0^{t_1}\frac{v(t_1,s)}{(t_1-s)^{1-q}}\,g(s,x_n(s),x_n(\eta(s)))\,ds\Big|\\ &\le \Big|\frac{1}{\Gamma(q)}\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}\,g(s,x_n(s),x_n(\eta(s)))\,ds\\ &\qquad -\frac{1}{\Gamma(q)}\int_0^{t_2}\frac{v(t_1,s)}{(t_2-s)^{1-q}}\,g(s,x_n(s),x_n(\eta(s)))\,ds\Big|\\ &\quad +\Big|\frac{1}{\Gamma(q)}\int_0^{t_2}\frac{v(t_1,s)}{(t_2-s)^{1-q}}\,g(s,x_n(s),x_n(\eta(s)))\,ds\\ &\qquad -\frac{1}{\Gamma(q)}\int_0^{t_1}\frac{v(t_1,s)}{(t_2-s)^{1-q}}\,g(s,x_n(s),x_n(\eta(s)))\,ds\Big|\\ &\quad +\Big|\frac{1}{\Gamma(q)}\int_0^{t_1}\frac{v(t_1,s)}{(t_2-s)^{1-q}}\,g(s,x_n(s),x_n(\eta(s)))\,ds\\ &\qquad -\frac{1}{\Gamma(q)}\int_0^{t_1}\frac{v(t_1,s)}{(t_1-s)^{1-q}}\,g(s,x_n(s),x_n(\eta(s)))\,ds\Big|\\ &\le \frac{1}{\Gamma(q)}\int_0^{t_2}\frac{|v(t_2,s)-v(t_1,s)|}{(t_2-s)^{1-q}}\,|g(s,x_n(s),x_n(\eta(s)))|\,ds\\ &\quad +\frac{1}{\Gamma(q)}\Big|\int_{t_1}^{t_2}\frac{|v(t_1,s)|}{(t_2-s)^{1-q}}\,|g(s,x_n(s),x_n(\eta(s)))|\,ds\Big|\\ &\quad +\frac{1}{\Gamma(q)}\int_0^{t_1}|v(t_1,s)|\,|(t_2-s)^{q-1}-(t_1-s)^{q-1}|\,|g(s,x_n(s),x_n(\eta(s)))|\,ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(q)}\int_0^T \frac{|v(t_2,s)-v(t_1,s)|}{(t_2-s)^{1-q}}\, h(s)ds+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2} \frac{|v(t,s)|}{(t_2-s)^{1-q}}\, h(s)ds \\
&\quad + \frac{1}{\Gamma(q)}\int_0^T |v(t_1,s)|\left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right| \, h(s)\, ds \\
&\leq \frac{T^{q-1}}{\Gamma(q)}\left(\int_0^T |v(t_2,s)-v(t_1,s)|^2\, ds\right)^{1/2}\left(\int_0^T h^2(s)\, ds\right)^{1/2} \\
&\quad + \frac{V}{\Gamma(q)}\left(\int_0^T \left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|^2 ds\right)^{1/2}\left(\int_0^T h^2(s) ds\right)^{1/2} \\
&\quad + \frac{VT^{q-1}}{\Gamma(q)}|p(t_1)-p(t_2)| \qquad (3.10)
\end{aligned}
$$

Since the functions p is continuous on compact interval J and v and $(t-s)^{q-1}$ are continuous on compact set $J \times J$, they are uniformly continuous there. Therefore, from the above inequality (3.10) it follows that

$$|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad n \to \infty$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and hence $\mathcal{B}$ is partially continuous on E.

Step IV: *$\mathcal{B}$ is uniformly partially compact operator on E.* Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ be such that $y = \mathcal{B}x$. Now, by hypothesis (B_2),

$$
\begin{aligned}
|y(t)| &= \left|\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds\right| \\
&\leq \frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, |g(s,x(s),x(\eta(s)))|\, ds \\
&\leq \frac{V\, T^{q-1}\, \|h\|_{L^2}}{\Gamma(q)} \\
&= r
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t, we obtain $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathcal{B}(C)\| \leq r$ for all chains C in E. Hence, $\mathcal{B}$ is a uniformly partially bounded operator on E.

Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then, for any $y \in \mathcal{B}(C)$, one has

$$\begin{aligned}
&\Big|Bx(t_2) - Bx(t_1)\Big| \\
&= \Big|\frac{1}{\Gamma(q)}\int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds \\
&\qquad - \frac{1}{\Gamma(q)}\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds\Big| \\
&\le \Big|\frac{1}{\Gamma(q)}\int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds \\
&\qquad - \frac{1}{\Gamma(q)}\int_0^{t_2} \frac{v(t_1,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(q)}\int_0^{t_2} \frac{v(t_1,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds \\
&\qquad - \frac{1}{\Gamma(q)}\int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds\Big| \\
&\quad + \Big|\frac{1}{\Gamma(q)}\int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds \\
&\qquad - \frac{1}{\Gamma(q)}\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\,ds\Big| \\
&\le \frac{1}{\Gamma(q)}\int_0^{t_2} \frac{|v(t_2,s)-v(t_1,s)|}{(t_2-s)^{1-q}}\, |g(s,x(s),x(\eta(s)))|\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2} \frac{|v(t_1,s)|}{(t_2-s)^{1-q}}\, |g(s,x(s),x(\eta(s)))|\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{t_1} |v(t_1,s)|\,|[(t_2-s)^{q-1}-(t_1-s)^{q-1}]|\,|g(s,x(s),x(\eta(s)))|\,ds \\
&\le \frac{1}{\Gamma(q)}\int_0^{T} \frac{|v(t_2,s)-v(t_1,s)|}{(t_2-s)^{1-q}}\, h(s)\,ds + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2} \frac{|v(t,s)|}{(t_2-s)^{1-q}}\, h(s)\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{T} |v(t_1,s)|\,\big|\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]\big|\; h(s)\,ds \\
&\le \frac{1}{\Gamma(q)}\int_0^{T} \frac{|v(t_2,s)-v(t_1,s)|}{(t_2-s)^{1-q}}\, h(s)\,ds + \frac{1}{\Gamma(q)}\left|\int_{t_1}^{t_2} \frac{|v(t,s)|}{(t_2-s)^{1-q}}\, h(s)\,ds\right| \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{T} |v(t_1,s)|\,\big|(t_2-s)^{q-1}-(t_1-s)^{q-1}\big|\; h(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{T^{q-1}}{\Gamma(q)} \left(\int_0^T |v(t_2,s) - v(t_1,s)|^2 \, ds \right)^{1/2} \left(\int_0^T h^2(s)\, ds \right)^{1/2} \\
&\quad + \frac{V}{\Gamma(q)} \left(\int_0^T \left| (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right|^2 \, ds \right)^{1/2} \left(\int_0^T h^2(s)\, ds \right)^{1/2} \\
&\quad + \frac{VT^{q-1}}{\Gamma(q)} |p(t_1) - p(t_2)| \\
&\longrightarrow 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}
$$

uniformly for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step V: *u satisfies the operator inequality $u \leq \mathcal{A}u\,\mathcal{B}u + \mathcal{C}u$.* By hypothesis ($C_4$), the QFIE (1.1) has a lower solution u defined on J. Then, we have

$$
\begin{aligned}
&u(t) \\
&\leq k(t, u(t), u(\alpha(t))) \\
&\quad + \left[f(t, u(t), u(\beta(t))) \right] \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} \; g(s, u(s), u(\eta(s)))\, ds \right)
\end{aligned}
\tag{3.11}
$$

for all $t \in J$. From the definitions of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ it follows that $u(t) \leq \mathcal{A}u(t)\,\mathcal{B}u(t) + \mathcal{C}u(t)$ for all $t \in J$. Hence $u \leq \mathcal{A}u\,\mathcal{B}u + \mathcal{C}u$.

Step VI: *The $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ satisfy the growth condition $M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$ for $r > 0$.* Finally, the $\mathcal{D}$-function $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ of the operator $\mathcal{A}$ and $\mathcal{C}$ satisfy the inequality given in hypothesis (d) of Theorem 2.10, viz.,

$$
M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) \leq \left(\frac{V\,T^{q-1}\,\|h\|_{L^2}}{\Gamma(q)} \right) \psi_f(r) + \psi_k(r) < r
$$

for all $r > 0$.

Thus $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.10 and we conclude that the operator equation $\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x$ has a solution. Consequently the QFIE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof. □

The conclusion of Theorems 3.4 also remains true if we replace the hypothesis (C_3) with the following one:

(C_3') The QFIE (1.1) has an upper solution $v \in C(J, \mathbb{R})$.

The proof of Theorem 3.4 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Remark 3.5. We note that if the QFIE (1.1) has a lower solution u as well as an upper solution v such that $u \leq v$, then under the given conditions of Theorem 3.4 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \leq x^*$. Hence they are the minimal and maximal solutions of the QFIE (1.1) in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set in $C^1(J, \mathbb{R})$ defined by

$$[u, v] = \{x \in C(J, \mathbb{R}) \mid u \leq x \leq v\}.$$

This is because the order relation $\leq$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \geq \theta\}$ which is a closed set in $C(J, \mathbb{R})$.

Remark 3.6. If the function k is nonnegative on $J \times \mathbb{R} \times \mathbb{R}$ in Theorem 3.4, then the QFIE (1.1) has a positive solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (3.5) converges to x^*.

4. Maximal and minimal solutions

We need the following definition in what follows.

Definition 4.1. A function $r \in C(J, \mathbb{R})$ is said be a maximal solution of the QFIE (1.1) if for any other solution x of the QFIE (1.1), one has $x(t) \leq r(t)$ for all $t \in J$. Similarly, a minimal solution ρ of the QFIE (1.1) can be defined in a similar way by reversing the above inequality.

The following lemma is fundamental in the proof of minimal and maximal solutions for the QFIE (1.1) on J.

Lemma 4.2. *Assume that hypotheses* (A_1)-(A_1), (B_1), (B_3) *and* (C_1) *hold. Suppose that there exist two functions* $y, z \in C(J, \mathbb{R})$ *satisfying*

$$\begin{aligned} y(t) &\leq k(t, y(t), y(\alpha(t))) \\ &\quad + \big[f(t, y(t), y(\beta(t)))\big]\left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s, y(s), y(\eta(s)))\, ds\right) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} z(t) &\geq k(t, z(t), z(\alpha(t))) \\ &\quad + \big[f(t, z(t), z(\beta(t)))\big]\left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s, z(s), z(\eta(s)))\, ds\right) \end{aligned} \tag{4.2}$$

for all $t \in J$. *If one of the inequalities* (4.1) *and* (4.2) *is strict, then*

$$y(t) < z(t) \tag{4.3}$$

for all $t \in J$.

Proof. Suppose that the inequality (4.2) is strict and let the conclusion (4.3) be false. Then there exists $t_1 \in J$ such that

$$y(t_1) = z(t_1), \quad t_1 > 0$$

and

$$y(t) < z(t), \quad 0 < t < t_1.$$

From the monotonicity of $f(t, x, y)$, $g(t, x, y)$ and $k(t, x, y)$ in x and y , we get

$$\begin{aligned} & y(t_1) \\ & \leq k(t_1, y(t_1), y(\alpha(t_1))) \\ & \quad + \big[f(t_1, y(t_1), y(\beta(t_1)))\big] \left(\frac{1}{\Gamma(q)} \int_0^{t_1} \frac{v(t_1, s)}{(t_1 - s)^{1-q}} \, g(s, y(s), y(\eta(s))) \, ds \right) \\ & = k(t_1, z(t_1), z(\alpha(t_2))) \\ & \quad + \big[f(t_1, z(t_1), z(\beta(t_1)))\big] \left(\frac{1}{\Gamma(q)} \int_0^{t_1} \frac{v(t_1, s)}{(t_1 - s)^{1-q}} \, g(s, z(s), z(\eta(s))) \, ds \right) \\ & < z(t_1), \end{aligned} \tag{4.4}$$

which contradicts the fact that $y(t_1) = z(t_1)$. Hence, $y(t) < z(t)$ for all $t \in J$. □

Theorem 4.3. *Suppose that all the hypotheses of Theorem* 3.4 *hold. Then the QFIE* (1.1) *has a minimal and a maximal solution on* J.

Proof. We shall prove the case of maximal solution only, because the case of minimal solution is similar and can be obtained with appropriate modifications. Let $\epsilon > 0$ be given. Now consider the quadratic fractional integral equation

$$\begin{aligned} & x_\epsilon(t) \\ & = k_\epsilon(t, x_\epsilon(t), x_\epsilon(\alpha(t))) \\ & \quad + \big[f_\epsilon(t, x_\epsilon(t), x_\epsilon(\beta(t)))\big]\left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t, s)}{(t-s)^{1-q}} \, g_\epsilon(s, x_\epsilon(s), x_\epsilon(\eta(s))) \, ds \right) \end{aligned} \tag{4.5}$$

for all $t \in J$, where

$$f_\epsilon(t, x_\epsilon(t), x_\epsilon(\beta(t))) = f(t, x_\epsilon(t), x_\epsilon(\beta(t))) + \epsilon,$$

$$g_\epsilon(s, x_\epsilon(s), x_\epsilon(\eta(s))) = g(s, x_\epsilon(s), x_\epsilon(\eta(s))) + \epsilon$$

and

$$k_\epsilon(t, x_\epsilon(t), x_\epsilon(\alpha(t))) = k(t, x_\epsilon(t), x_\epsilon(\alpha(t))) + \epsilon.$$

Clearly the functions $f_\epsilon(t, x_\epsilon, x_\epsilon(\beta))$, $g_\epsilon(t, x_\epsilon, x_\epsilon(\eta))$ and $k_\epsilon(t, x_\epsilon, x_\epsilon(\alpha))$ satisfy all the hypotheses (A_0)-(A_3), (B_1)-(B_4) and (C_1)-(C_3), and therefore, by Theorem 3.4, QFIE (4.5) has at least a solution $x_\epsilon(t) \in C(J, \mathbb{R})$.

Let ϵ_1 and ϵ_2 be two real numbers such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then, we have

$$\begin{aligned} x_{\epsilon_2}(t) &= k_{\epsilon_2}(t, x_{\epsilon_2}(t), x_{\epsilon_2}(\alpha(t))) \\ &\quad + \left[f_{\epsilon_2}(t, x_{\epsilon_2}(t), x_{\epsilon_2}(\beta(t)))\right] \\ &\quad \times \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g_{\epsilon_2}(s, x_{\epsilon_2}(s), x_{\epsilon_2}(\eta(s)))\, ds\right) \\ &= k(t, x_{\epsilon_2}(t), x_{\epsilon_2}(\alpha(t))) + \epsilon_2 \\ &\quad + \left[f(t, x_{\epsilon_2}(t), x_{\epsilon_2}(\beta(t))) + \epsilon_2\right] \\ &\quad \times \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} \left[g(s, x_{\epsilon_2}(s), x_{\epsilon_2}(\eta(s))) + \epsilon_2\right] ds\right) \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} x_{\epsilon_1}(t) &= k_{\epsilon_1}(t, x_{\epsilon_1}(t), x_{\epsilon_1}(\alpha(t))) \\ &\quad + \left[f_{\epsilon_1}(t, x_{\epsilon_1}(t), x_{\epsilon_1}(\beta(t)))\right] \\ &\quad \times \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g_{\epsilon_1}(s, x_{\epsilon_1}(s), x_{\epsilon_1}(\eta(s)))\, ds\right) \\ &= k(t, x_{\epsilon_1}(t), x_{\epsilon_1}(\alpha(t))) + \epsilon_1 \\ &\quad + \left[f(t, x_{\epsilon_1}(t), x_{\epsilon_1}(\beta(t))) + \epsilon_1\right] \\ &\quad \times \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} \left[g(s, x_{\epsilon_1}(s), x_{\epsilon_1}(\eta(s))) + \epsilon_1\right] ds\right) \\ &> k(t, x_{\epsilon_1}(t), x_{\epsilon_1}(\alpha(t))) + \epsilon_2 \\ &\quad + \left[f(t, x_{\epsilon_1}(t), x_{\epsilon_1}(\beta(t))) + \epsilon_2\right] \\ &\quad \times \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} \left[g(s, x_{\epsilon_1}(s), x_{\epsilon_1}(\eta(s))) + \epsilon_2\right] ds\right) \end{aligned} \tag{4.7}$$

for all $t \in J$. Now, applying the Lemma 4.2 to the inequalities (4.6) and (4.7), we obtain

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \tag{4.8}$$

for all $t \in J$.

Let $\epsilon_0 = \epsilon$ and define a decreasing sequence $\{\epsilon_n\}_{n=0}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then in view of the above facts $\{x_{\epsilon_n}\}$ is a decreasing sequence of functions in $C(J,\mathbb{R})$. We show that is is uniformly bounded and equicontinuous. Now, by hypotheses,

$$\begin{aligned} |x_{\epsilon_n}(t)| &\leq |k_{\epsilon_n}(t, x_{\epsilon_n}(t), x_{\epsilon_n}(\alpha(t))|) \\ &\quad + \big|\big[k_{\epsilon_n}(t, x_{\epsilon_n}(t), x_{\epsilon_n}(\alpha(t))|)\big]\big| \\ &\qquad \times \left(\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\ |g_{\epsilon_n}(s, x_{\epsilon_n}(s), x_{\epsilon_n}(\eta(s)))|\,ds\right) \\ &\leq M_k + \epsilon + (M_f + \epsilon)\left(\frac{V\,T^{q-1}\,(\|h\|_{L^2} + T\,\epsilon)}{\Gamma(q)}\right) \\ &= r \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|x_{\epsilon_n}\| \leq r$ for all $n \in \mathbb{N}$. This shows that the sequence $\{x_{\epsilon_n}\}$ is uniformly bounded.

Next we show that $\{x_{\epsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned} &|x_{\epsilon_n}(t_1) - x_{\epsilon_n}(t_2)| \\ &\leq |k(t_1, x_{\epsilon_n}(t_1), x_{\epsilon_n}(\alpha(t_1))) - k(t_2, x_{\epsilon_n}(t_2), x_{\epsilon_n}(\alpha(t_2)))| \\ &\quad + |f(t_1, x_{\epsilon_n}(t_1), x_{\epsilon_n}(\alpha(t_1))) - f(t_2, x_{\epsilon_n}(t_2), x_{\epsilon_n}(\alpha(t_2)))| \\ &\qquad \times \left(\frac{1}{\Gamma(q)}\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}}\ |g_{\epsilon_n}(s, x_{\epsilon_n}(s), x_{\epsilon_n}(\eta(s)))|\,ds\right) \\ &\quad + |f(t_2, x_{\epsilon_n}(t_2), x_{\epsilon_n}(\alpha(t_2)))| \\ &\qquad \times \frac{1}{\Gamma(q)}\bigg|\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}}\ |g_{\epsilon_n}(s, x_{\epsilon_n}(s), x_{\epsilon_n}(\eta(s)))|\,ds \\ &\qquad\qquad - \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}}\ |g_{\epsilon_n}(s, x_{\epsilon_n}(s), x_{\epsilon_n}(\eta(s)))|\,ds\bigg| \\ &\leq |k(t_1, x_{\epsilon_n}(t_1), x_{\epsilon_n}(\alpha(t_1))) - k(t_2, x_{\epsilon_n}(t_2), x_{\epsilon_n}(\alpha(t_2)))| \\ &\quad + \big|f(t_1, x_{\epsilon_n}(t_1), x_{\epsilon_n}(\alpha(t_1))) - f(t_2, x_{\epsilon_n}(t_2), x_{\epsilon_n}(\alpha(t_2)))\big| \\ &\qquad \times \left(\frac{V\,T^{q-1}\,(\|h\|_{L^2} + T\,\epsilon)}{\Gamma(q)}\right) \\ &\quad + M_f\,\frac{1}{\Gamma(q)}\bigg|\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}}\ g_{\epsilon_n}(s, x_{\epsilon_n}(s), x_{\epsilon_n}(\eta(s)))\,ds \\ &\qquad\qquad - \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}}\ g_{\epsilon_n}(s, x_{\epsilon_n}(s), x_{\epsilon_n}(\eta(s)))\,ds\bigg| \\ &\leq |k(t_1, x_{\epsilon_n}(t_1), x_{\epsilon_n}(\alpha(t_1))) - k(t_2, x_{\epsilon_n}(t_2), x_{\epsilon_n}(\alpha(t_2)))| \\ &\quad + \big|f(t_1, x_{\epsilon_n}(t_1), x_{\epsilon_n}(\alpha(t_1))) - f(t_2, x_{\epsilon_n}(t_2), x_{\epsilon_n}(\alpha(t_2)))\big| \\ &\qquad \times \left(\frac{V\,T^{q-1}\,(\|h\|_{L^2} + T\,\epsilon)}{\Gamma(q)}\right) \end{aligned}$$

$$
\begin{aligned}
&+ M_f\,\frac{1}{\Gamma(q)}\left|\int_{t_2}^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}}\;|g_{\epsilon_n}(s,x_{\epsilon_n}(s),x_{\epsilon_n}(\eta(s)))|\,ds\right| \\
&+ M_f\,\frac{1}{\Gamma(q)}\int_0^T \left|\frac{v(t_1,s)}{(t_1-s)^{1-q}} - \frac{v(t_2,s)}{(t_2-s)^{1-q}}\right|\;|g_{\epsilon_n}(s,x_{\epsilon_n}(s),x_{\epsilon_n}(\eta(s)))|\,ds \\
\le\; & |k(t_1,x_{\epsilon_n}(t_1),x_{\epsilon_n}(\alpha(t_1))) - k(t_2,x_{\epsilon_n}(t_2),x_{\epsilon_n}(\alpha(t_2)))| \\
&+ \big|f(t_1,x_{\epsilon_n}(t_1),x_{\epsilon_n}(\alpha(t_1))) - f(t_2,x_{\epsilon_n}(t_2),x_{\epsilon_n}(\alpha(t_2)))]\big| \\
&\quad\times\left(\frac{V\,T^{q-1}\,(\|h\|_{L^2} + T\,\epsilon)}{\Gamma(q)}\right) \\
&+ M_f\,\frac{T^{q-1}}{\Gamma(q)}\left(\int_0^T \left|\frac{v(t_1,s)}{(t_1-s)^{1-q}} - \frac{v(t_2,s)}{(t_2-s)^{1-q}}\right|^2 ds\right)^{1/2} \\
&\quad\times\left(\int_0^T h^2(s)\,ds\right)^{1/2} \\
&+ \frac{VT^{q-1}}{\Gamma(q)}|p(t_1)-p(t_2)|.
\end{aligned}
\tag{4.9}
$$

Since the functions k and f are continuous on compact $[0,T]\times[-r,r]\times[-r,r]$, $\frac{v(t,s)}{(t-s)^{1-q}}$ is continuous on compact $[0,T]\times[0,T]$ and p is continuous on compact $[0,T]$, they are uniformly continuous there. Hence, from (4.9) it follows that

$$
|x_{\epsilon_n}(t_1) - x_{\epsilon_n}(t_2)| \to 0 \quad \text{as} \;\; t_1 \to t_2
$$

uniformly for all $n \in \mathbb{N}$. As a result $\{x_{\epsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,\mathbb{R})$. Now the sequence $\{x_{\epsilon_n}\}$ is uniformly bounded and equicontinuous, so it is compact in view of Arzelá-Ascoli theorem. By Lemma 3.1, $\{x_{\epsilon_n}\}$ converges uniformly to a function, say $r \in C(J,\mathbb{R})$, that is, $\lim_{n\to\infty} x_{\epsilon_n}(t) = r(t)$ uniformly on J.

We show that the function r is a solution of the QFIE (1.1) on J. Now, $\{x_{\epsilon_n}\}$ is a solution of the QFIE

$$
\begin{aligned}
x_{\epsilon_n}(t) &= k_{\epsilon_n}(t,x_{\epsilon_n}(t),x_{\epsilon_n}(\alpha(t))) \\
&\quad + \big[f_{\epsilon_n}(t,x_{\epsilon_n}(t),x_{\epsilon_n}(\beta(t)))\big] \\
&\qquad \times\left(\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\; g_{\epsilon_n}(s,x_{\epsilon_n}(s),x_{\epsilon_n}(\eta(s)))\,ds\right) \\
&= k(t,x_{\epsilon_n}(t),x_{\epsilon_n}(\alpha(t))) + \epsilon_n \\
&\quad + \big[f(t,x_{\epsilon_n}(t),x_{\epsilon_n}(\beta(t))) + \epsilon_n\big] \\
&\qquad \times\left(\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\;\big[g(s,x_{\epsilon_n}(s),x_{\epsilon_n}(\eta(s))) + \epsilon_n\big]\,ds\right)
\end{aligned}
\tag{4.10}
$$

for all $t \in J$. Now, taking the limit as by hypotheses $n \to \infty$ in the above inequality (4.10), we obtain

$$\begin{aligned} r(t) &= k(t, r(t), r(\alpha(t))) \\ &\quad + \big[f(t, r(t), r(\beta(t)))\big] \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s, r(s), r(\eta(s)))\, ds \right) \end{aligned}$$

for all $t \in J$. This shows that r is a solution of the QFIE (1.1) defined on J.

Finally, we shall show that $r(t)$ is the maximal solution of the QFIE (1.1) defined on J. To do this, let $x(t)$ be any solution of the QFIE (1.1) defined on J. Then,we have

$$\begin{aligned} x(t) &= k(t, x(t), r(\alpha(t))) \\ &\quad + \big[f(t, x(t), x(\beta(t)))\big] \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s, x(s), x(\eta(s)))\, ds \right) \end{aligned} \tag{4.11}$$

for all $t \in J$. Similarly, if x_ϵ is any solution of the QFIE

$$\begin{aligned} x_\epsilon(t) &= k(t, x_\epsilon(t), x_\epsilon(\alpha(t))) + \epsilon \\ &\quad + \big[f(t, x_\epsilon(t), x_\epsilon(\beta(t))) + \epsilon\big] \\ &\qquad \times \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, \big[g(s, x_\epsilon(s), x_\epsilon(\eta(s))) + \epsilon\big]\, ds \right), \end{aligned} \tag{4.12}$$

then

$$\begin{aligned} x_\epsilon(t) &> k(t, x_\epsilon(t), x_\epsilon(\alpha(t))) \\ &\quad + \big[f(t, x_\epsilon(t), x_\epsilon(\beta(t)))\big] \\ &\qquad \times \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, \big[g(s, x_\epsilon(s), x_\epsilon(\eta(s)))\big]\, ds \right) \end{aligned} \tag{4.13}$$

for all $t \in J$. From the inequalities (4.11) and (4.13) it follows that $x(t) \leq x_\varepsilon(t),\ t \in J$. Taking the limit as $\epsilon \to 0$, we obtain $x(t) \leq r(t)$ for all $t \in J$. Hence r is a maximal solution of the QFIE (1.1) defined on J. This completes the proof. □

5. Comparison principle

The main problem of the integral inequalities is to estimate a bound for the solution set of the integral inequality related to the QFIE (1.1). In this section we prove that the maximal and minimal solutions serve as the bounds for the solutions of the related differential inequality to QFIE (1.1) on $J = [0, T]$.

Theorem 5.1. *Suppose that all the hypotheses of Theorem* 3.4 *hold. Further, if there exists a function* $u \in C(J, \mathbb{R})$ *such that*

$$\begin{aligned} u(t) &\leq k(t, u(t), u(\alpha(t))) \\ &\quad + \big[f(t, u(t), u(\beta(t)))\big]\left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\ g(s, u(s), u(\eta(s)))\, ds\right) \end{aligned} \tag{5.1}$$

for all $t \in J$*, then*

$$u(t) \leq r(t) \tag{5.2}$$

for all $t \in J$*, where* r *is a maximal solution of the QFIE* (1.1) *on* J*.*

Proof. Let $\epsilon > 0$ be arbitrary small. Then, by Theorem 3.4, $r_\epsilon(t)$ is a solution of the QFIE (4.5) and that the limit

$$r(t) = \lim_{\epsilon \to 0} r_\epsilon(t) \tag{5.3}$$

is uniform on J and is a maximal solution of the QFIE (1.1) on J. Hence, we obtain

$$\begin{aligned} r_\epsilon(t) &= k(t, r_\epsilon(t), r_\epsilon(\alpha(t))) + \epsilon \\ &\quad + \big[f(t, r_\epsilon(t), r_\epsilon(\beta(t))) + \epsilon\big] \\ &\qquad \times \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\ \big[g(s, r_\epsilon(s), r_\epsilon(\eta(s))) + \epsilon\big]\, ds\right) \end{aligned}$$

for all $t \in J$. From the above inequality it follows that

$$\begin{aligned} &r_\epsilon(t) \\ &> k(t, r_\epsilon(t), r_\epsilon(\alpha(t))) \\ &\quad + \big[f(t, r_\epsilon(t), r_\epsilon(\beta(t)))\big]\left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\ g(s, r_\epsilon(s), r_\epsilon(\eta(s)))\, ds\right) \end{aligned} \tag{5.4}$$

for all $t \in J$. Now we apply Lemma 4.2 to the inequalities (5.1) and (5.4) and conclude that

$$u(t) < r_\epsilon(t) \tag{5.5}$$

for all $t \in J$. This further in view of limit (5.3) implies that the inequality (5.2) holds on J. This completes the proof. □

Similarly, we have the following comparison result for the QFIE (1.1) on J.

Theorem 5.2. *Suppose that all the hypotheses of Theorem* 3.4 *hold. Further, if there exists a function* $v \in C(J, \mathbb{R})$ *such that*

$$\begin{aligned} v(t) &\geq k(t, v(t), v(\alpha(t))) \\ &\quad + \big[f(t, v(t), v(\beta(t)))\big]\left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\ g(s, v(s), v(\eta(s)))\, ds\right) \end{aligned} \tag{5.6}$$

for all $t \in J$, *then*

$$v(t) \geq \rho(t) \tag{5.7}$$

for all $t \in J$, *where* ρ *is a minimal solution of the QFIE* (1.1) *on* J.

Finally, we give a couple of examples to illustrate the hypotheses imposed on the nonlinearities and the main existence and approximation result proved in this paper.

Example 5.3. Given a closed and bounded interval $J = [0,1]$, consider the QFIE,

$$x(t) = \frac{1}{2}\big[2 + \arctan x(t)\big]\left(\frac{1}{\Gamma(3/2)}\int_0^t \frac{(t-s)^{1/2}}{t^2+1}\cdot\frac{[1+\tanh x(s)]}{4}\,ds\right) + \frac{1}{2}\arctan x(t) \tag{5.8}$$

for $t \in J$.

Here, $v(t,s) = \dfrac{1}{t^2+1}$ which is continuous and $V = 1$. Similarly, the functions k, f and g are defined by $k(t,x,y) = k(t,x) = \dfrac{1}{2}\arctan x$, $f(t,x,y) = f(t,x) = \dfrac{1}{2}\big[2 + \arctan x(t)\big]$ and $g(t,x,y) = g(t,x) = \dfrac{1+\tanh x}{4}$.

The function f satisfies the hypothesis (A_3) with $\psi_f(r) = \dfrac{1}{2}\cdot\dfrac{r}{1+\xi^2}$ for each $0 < \xi < r$. To see this, we have

$$0 \leq f(t,x_1,x_2) - f(t,y_1,y_2) \leq \frac{1}{2}\cdot\frac{1}{1+\xi^2}\cdot(x_1 - y_1)$$

for all $x_1, y_1 \in \mathbb{R}$, $x_1 \geq y_1$ and $x_1 > \xi > y_1$. Moreover, the function f is nonnegative and bounded on $J\times\mathbb{R}\times$ with bound $M_f = 2$ and so the hypothesis (A_2) is satisfied. Again, since g is nonnegative and bounded on $J\times\mathbb{R}\times\mathbb{R}$ with bound $\|h\|_{L^2} = \dfrac{1}{2}$, the hypothesis ($B_2$) holds. Furthermore, $g(t,x,y) = g(t,x)$ is nondecreasing in x and y for all $t \in J$, and thus hypothesis (B_3) is satisfied.

Similarly, the function k satisfies the hypothesis (C_2) with $\psi_k(r) = \dfrac{1}{2}\cdot\dfrac{r}{1+\xi^2}$ for every $0 < \xi < r$. To see this, we have

$$0 \leq k(t,x_1,x_2) - k(t,y_1,y_2) \leq \frac{1}{2}\cdot\frac{1}{1+\xi^2}\cdot(x_1 - y_1)$$

for all $x_1, y_1 \in \mathbb{R}$, $x_1 \geq y_1$ and $x_1 > \xi > y_1$. Moreover, the function k is bounded on $J\times\mathbb{R}$ with bound $M_k = \dfrac{\pi}{4}$ and so the hypothesis (C_1) is satisfied.

Also we have

$$\left(\frac{V\,T^q\,\|h\|_{L^2}}{\Gamma(q)}\right)\psi_f(r)+\psi_k(r)\le\frac{r}{1+\xi^2}<r$$

for every $r>0$. Thus, condition (3.4) of Theorem 3.4 is held. Finally, the QFIE (5.8) has a lower solution $u(t)=0$ on J. Thus all the hypotheses of Theorem 3.4 are satisfied. Hence we apply Theorem 3.4 and conclude that the QFIE (5.8) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by

$$\begin{aligned} x_{n+1}(t)&=\frac{1}{2}\big[2+\arctan x_n(t)\big]\left(\int_0^t\frac{(t-s)^{1/2}}{t^2+1}\cdot\frac{[1+\tanh x_n(s)]}{4}\,ds\right)\\ &\quad+\frac{1}{2}\arctan x_n(t), \end{aligned} \tag{5.9}$$

for all $t\in J$, where $x_0=0$, converges monotonically to x^*.

Example 5.4. Given a closed and bounded interval $J=[0,1]$, consider the QFIE,

$$\begin{aligned} x(t)&=\frac{1}{2}\big[2+\arctan x(t)\big]\left(\frac{1}{\Gamma(3/2)}\int_0^t\frac{(t-s)^{1/2}}{t^2+1}\cdot\frac{[1+\tanh x(s)]}{4}\,ds\right)\\ &\quad+\arctan x(t)+1 \end{aligned} \tag{5.10}$$

for $t\in J$.

Now following the arguments similar to those given in Example 5.3 it is proved that the nonlinear quadratic fractional integral equation (5.10) has a positive x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by

$$\begin{aligned} x_{n+1}(t)&=\frac{1}{2}\big[2+\arctan x_n(t)\big]\left(\int_0^t\frac{(t-s)^{1/2}}{t^2+1}\cdot\frac{[1+\tanh x_n(s)]}{4}\,ds\right)\\ &\quad+\arctan x_n(t)+1, \end{aligned} \tag{5.11}$$

for all $t\in J$, where $x_0=0$, converges monotonically to x^*.

Remark 5.5. The conclusion of Examples 5.3 and 5.4 also remains true if we replace the lower solution u of the nonlinear quadratic fractional integral equations with the upper solution $v(t)=t+1$, $t\in[0,1]$.

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