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FIXED POINT THEOREMS UNDER RATIONAL CONTRACTION IN COMPLEX VALUED METRIC SPACES

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Abstract. In this paper, we establish some fixed point theorems under rational contraction in the setting of complex valued metric spaces. The results presented in this paper extend and generalize several results from the current existing literature.

1. Introduction

Fixed point theory plays a very important role in the development of nonlinear analysis. The Banach contraction principle [4] is a very popular tool in solving existence problems in many branches of mathematics. The Banach contraction principle with rational expressions have been expanded and some fixed point and common fixed point theorems have been obtained in [6], [7].

In the existing literature, there are a great number of generalizations of the Banach contraction principle (see [1, 2] and others).

In 2011, Azam *et al.* [3] (Numer. Funct. Anal. Optim **3**(3) (2011), 243–253) introduced the concept of complex valued metric space and established some fixed point results for mappings satisfying a rational inequality. In this paper, we establish some fixed point theorems under rational contraction in the framework of complex valued metric spaces.

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2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), or (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \lesssim z_1 \lnsim z_2 \;\Rightarrow\; |z_1| < |z_2|,$$
$$z_1 \precsim z_2,\; z_2 \prec z_3 \;\Rightarrow\; z_1 \prec z_3.$$

The following definition was introduced by Azam *et al.* in 2011 (see, [3]).

Definition 2.1. ([3]) Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to \mathbb{C}$ satisfies:

(i) $0 \precsim d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 2.2. Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d\colon X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{it}|z_1 - z_2|$ where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $t \in [0, \frac{\pi}{2}]$. Then (X, d) is a complex valued metric space.

Definition 2.3. A point $x \in X$ is called an interior point of a subset $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

Definition 2.4. A point $x \in X$ is called a limit of A whenever for every $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) \cap \Big(A - \{X\}\Big) \neq \emptyset.$$

Definition 2.5. The set A is called open whenever each element of A is an interior point of A. A subset B is called closed whenever each limit point of B belongs to B.

The family $\mathcal{F} := \{B(x,r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.

Definition 2.6. ([3]) Let (X,d) be a complex valued metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is called convergent, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. Also, $\{x_n\}$ converges to x (written as, $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) and x is the limit of $\{x_n\}$.

(ii) $\{x_n\}$ is called a Cauchy sequence in X, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$. If every Cauchy sequence converges in X, then X is called a complete complex valued metric space.

Lemma 2.7. ([3]) *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if* $\lim_{n\to\infty} |d(x_n, x)| = 0$.

Lemma 2.8. ([3]) *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if* $\lim_{n\to\infty} |d(x_n, x_{n+m})| = 0$.

3. Main Results

In this section we shall prove some fixed point theorems under rational contraction in the framework of complex valued metric spaces.

Theorem 3.1. *Let (X,d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$d(Tx,Ty) \precsim a\, d(x,y) + b\,\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + c\,\frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)} \tag{3.1}$$

for all $x, y \in X$, where a, b, c are nonnegative reals with $a+b+c<1$. Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, that is, $x_{n+1} = Tx_n = T^{n+1}x_0$. From (3.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\precsim a\, d(x_{n-1}, x_n) + b\,\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{1+d(x_{n-1}, x_n)} \\ &\quad + c\,\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{1+d(Tx_{n-1}, Tx_n)} \end{aligned}$$

$$\begin{aligned} &= a\, d(x_{n-1}, x_n) + b\, \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \\ &\quad + c\, \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})} \\ &\precsim (a + b + c) d(x_{n-1}, x_n) \\ &= \alpha\, d(x_{n-1}, x_n), \end{aligned}$$

where $\alpha = a + b + c$. As $a + b + c < 1$, it follows that $0 < \alpha < 1$. By induction, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\precsim \alpha\, d(x_{n-1}, x_n) \precsim \alpha^2\, d(x_{n-2}, x_{n-1}) \precsim \dots \\ &\precsim \alpha^n\, d(x_0, x_1). \end{aligned}$$

Let $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\ &\quad + \dots + d(x_{n+m-1}, x_m) \\ &\precsim [\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \dots + \alpha^{n+m-1}] d(x_1, x_0) \\ &\precsim \Big[\frac{\alpha^n}{1 - \alpha}\Big] d(x_1, x_0) \end{aligned}$$

and so

$$|d(x_n, x_m)| \leq \Big[\frac{\alpha^n}{1 - \alpha}\Big] |d(x_1, x_0)| \to 0 \quad \text{as} \quad m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. It follows that $v = Tv$, otherwise $d(v, Tv) = z > 0$ and we would then have

$$\begin{aligned} z &\precsim d(v, x_{n+1}) + d(x_{n+1}, Tv) \\ &= d(v, x_{n+1}) + d(Tx_n, Tv) \\ &\precsim d(v, x_{n+1}) + a\, d(x_n, v) + b\, \frac{d(x_n, Tx_n) d(v, Tv)}{1 + d(x_n, v)} \\ &\quad + c\, \frac{d(x_n, Tx_n) d(v, Tv)}{1 + d(Tx_n, Tv)} \\ &\precsim d(v, x_{n+1}) + a\, d(x_n, v) + b\, \frac{d(x_n, x_{n+1}) d(v, Tv)}{1 + d(x_n, v)} \\ &\quad + c\, \frac{d(x_n, x_{n+1}) d(v, Tv)}{1 + d(x_{n+1}, Tv)}. \end{aligned}$$

This implies that

$$\begin{aligned}|z| &\leq |d(v, x_{n+1})| + a\,|d(x_n, v)| + b\,\frac{|d(x_n, x_{n+1})||d(v, Tv)|}{1 + |d(x_n, v)|}\\ &\quad + c\,\frac{|d(x_n, x_{n+1})||d(v, Tv)|}{1 + |d(x_{n+1}, Tv)|}.\end{aligned}$$

Letting $n \to \infty$, it follows that

$$|z| \leq 0$$

which is a contradiction and so $|z| = 0$, that is, $v = Tv$.

To prove the uniqueness of fixed point of T, assume that v^* is another fixed point of T, that is, $Tv^* = v^*$ such that $v \neq v^*$. Then

$$\begin{aligned}d(v, v^*) &= d(Tv, Tv^*)\\ &\precsim a\,d(v, v^*) + b\,\frac{d(v, Tv)d(v^*, Tv^*)}{1 + d(v, v^*)} + c\,\frac{d(v, Tv)d(v^*, Tv^*)}{1 + d(Tv, Tv^*)}\\ &= a\,d(v, v^*) + b\,\frac{d(v, v)d(v^*, v^*)}{1 + d(v, v^*)} + c\,\frac{d(v, v)d(v^*, v^*)}{1 + d(v, v^*)}\\ &= a\,d(v, v^*),\end{aligned}$$

so that $|d(v, v^*)| \leq a\,|d(v, v^*)| < |d(v, v^*)|$, since $0 < a < 1$, which is a contradiction and hence $d(v, v^*) = 0$. Thus $v = v^*$, which proves the uniqueness of fixed point of T. This completes the proof. □

Corollary 3.2. *Let (X, d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfies (for fixed n):*

$$\begin{aligned}d(T^n x, T^n y) &\precsim a\,d(x, y) + b\,\frac{d(x, T^n x)d(y, T^n y)}{1 + d(x, y)}\\ &\quad + c\,\frac{d(x, T^n x)d(y, T^n y)}{1 + d(T^n x, T^n y)}\end{aligned}$$

for all $x, y \in X$, where a, b, c are nonnegative reals with $a + b + c < 1$. Then T has a unique fixed point in X.

Proof. By Theorem 3.1, there exists $u \in X$ such that $T^n u = u$. Then

$$\begin{aligned}d(Tu, u) &= d(TT^n u, T^n u) = d(T^n Tu, T^n u)\\ &\precsim a\,d(Tu, u) + b\,\frac{d(Tu, T^n Tu)d(u, T^n u)}{1 + d(Tu, u)}\\ &\quad + c\,\frac{d(Tu, T^n Tu)d(u, T^n u)}{1 + d(T^n Tu, T^n u)}\end{aligned}$$

$$
\begin{aligned}
&= a\,d(Tu,u) + b\,\frac{d(Tu,TT^nu)d(u,T^nu)}{1+d(Tu,u)} \\
&\quad +c\,\frac{d(Tu,TT^nu)d(u,T^nu)}{1+d(TT^nu,T^nu)} \\
&= a\,d(Tu,u) + b\,\frac{d(Tu,Tu)d(u,u)}{1+d(Tu,u)} \\
&\quad +c\,\frac{d(Tu,Tu)d(u,u)}{1+d(Tu,u)} \\
&= a\,d(Tu,u),
\end{aligned}
$$

so that $|d(Tu,u)| \le a\,|d(Tu,u)| < |d(Tu,u)|$, since $0 < a < 1$, which is a contradiction and hence $d(Tu,u) = 0$. Thus $Tu = u$. This shows that T has a unique fixed point in X. This completes the proof. □

If we put $b = c = 0$ in Corollary 3.2, we draw following corollary which can be viewed as an extension of Bryant (see, [5]) theorem to complex valued metric space.

Corollary 3.3. *Let (X,d) be a complete complex valued metric space. Suppose that the mapping $T\colon X \to X$ satisfying the condition:*

$$d(T^nx,T^ny) \precsim a\,d(x,y)$$

for all $x,y \in X$ and $a \in [0,1)$ is a constant. Then T has a unique fixed point in X.

The following example demonstrates the superiority of Bryant (see, [5]) theorem over Banach contraction theorem.

Example 3.4. ([8]) Let $X = \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. Define a mapping $d\colon \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ by $d(z_1,z_2) = |x_1-x_2| + i|y_1-y_2|$ where $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$. Then $(\mathbb{C},d)$ is a complex valued metric space. Define $T\colon \mathbb{C} \to \mathbb{C}$ as

$$
T(x+iy) = \begin{cases}
0, & \text{if } x,y \in Q, \\
i, & \text{if } x,y \in Q^c, \\
1, & \text{if } x \in Q^c, y \in Q, \\
1+i, & \text{if } x \in Q, y \in Q^c.
\end{cases}
$$

Now for $x = \frac{1}{\sqrt{2}}$ and $y = 0$, we get

$$d(T(\frac{1}{\sqrt{2}}), T(0)) = d(1, 0) \precsim \lambda d(\frac{1}{\sqrt{2}}, 0) = \frac{\lambda}{\sqrt{2}}.$$

Thus $\lambda \geq \sqrt{2}$ which is a contradiction that $0 \leq \lambda < 1$. However, we notice that $T^2(z) = 0$, so that

$$0 = d(T^2(z_1), T^2(z_2)) \precsim \lambda d(z_1, z_2),$$

which shows that T^2 satisfies the requirement of Bryant theorem and $z = 0$ is a unique fixed point of T.

Example 3.5. Let $X = \{0, \frac{1}{2}, 2\}$ and partial order $'\precsim'$ is defined as $x \precsim y$ iff $x \geq y$. Let the complex valued metric d be given as

$$d(x, y) = |x - y|\sqrt{2}e^{i\frac{\pi}{4}} = |x - y|(1 + i) \quad \text{for } \ x, y \in X.$$

Let $T \colon X \to X$ be defined as follows:

$$T(0) = 0, T(\frac{1}{2}) = 0, T(2) = \frac{1}{2}.$$

Case I. Take $x = \frac{1}{2}$, $y = 0$, $T(0) = 0$ and $T(\frac{1}{2}) = 0$ in Theorem 3.1, then we have

$$d(Tx, Ty) = 0 \leq a.(\frac{1 + i}{2}) + b.0 + c.0.$$

This implies that $a \geq 0$. If we take $a = \frac{1}{3}$ and $b = c = \frac{1}{4}$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the unique fixed point of T.

Case II. Take $x = 2$, $y = \frac{1}{2}$, $T(2) = \frac{1}{2}$ and $T(\frac{1}{2}) = 0$ in Theorem 3.1, then we have

$$d(Tx, Ty) = \frac{1 + i}{2} \leq a.\Big(\frac{3(1 + i)}{2}\Big) + b.\Big(\frac{3/4(1 + i)}{1 + 3/2(1 + i)}\Big) + c.\Big(\frac{3/4(1 + i)}{1 + 1/2(1 + i)}\Big).$$

This implies that $a = \frac{1}{6}$ and $b = c = \frac{1}{3}$ satisfied all the conditions of Theorem 3.1 and of course 0 is the unique fixed point of T.

Case III. Take $x = 2$, $y = 0$, $T(2) = \frac{1}{2}$ and $T(0) = 0$ in Theorem 3.1, then we have

$$d(Tx, Ty) = \frac{1 + i}{2} \leq a.2(1 + i) + b.0 + c.0.$$

This implies that $a \geq \frac{1}{4}$. If we take $a = \frac{1}{2}$ and $b = c = \frac{1}{5}$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the unique fixed point of T.

4. Conclusion

In this paper, we establish some fixed point theorems using rational contraction in the setting of complex-valued metric spaces and give an example in support of our results. Our results extend and generalize several known results from the current existing literature.

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