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APPROXIMATE GENERALIZED JORDAN DERIVATIONS ON BANACH MODULES

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Abstract. In this paper, we prove the generalized Hyers–Ulam–Rassias stability for generalized Jordan derivations on a ring $\mathcal{R}$ into a Banach $\mathcal{R}$-bimodule $\mathcal{M}$ following Th.M. Rassias' stability theory approach.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [34] concerning the stability of group homomorphisms : Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric d. Given $\epsilon > 0$, does there exist $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations where homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a homomorphism near it. Hyers [10] gave a first partial affirmative answer to the

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question of Ulam for Banach spaces. Let X and Y be Banach spaces and assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for some $\varepsilon \geq 0$ and all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$.

Aoki [1] generalized the Hyers' theorem for additive mappings and Rassias [28] provided a generalization of the Hyers' theorem for linear mappings by allowing the Cauchy difference to be unbounded.

Theorem 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ε and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The inequality (1.1) has provided a lot of influence in the development of what is now known as the *Hyers–Ulam–Rassias stability* of functional equations. Isac and Rassias [13] by introducing the concept of ψ-additive mappings between Banach spaces provided the first generalization of Rassias' theorem for approximate homomorphisms. One year later, Găvruta [7] provided another generalization of Rassias theorem by replacing the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Isac and Rassias [14] were the first to prove new fixed point theorems of mappings by applying the stability approach that has been influenced by the Ulam's problem. In addition they indicated new applications to nonlinear eigenvalue problems. In 1998, Hyers, Isac and Rassias [12] were the first to prove an asymptotic analogue of Rassias' theorem for the Hyers–Ulam stability of mappings. Since then the stability problems of various functional equations and mappings and their Pexiderized versions with more general domains and ranges have been investigated by a number of authors (see [20]–[27]). We also refer the readers to the books [6], [11], [17] and [29].

Let A be a real or complex algebra. A mapping $D : A \to A$ is said to be a *(ring) derivation* if

$$D(a+b) = D(a) + D(b), \qquad D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. If, in addition, $D(\lambda a) = \lambda D(a)$ for all $a \in A$ and all $\lambda \in \mathbb{F}$, then D is called a *linear derivation,* where $\mathbb{F}$ denotes the scalar field of A. Singer and Wermer [32] proved that if A is a commutative Banach algebra and $D : A \to A$ is a continuous linear derivation, then $D(A) \subseteq \mathrm{rad}(A)$. They also conjectured that the same result holds even D is a discontinuous linear derivation. Thomas [33] proved the conjecture. As a direct consequence, we see that there are no non-zero linear derivations on a semi-simple commutative Banach algebra, which had been proved by Johnson [16]. On the other hand, it is not the case for ring derivations. Hatori and Wada [8] determined a representation of ring derivations on a semi-simple commutative Banach algebra (see also [31]) and they proved that only the zero operator is a ring derivation on a semi-simple commutative Banach algebra with the maximal ideal space without isolated points. The stability of derivations between operator algebras was first obtained by Šemrl [30]. Badora [2] and Miura *et al.* [21] proved the Hyers–Ulam–Rassias stability of ring derivations on Banach algebras.

Let $\mathcal{R}$ be an associative ring and $\mathcal{N}$ be a $\mathcal{R}$-bimodule. An additive mapping $D : \mathcal{R} \to \mathcal{N}$ is called a *derivation* if

$$D(ab) = D(a)b + aD(b)$$

holds for all pairs $a, b \in \mathcal{R}$ and is called a *Jordan derivation* in case

$$D(a^2) = D(a)a + aD(a)$$

is fulfilled for all $a \in \mathcal{R}$. Every derivation is a Jordan derivation. The converse is in general not true (see [5, 9]). An additive mapping $F : \mathcal{R} \to \mathcal{N}$ is called a *generalized derivation* in case

$$F(ab) = F(a)b + aD(b)$$

holds for all pairs $a, b \in \mathcal{R}$, where $D : \mathcal{R} \to \mathcal{N}$ is a derivation. The concept of generalized derivation has been introduced by Brešar [3]. Jing and Lu [15] introduced a concept of generalized Jordan derivation. An additive mapping $F : \mathcal{R} \to \mathcal{N}$ is called a *generalized Jordan derivation* in case

$$F(a^2) = F(a)a + aD(a)$$

holds for all $a \in \mathcal{R}$, where $D : \mathcal{R} \to \mathcal{N}$ is a Jordan derivation. It is clear that every generalized derivation is a generalized Jordan derivation. For the converse see [35].

The aim of the present paper is to establish the stability problem of generalized Jordan derivations by using the fixed point method (see [4, 18, 20]).

Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a *generalized metric* on E if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in E$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.2. [19] *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in E$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all non-negative integers n or there exists a non-negative integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set*

$$Y = \{\, y \in E : d(J^{n_0} x, y) < \infty \,\};$$

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

2. Main Results

In this section, we assume that $\mathcal{R}$ is a 2-divisible associative ring, and $\mathcal{M}$ is a Banach $\mathcal{R}$-bimodule. For convenience, we use the following abbreviation for given mappings $f, g : \mathcal{R} \to \mathcal{M}$,

$$D_{f,g}(a, b, c) := f(a^2 + b + c) - f(a)a - ag(a) - f(b) - f(c)$$

for all $a, b, c \in \mathcal{R}$.

Now we prove the generalized Hyers–Ulam–Rassias stability of generalized Jordan derivations in Banach $\mathcal{R}$-bimodules.

Theorem 2.1. *Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings for which there exist functions $\varphi, \psi : \mathcal{R}^3 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} 4^n \varphi\big(\frac{a}{2^n}, 0, 0\big) = \lim_{n\to\infty} 2^n \varphi\big(0, \frac{b}{2^n}, \frac{c}{2^n}\big) = 0, \tag{2.1}$$

$$\|D_{f,g}(a, b, c)\| \leq \varphi(a, b, c), \tag{2.2}$$

$$\lim_{n\to\infty} 4^n \psi\big(\frac{a}{2^n}, 0, 0\big) = \lim_{n\to\infty} 2^n \psi\big(0, \frac{b}{2^n}, \frac{c}{2^n}\big) = 0, \tag{2.3}$$

$$\|D_{g,g}(a, b, c)\| \leq \psi(a, b, c) \tag{2.4}$$

for all $a, b, c \in \mathcal{R}$*. If there exist constants* $0 < L, K < 1$ *such that*

$$2\varphi(0, a, a) \leq L\varphi(0, 2a, 2a), \qquad 2\psi(0, a, a) \leq K\psi(0, 2a, 2a)$$

for all $a \in \mathcal{R}$*, then there exist a unique Jordan derivation* $G : \mathcal{R} \to \mathcal{M}$ *and a unique generalized Jordan derivation* $F : \mathcal{R} \to \mathcal{M}$ *satisfying*

$$\|f(a) - F(a)\| \leq \frac{L}{2 - 2L}\varphi(0, a, a), \tag{2.5}$$

$$\|g(a) - G(a)\| \leq \frac{K}{2 - 2K}\psi(0, a, a), \tag{2.6}$$

for all $a \in \mathcal{R}$*.*

Proof. It follows from (2.1) and (2.3) that $\varphi(0, 0, 0) = 0 = \psi(0, 0, 0)$ and so $f(0) = g(0) = 0$. Letting $a = 0$ and $b = c$ in (2.2), we get

$$\|f(2c) - 2f(c)\| \leq \varphi(0, c, c) \tag{2.7}$$

for all $c \in \mathcal{R}$. Let $E := \{ h : \mathcal{R} \to \mathcal{M} \mid h(0) = 0 \}$. We introduce a generalized metric on E as follows:

$$d_\varphi(h, k) := \inf\{ C \in [0, \infty] : \|h(a) - k(a)\| \leq C\varphi(0, a, a) \text{ for all } a \in \mathcal{R} \}.$$

It is easy to show that (E, d_φ) is a generalized complete metric space [4].

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda h)(a) = 2h\big(\frac{a}{2}\big), \quad \text{for all } h \in E \text{ and } a \in \mathcal{R}.$$

Let $h, k \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d_\varphi(h, k) \leq C$. From the definition of d_φ, we have

$$\|h(a) - k(a)\| \leq C\varphi(0, a, a)$$

for all $a \in \mathcal{R}$. By the assumption and the last inequality, we have

$$\|(\Lambda h)(a) - (\Lambda k)(a)\| = 2\Big\|h\big(\frac{a}{2}\big) - k\big(\frac{a}{2}\big)\Big\| \leq 2C\varphi\big(0, \frac{a}{2}, \frac{a}{2}\big) \leq CL\varphi(0, a, a)$$

for all $a \in \mathcal{R}$. So

$$d_\varphi(\Lambda h, \Lambda k) \leq Ld_\varphi(h, k)$$

for any $h, k \in E$. It follows from the assumption and (2.7) that $d_\varphi(\Lambda f, f) \leq L/2$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point F of Λ, i.e.,

$$F : \mathcal{R} \to \mathcal{M}, \quad F(a) = \lim_{n\to\infty}(\Lambda^n f)(a) = \lim_{n\to\infty} 2^n f\big(\frac{a}{2^n}\big)$$

and $F(2a) = 2F(a)$ for all $a \in \mathcal{R}$. Also F is the unique fixed point of Λ in the set $E_\varphi = \{h \in E : d_\varphi(f, h) < \infty \}$ and

$$d_\varphi(F, f) \leq \frac{1}{1 - L}d_\varphi(\Lambda f, f) \leq \frac{L}{2 - 2L},$$

i.e., inequality (2.5) holds true for all $a \in \mathcal{R}$. Similarly, we obtain that

$$d_\psi(\Lambda h, \Lambda k) \le K d_\psi(h,k), \quad d_\psi(\Lambda g, g) \le K/2$$

for any $h, k \in E$, where

$$d_\psi(h,k) := \inf\{\, C \in [0,\infty] : \|h(a) - k(a)\| \le C\psi(0,a,a) \text{ for all } a \in \mathcal{R}\,\}.$$

So according to Theorem 1.2, the sequence $\{\Lambda^n g\}$ converges to a fixed point G of Λ, i.e.,

$$G : \mathcal{R} \to \mathcal{M}, \quad G(a) = \lim_{n\to\infty} (\Lambda^n g)(a) = \lim_{n\to\infty} 2^n g\big(\frac{a}{2^n}\big)$$

and $G(2a) = 2G(a)$ for all $a \in \mathcal{R}$. Also G is the unique fixed point of Λ in the set $E_\psi = \{h \in E : d_\psi(g,h) < \infty\,\}$ and

$$d_\psi(G,g) \le \frac{1}{1-K} d_\psi(\Lambda g, g) \le \frac{K}{2-2K},$$

i.e., inequality (2.6) holds true for all $a \in \mathcal{R}$. It follows from the definitions of F, G, (2.1) and (2.2) that

$$\begin{aligned} \|D_{F,G}(a,0,0)\| &= \lim_{n\to\infty} 4^n \Big\| D_{f,g}\big(\frac{a}{2^n}, 0, 0\big) \Big\| \\ &\le \lim_{n\to\infty} 4^n \varphi\big(\frac{a}{2^n}, 0, 0\big) = 0, \\ \|D_{F,G}(0,b,c)\| &= \lim_{n\to\infty} 2^n \Big\| D_{f,g}\big(0, \frac{b}{2^n}, \frac{c}{2^n}\big) \Big\| \\ &\le \lim_{n\to\infty} 2^n \varphi\big(0, \frac{b}{2^n}, \frac{c}{2^n}\big) = 0 \end{aligned}$$

for all $a, b, c \in \mathcal{R}$. Hence

$$F(a^2) = F(a)a + aG(a), \quad F(b+c) = F(b) + F(c) \tag{2.8}$$

for all $a, b, c \in \mathcal{R}$. Similarly, it follows from the definition of G, (2.3) and (2.4) that

$$G(a^2) = G(a)a + aG(a), \quad G(b+c) = G(b) + G(c) \tag{2.9}$$

for all $a, b, c \in \mathcal{R}$. Hence G is a Jordan derivation. So we infer from (2.8) and (2.9) that F is a generalized Jordan derivation.

Finally it remains to prove the uniqueness of F and G. Let $F_1, G_1 : \mathcal{R} \to \mathcal{M}$ be another additive mappings satisfying (2.5) and (2.6), respectively. Since $d_\varphi(f, F_1) \le \frac{L}{2-2L}$, $d_\psi(g, G_1) \le \frac{K}{2-2K}$ and F_1, G_1 are additive, we get $F_1 \in E_\varphi$, $G_1 \in E_\psi$ and $(\Lambda F_1)(a) = 2F_1(a/2) = F_1(a)$, $(\Lambda G_1)(a) = 2G_1(a/2) = G_1(a)$ for all $a \in \mathcal{R}$, i.e., F_1, G_1 are fixed points of Λ. Since F and G are the unique fixed points of Λ in E_φ and E_ψ, respectively, we get $F_1 = F$ and $G_1 = G$. □

Theorem 2.2. *Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings with $f(0) = g(0) = 0$ for which there exist functions $\Phi, \Psi : \mathcal{R}^3 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{4^n}\Phi(2^n a, 0, 0) = \lim_{n\to\infty} \frac{1}{2^n}\Phi(0, 2^n b, 2^n c) = 0, \tag{2.10}$$

$$\|D_{f,g}(a, b, c)\| \le \Phi(a, b, c), \tag{2.11}$$

$$\lim_{n\to\infty} \frac{1}{4^n}\Psi(2^n a, 0, 0) = \lim_{n\to\infty} \frac{1}{2^n}\Psi(0, 2^n b, 2^n c) = 0, \tag{2.12}$$

$$\|D_{g,g}(a, b, c)\| \le \Psi(a, b, c) \tag{2.13}$$

for all $a, b, c \in \mathcal{R}$. If there exist constants $L, K < 1$ such that

$$\Phi(0, 2a, 2a) \le 2L\Phi(0, a, a), \qquad \Psi(0, 2a, 2a) \le 2K\Psi(0, a, a)$$

for all $a \in \mathcal{R}$, then there exist a unique Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \le \frac{1}{2 - 2L}\Phi(0, a, a), \tag{2.14}$$

$$\|g(a) - G(a)\| \le \frac{1}{2 - 2K}\Psi(0, a, a), \tag{2.15}$$

for all $a \in \mathcal{R}$.

Proof. Using the same method as in the proof of Theorem 2.1, we have

$$\left\|\frac{1}{2}f(2c) - f(c)\right\| \le \frac{1}{2}\Phi(0, c, c), \quad \left\|\frac{1}{2}g(2c) - g(c)\right\| \le \frac{1}{2}\Psi(0, c, c) \tag{2.16}$$

for all $c \in \mathcal{R}$. We introduce the same definitions for E, d_Φ and d_Ψ as in the proof of Theorem 2.1 such that (E, d_Φ) and (E, d_Ψ) become generalized complete metric spaces. Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda h)(a) = \frac{1}{2}h(2a), \quad \text{for all } h \in E \text{ and } a \in \mathcal{R}.$$

One can show that

$$d_\Phi(\Lambda h, \Lambda k) \le L d_\Phi(h, k), \quad d_\Psi(\Lambda h, \Lambda k) \le K d_\Psi(h, k)$$

for any $h, k \in E$. It follows from (2.16) that $d_\Phi(\Lambda f, f) \le \frac{1}{2}$ and $d_\Psi(\Lambda g, g) \le \frac{1}{2}$. Due to Theorem 1.2, the sequences $\{\Lambda^n f\}$ and $\{\Lambda^n g\}$ converge to fixed points F and G of Λ, respectively, i.e., $F, G : \mathcal{R} \to \mathcal{M}$,

$$F(a) = \lim_{n\to\infty} (\Lambda^n f)(a) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n a),$$

$$G(a) = \lim_{n\to\infty} (\Lambda^n g)(a) = \lim_{n\to\infty} \frac{1}{2^n} g(2^n a),$$

$F(2a) = 2F(a)$ and $G(2a) = 2G(a)$ for all $a \in \mathcal{R}$. Also

$$d_\Phi(F, f) \le \frac{1}{1-L} d_\Phi(\Lambda f, f) \le \frac{1}{2-2L},$$
$$d_\Psi(G, g) \le \frac{1}{1-K} d_\Psi(\Lambda g, g) \le \frac{1}{2-2K},$$

i.e., the inequalities (2.14) and (2.15) hold true for all $a \in \mathcal{R}$.

The rest of the proof is similar to the proof of Theorem 2.1 and we omit the details. □

Corollary 2.3. *Let $\varepsilon, \delta, p, q$ be non-negative real numbers with $0 < p, q < 1$ or $p, q > 2$. If $\mathcal{R}$ is a normed ring and $f, g : \mathcal{R} \to \mathcal{M}$ are mappings satisfy the inequalities*

$$\|D_{f,g}(a, b, c)\| \le \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p)$$

and

$$\|D_{g,g}(a, b, c)\| \le \delta(\|a\|^q + \|b\|^q + \|c\|^q)$$

for all $a, b, c \in \mathcal{R}$, then there exist a unique Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \le \frac{2\varepsilon}{|2 - 2^p|}\|a\|^p, \quad \|g(a) - G(a)\| \le \frac{2\delta}{|2 - 2^q|}\|a\|^q$$

for all $a \in \mathcal{R}$.

Proof. Let

$$L := \begin{cases} 2^{p-1}, & 0 < p < 1; \\ 2^{1-p}, & p > 2. \end{cases} \qquad K := \begin{cases} 2^{q-1}, & 0 < q < 1; \\ 2^{1-q}, & q > 2. \end{cases}$$

So the result follows from Theorems 2.1 and 2.2. □

Corollary 2.4. *Let ε and δ be non-negative real numbers and let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings satisfying $f(0) = g(0) = 0$ and the inequalities*

$$\|D_{f,g}(a, b, c)\| \le \varepsilon, \quad \|D_{g,g}(a, b, c)\| \le \delta$$

for all $a, b, c \in \mathcal{R}$. Then there exist a unique Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \le \varepsilon, \quad \|g(a) - G(a)\| \le \delta$$

for all $a \in \mathcal{R}$.

Proof. The proof follows from Theorem 2.2 by taking

$$\Phi(a, b, c) := \varepsilon, \quad \Psi(a, b, c) := \delta$$

for all $a, b, c \in \mathcal{R}$. Then we can choose $L = K = 1/2$ and we get the desired results. □

Theorem 2.5. *Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings with $f(0) = g(0) = 0$ for which there exists a function $\Phi : \mathcal{R}^4 \to [0, \infty)$ satisfying*

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{2^n}\Phi(2^n a, 2^n b, 2^n c, 2^n d) &= \lim_{n\to\infty} \frac{1}{2^n}\Phi(2^n a, b, 0, 0) \\ &= \lim_{n\to\infty} \frac{1}{2^n}\Phi(a, 2^n b, 0, 0) = 0, \end{aligned} \tag{2.17}$$

$$\|f(ab + c + d) - f(a)b - ag(b) - f(c) - f(d)\| \le \Phi(a, b, c, d), \tag{2.18}$$

for all $a, b, c, d \in \mathcal{R}$. If $\mathcal{R}$ has the identity e, $\mathcal{M}$ is unit linked and there exists a constant $0 < L < 1$ such

$$\Phi(0, 0, 2a, 2a) \le 2L\Phi(0, 0, a, a)$$

for all $a \in \mathcal{R}$, then g is a derivation and f is a generalized derivation. Moreover, $f(a) = ba + g(a)$ for all $a \in \mathcal{R}$, where $b = \lim_{n\to\infty} \frac{1}{2^n} f(2^n e)$.

Proof. Letting $a = b = 0$ and $c = d$ in (2.18), we get

$$\|f(2c) - 2f(c)\| \le \Phi(0, 0, c, c)$$

for all $c \in \mathcal{R}$. Using the same method as in the proof of Theorem 2.2, we infer that the limit

$$F(a) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n a) \tag{2.19}$$

exists for all $a \in \mathcal{R}$ and the mapping $F : \mathcal{R} \to \mathcal{M}$ is additive. Letting $c = d = 0$ and replacing a and b by $2^n e$ and $2^n b$, respectively, in (2.18), we get

$$\|f(4^n b) - 2^n f(2^n e)b - 2^n g(2^n b)\| \le \Phi(2^n e, 2^n b, 0, 0)$$

for all $b \in \mathcal{R}$ and all $n \in \mathbb{N}$. Then

$$\Big\|\frac{1}{4^n} f(4^n b) - \frac{1}{2^n} f(2^n e)b - \frac{1}{2^n} g(2^n b)\Big\| \le \frac{1}{4^n}\Phi(2^n e, 2^n b, 0, 0) \tag{2.20}$$

for all $b \in \mathcal{R}$ and all $n \in \mathbb{N}$. It follows from (2.17), (2.19) and (2.20) that the limit

$$G(b) := \lim_{n\to\infty} \frac{1}{2^n} g(2^n b)$$

exists and $G(b) = F(b) - F(e)b$ for all $b \in \mathcal{R}$. Hence G is additive. It follows from the definitions of F, G, (2.17) and (2.18) that

$$\begin{aligned} &\|F(ab) - F(a)b - aG(b)\| \\ &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n ab) - 2^n f(2^n a)b - 2^n ag(2^n b)\| \\ &\le \lim_{n\to\infty} \frac{1}{4^n}\Phi(2^n a, 2^n b, 0, 0) = 0 \end{aligned}$$

for all $a, b \in \mathcal{R}$. Therefore

$$F(ab) = F(a)b + aG(b) \tag{2.21}$$

for all $a, b \in \mathcal{R}$. Further, by (2.21) we have

$$\begin{aligned} G(ab) &= F(ab) - F(e)ab \\ &= F(a)b + aG(b) - F(e)ab \\ &= [F(a) - F(e)a]b + aG(b) \\ &= G(a)b + aG(b) \end{aligned}$$

for all $a, b \in \mathcal{R}$. Thus G is a derivation and (2.21) shows that F is a generalized derivation.

By (2.17), (2.18) and the definitions of F, G, we have

$$F(ab) - F(a)b = ag(b), \tag{2.22}$$

$$F(ab) - aG(b) = f(a)b \tag{2.23}$$

for all $a, b \in \mathcal{R}$. Since $G(e) = 0$, letting $a = e$ in (2.22) and $b = e$ in (2.23), we get $g(b) = F(b) - F(e)b = G(b)$ and $F(a) = f(a)$, respectively, for all $a, b \in \mathcal{R}$. So g is a derivation and f is a generalized derivation. Moreover, $f(a) = F(e)a + g(a)$ for all $a \in \mathcal{R}$. □

Corollary 2.6. *Let ε, δ, p be non-negative real numbers with $0 < p < 1$. If $\mathcal{R}$ is a normed ring with the identity e, $\mathcal{M}$ is unit linked and $f, g : \mathcal{R} \to \mathcal{M}$ are mappings with $f(0) = g(0) = 0$ and satisfy the inequality*

$$\begin{aligned} &\|f(ab + c + d) - f(a)b - ag(b) - f(c) - f(d)\| \\ &\le \delta + \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p), \end{aligned}$$

for all $a, b, c, d \in \mathcal{R}$, then g is a derivation and f is a generalized derivation. Moreover, $f(a) = ba + g(a)$ for all $a \in \mathcal{R}$, where $b = \lim_{n\to\infty} \frac{1}{2^n} f(2^n e)$.

Theorem 2.7. *Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings for which there exist functions $\varphi, \psi : \mathcal{R}^2 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{n}\varphi(na, b) = \lim_{n\to\infty} \frac{1}{n}\varphi(a, nb) = 0, \tag{2.24}$$

$$\lim_{n\to\infty} \frac{1}{n}\psi(na, b) = \lim_{n\to\infty} \frac{1}{n}\psi(a, nb) = 0, \tag{2.25}$$

$$\|f(ab) - f(a)b - ag(b)\| \le \varphi(a, b), \tag{2.26}$$

$$\|g(ab) - g(a)b - ag(b)\| \le \psi(a, b) \tag{2.27}$$

for all $a, b, c \in \mathcal{R}$. If $\mathcal{R}$ is normed with the identity e and $\mathcal{M}$ is unit linked, then

$$g(ab) = g(a)b + ag(b), \qquad f(ab) = f(a)b + ag(b) \tag{2.28}$$

for all $a, b \in \mathcal{R}$.

Proof. By (2.25) and (2.27), we get

$$\lim_{n\to\infty}\frac{1}{n}[g(nab)-g(na)b]=ag(b),\qquad \lim_{n\to\infty}\frac{1}{n}[g(nab)-ag(nb)]=g(a)b \tag{2.29}$$

for all $a, b \in \mathcal{R}$. Using the Badora's method [2] on the inequality (2.27), we get that the mapping g satisfies

$$g(ab)=g(a)b+ag(b) \tag{2.30}$$

for all $a, b \in \mathcal{R}$. By (2.24) and (2.26), we have

$$\lim_{n\to\infty}\frac{1}{n}[f(nab)-f(na)b]=ag(b),\qquad \lim_{n\to\infty}\frac{1}{n}[f(nab)-ag(nb)]=f(a)b \tag{2.31}$$

for all $a, b \in \mathcal{R}$. Let $a, b \in \mathcal{R}$ and $n \in \mathbb{N}$ be fixed. Since g satisfies (2.30), we have $g(nb) = g(bne) = bg(ne) + ng(b)$. Using (2.26), we have

$$\begin{aligned}
\|f(ab)-f(a)b-ag(b)\| &\le \Big\|f(ab)-\frac{1}{n}f(nabe)+\frac{1}{n}abg(ne)\Big\| \\
&\quad+\Big\|\frac{1}{n}f(nab)-\frac{1}{n}ag(nb)-f(a)b\Big\| \\
&\quad+\Big\|\frac{1}{n}f(nab)-\frac{1}{n}f(na)b-ag(b)\Big\| \\
&\quad+\frac{1}{n}\|ag(nb)-abg(ne)+f(na)b-f(nab)\| \\
&=\Big\|f(ab)-\frac{1}{n}f(nabe)+\frac{1}{n}abg(ne)\Big\| \\
&\quad+\Big\|\frac{1}{n}f(nab)-\frac{1}{n}ag(nb)-f(a)b\Big\| \\
&\quad+\Big\|\frac{1}{n}f(nab)-\frac{1}{n}f(na)b-ag(b)\Big\| \\
&\quad+\frac{1}{n}\|nag(b)+f(na)b-f(nab)\| \\
&\le \Big\|f(ab)-\frac{1}{n}f(nabe)+\frac{1}{n}abg(ne)\Big\| \\
&\quad+\Big\|\frac{1}{n}f(nab)-\frac{1}{n}ag(nb)-f(a)b\Big\| \\
&\quad+\Big\|\frac{1}{n}f(nab)-\frac{1}{n}f(na)b-ag(b)\Big\|+\frac{1}{n}\varphi(na,b).
\end{aligned}$$

Applying (2.24) and (2.31) we observe that the right side of the last inequality tends to 0 when n tends to infinity. Therefore

$$f(ab) = f(a)b + ag(b).$$

$\square$

Corollary 2.8. *Let $\varepsilon, \delta, p, q$ be non-negative real numbers with $0 < p, q < 1$. If $\mathcal{R}$ is a normed ring with the identity e, $\mathcal{M}$ is unit linked and $f, g : \mathcal{R} \to \mathcal{M}$ are mappings satisfy the inequalities*

$$\|f(ab) - f(a)b - ag(b)\| \leq \delta + \varepsilon(\|a\|^p + \|b\|^q)$$
$$\|g(ab) - g(a)b - ag(b)\| \leq \delta + \varepsilon(\|a\|^p + \|b\|^q)$$

for all $a, b \in \mathcal{R}$, then f and g satisfy (2.28) for all $a, b \in \mathcal{R}$.

References

[1] T. Aoki, *On the stability of the linear transformationin Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] R. Badora, *On approximate derivations,* Math. Inequal. Appl. **9** (2006), 167–173.

[3] M. Brešar, *Jordan derivations on semiprime rings,* Proc. Amer. Math. Soc. **104** (1988), 1003–1006.

[4] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: A fixed point approach,* Grazer Math. Ber. **346** (2004), 43–52.

[5] J. Cusak, *Jordan derivations on rings,* Proc. Amer. Math. Soc. **53** (1975), 321–324.

[6] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Co. Inc., River Edge, NJ, 2002.

[7] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings,* J. Math. Anal. Appl. **184** (1994), 431–436.

[8] O. Hatori and J. Wada, *Ring derivations on semi-simple commutative Banach algebras,* Tokyo J. Math. **15** (1992), 223–229.

[9] I. N. Herstein, *Jordan derivations of prime rings,* Proc. Amer. Math. Soc. **8** (1957), 1104–1119.

[10] D.H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[11] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[12] D.H. Hyers, G. Isac and Th.M. Rassias, *On the asymptoticity aspect of Hyers–Ulam stability of mappings,* Proc. Amer. Math. Soc. **126** (1998), 425–430.

[13] G. Isac, Th.M. Rassias, *On the Hyers–Ulam stability of ψ-additive mappings,* J. Approx. Theory **72** (1993), 131–137.

[14] G. Isac, Th.M. Rassias, *Stability of Ψ-additive mappings: Applications to nonlinear analysis,* Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[15] W. Jing and S. Lu, *Generalized Jordan derivations on prime rings and standard operator algebras,* Taiwanese J. Math. **7** (2003), 605–613.

[16] B.E. Johnson, *Continuity of derivations on commutative algebras,* Amer. J. Math. **91** (1969) 1–10.

[17] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.

[18] S.-M. Jung and T.-S. Kim, *A fixed point approach to stability of cubic functional equation*, Bol. Soc. Mat. Mexicana **12** (2006), 51–57.
[19] B. Margolis and J.B. Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space,* Bull. Amer. Math. Soc. **74** (1968), 305–309.
[20] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
[21] T. Miura, G. Hirasawa and S.-E. Takahasi, *A perturbation of ring derivations on Banach algebras,* J. Math. Anal. Appl. **319** (2006) 522–530.
[22] M.S. Moslehian , *Hyers–Ulam–Rassias stability of generalized derivations*, Int. J. Math. Math. Sci. (2006), Article ID 93942, Pages 1–8.
[23] A. Najati, *Hyers–Ulam stability of an n-Apollonius type quadratic mapping,* Bull. Belgian Math. Soc. Simon-Stevin **14** (2007), 755–774.
[24] A. Najati, *On the stability of a quartic functional equation,* J. Math. Anal. Appl. **340** (2008), 569–574.
[25] A. Najati and M.B. Moghimi, *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces,* J. Math. Anal. Appl. **337** (2008), 399–415.
[26] A. Najati and C. Park, *Hyers–Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation,* J. Math. Anal. Appl. **335** (2007), 763–778.
[27] C. Park, *On the stability of the linear mapping in Banach modules,* J. Math. Anal. Appl. **275** (2002), 711–720.
[28] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc. **72** (1978), 297–300.
[29] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers Co., Dordrecht, Boston, London, 2003.
[30] P. Šemrl, *The functional equation of multiplicative derivation is superstable on standard operator algebras,* Integr. Equat. Oper. Theory **18** (1994), 118–122.
[31] P. Šemrl, *On ring derivations and quadratic functionals,* Aequationes Math. **42** (1991), 80–84.
[32] I.M. Singer and J. Wermer, *Derivations on commutative normed algebras,* Math. Ann. **129** (1955) 260–264.
[33] M.P. Thomas, *The image of a derivation is contained in the radical,* Ann. of Math. **128** (1988), 435–460.
[34] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[35] J. Vukman, *A note on generalized derivations of semiprime rings,* Taiwanese J. Math. **11** (2007), 367–370.