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COUPLED FIXED POINT THEOREMS IN S_b-METRIC SPACES

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Abstract. Bhaskar and Lakshmikantham [1], Lakshmikantham and Ćiric [3] introduced the concept of a coupled coincidence point of a mapping F from $X \times X$ into X and a mapping g from X into X. In this paper, we introduce S_b-metric spaces and give some of their properties. Also we prove a coupled coincidence fixed point theorem in S_b-metric spaces. Using a similar method as in [2] coupled fixed point theorems in G_b-metric spaces is obtained in S_b-metric spaces. One example is presented to verify the effectiveness and applicability of our main result.

1. Introduction

The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces with a partially order have been studied by meny authors. Bhaskar and Lakshmikantham [1], Lakshmikantham and Ćiric [3] introduced the concept of a coupled coincidence point and studied the problems of a uniqueness of a coupled fixed point in partially ordered metric spaces. They applied their theorems to problems of the existence of solution for periodic boundary value problem. Lakshmikantham [1], Lakshmikantham

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and Ćiric [3] established some coincidence and common coupled fixed point theorems under nonlinear contractions in partially ordered metric spaces.

In this paper, we prove a coupled coincidence fixed point theorem in S_b-metric spaces.

First we recall some notions, lemmas and examples which will be useful later.

Definition 1.1. ([6]) Let X be a nonempty set. A S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$,

(S1) $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,
(S2) $S(x, y, z) = 0$ if $x = y = z$,
(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space.

Example 1.1. ([6]) Let $X = \mathbb{R}^2$ and d be an ordinary metric on X. Put $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given x, y, z. Then S is an S-metric on X.

Lemma 1.1. ([5]) *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Definition 1.2. ([7]) Let (X, S) be an S-metric space and $A \subseteq X$.

(1) If for every $x \in X$ there exists $r > 0$ such that $B_s(x, r) \subseteq A$, then the subset A is called open subset of X.
(2) Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.
(3) A sequence $\{x_n\}$ in X convergents to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote by $\lim_{n \longrightarrow \infty} x_n = x$.
(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$.
(5) The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.
(6) Let τ be the of all $A \subseteq X$ which $x \in A$ if and only if there exists $r > 0$ such that $B_s(x, r) \subseteq A$. Then τ is a topology on X.

Lemma 1.2. ([7]) *Let (X, S) be an S-metric space. If there exist sequence $\{x_n\}, \{y_n\}$ such that $\lim_{n \longrightarrow \infty} x_n = x$ and $\lim_{n \longrightarrow \infty} y_n = y$, then*

$$\lim_{n \longrightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Lemma 1.3. ([2]) *Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$

and

$$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$$

for all $x, y, z \in X$.

Following we give our definitions and examples of S_b-metric spaces.

Definition 1.3. Let X be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $S : X^3 \to [0, \infty)$ satisfies:

(S$_b$1) $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,
(S$_b$2) $S(x, y, z) = 0$ if $x = y = z$,
(S$_b$3) $S(x, y, z) \leq b(S(x, x, a) + S(y, y, a) + S(z, z, a))$ for all $x, y, z, a \in X$.

Then S is called a S_b-metric and the pair (X, S) is called a S_b-metric space.

It should be noted that, the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is a S_b-metric space with $b = 1$.

Following example shows that a S_b-metric on X need not be a S-metric on X.

Example 1.2. Let (X, S) be a S-metric space and $S_*(x, y, z) = S(x, y, z)^p$, where $p > 1$ is a real number. Note that S_* is a S_b-metric with $b = 2^{2(p-1)}$. Obviously, S_* satisfies condition $(S_b1), (S_b2)$ of Definition 1.3, so it suffice to show (S_b3) holds. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$, $(x > 0)$ implies that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$. Thus, for each $x, y, z, a \in X$, we obtain

$$\begin{aligned}
S_*(x, y, z) &= S(x, y, z)^p \\
&\leq ([S(x, x, a) + S(y, y, a)] + S(z, z, a))^p \\
&\leq 2^{p-1}([S(x, x, a) + S(y, y, a)]^p + S(z, z, a)^p) \\
&\leq 2^{p-1}(2^{p-1}(S(x, x, a)^p + S(y, y, a)^p) + S(z, z, a)^p) \\
&\leq 2^{2(p-1)}(S(x, x, a)^p + S(y, y, a)^p) + 2^{p-1}S(z, z, a)^p \\
&\leq 2^{2(p-1)}(S(x, x, a)^p + S(y, y, a)^p + S(z, z, a)^p) \\
&\leq 2^{2(p-1)}(S_*(x, x, a) + S_*(y, y, a) + S_*(z, z, a))
\end{aligned}$$

so, S_* is a S_b-metric with $b = 2^{2(p-1)}$.

Also in the above example, (X, S_*) is not necessarily a S-metric space. For example, let $X = \mathbb{R}$, $S_*(x, y, z) = (|y + z - 2x| + |y - z|)^2$ is a S_b-metric on $\mathbb{R}$, with $p = 2$, $b = 2^{2(2-1)} = 4$, for all $x, y, z \in \mathbb{R}$. But it is not a S-metric on $\mathbb{R}$.

To see this, let $x=3$, $y=5$, $z=7$, $a=\frac{7}{2}$. Hence, we get

$$\begin{aligned} S_*(3,5,7) &= (|5+7-6|+|5-7|)^2 = 8^2 = 64,\\ S_*\left(3,3,\frac{7}{2}\right) &= \left(\left|3+\frac{7}{2}-6\right|+\left|3-\frac{7}{2}\right|\right)^2 = 1^2 = 1,\\ S_*\left(5,5,\frac{7}{2}\right) &= \left(\left|5+\frac{7}{2}-10\right|+\left|5-\frac{7}{2}\right|\right)^2 = 3^2 = 9,\\ S_*\left(7,7,\frac{7}{2}\right) &= \left(\left|7+\frac{7}{2}-14\right|+\left|7-\frac{7}{2}\right|\right)^2 = 7^2 = 49. \end{aligned}$$

Therefore,

$$\begin{aligned} S_*(3,5,7) &= 64\\ &\not\leq 59 = S_*\left(3,3,\frac{7}{2}\right)+S_*\left(5,5,\frac{7}{2}\right)+S_*\left(7,7,\frac{7}{2}\right). \end{aligned}$$

Now we present some definitions and propositions in S_b-metric space.

Definition 1.4. Let (X,S) be a S_b-metric space. Then, for $x\in X$, $r>0$ we defined the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r as follows respectively:

$$\begin{aligned} B_S(x,r) &= \{y\in X : S(y,y,x) < r\},\\ B_S[x,r] &= \{y\in X : S(y,y,x) \leq r\}. \end{aligned}$$

Example 1.3. Let $X=\mathbb{R}$. Denote $S(x,y,z)=(|y+z-2x|+|y-z|)^2$ is a S_b-metric on $\mathbb{R}$ with $b=2^{2(2-1)}=4$, for all $x,y,z\in\mathbb{R}$. Thus

$$\begin{aligned} B_S(1,2) &= \{y\in\mathbb{R} : S(y,y,1) < 2\}\\ &= \left\{y\in\mathbb{R} : |y-1| < \frac{\sqrt{2}}{2}\right\}\\ &= \left\{y\in\mathbb{R} : 1-\frac{\sqrt{2}}{2} < y < 1+\frac{\sqrt{2}}{2}\right\}\\ &= \left(1-\frac{\sqrt{2}}{2}, 1+\frac{\sqrt{2}}{2}\right). \end{aligned}$$

Lemma 1.4. *In an S_b-metric space, we have*

$$S(x,x,y) \leq bS(y,y,x)$$

and

$$S(y,y,x) \leq bS(x,x,y).$$

Proof. By third condition of S_b-metric, we have

$$\begin{aligned} S(x,x,y) &\leq b(2S(x,x,x)+S(y,y,x)) \\ &= bS(y,y,x) \end{aligned}$$

and similarly

$$\begin{aligned} S(y,y,x) &\leq b(2S(y,y,y)+S(x,x,y)) \\ &= bS(x,x,y). \end{aligned}$$

□

Lemma 1.5. *Let (X,S) be an S_b-metric space. Then*

$$S(x,x,z) \leq 2bS(x,x,y)+bS(z,z,y)$$

and

$$S(x,x,z) \leq 2bS(x,x,y)+b^2S(y,y,z).$$

Proof. By third condition of S_b-metric, we have

$$\begin{aligned} S(x,x,z) &\leq b(S(x,x,y)+S(x,x,y)+S(z,z,y)) \\ &= 2bS(x,x,y)+bS(z,z,y) \end{aligned}$$

and

$$\begin{aligned} S(x,x,z) &\leq b(S(x,x,y)+S(x,x,y)+S(z,z,y)) \\ &\leq b(2S(x,x,y)+bS(y,y,z)) \\ &= 2bS(x,x,y)+b^2S(y,y,z). \end{aligned}$$

□

Definition 1.5. Let (X,S) be a S_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) S_b-Cauchy sequence if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $m,n \geq n_0$.

(2) S_b-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that $S(x_n,x_n,x) < \varepsilon$ or $S(x,x,x_n) < \varepsilon$ for all $n \geq n_0$ and we denote by $\lim_{n\longrightarrow\infty} x_n = x$.

Definition 1.6. A S_b-metric space (X,S) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Definition 1.7. ([3]) Let X be a nonempty set. Then we say that the mappings $F : X\times X \to X$ and $g : X \to X$ are commutative if $gF(x,y) = F(gx,gy)$.

Definition 1.8. ([3]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.9. Let (X, S) and (X', S') be S_b-metric spaces and let $f : (X, S) \to (X', S')$ be a function. Then f is said to be continuous at a point $a \in X$ if and only if for every sequence x_n in X, $S(x_n, x_n, a) \to 0$ implies $S'(f(x_n), f(x_n), f(a)) \to 0$. A function f is continuous at X if and only if it is continuous at all $a \in X$.

Definition 1.10. ([1]) An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

2. Common fixed point results

Let Φ denote the class of all functions $\phi : R^+ \to R^+$ such that ϕ is increasing, continuous, $\phi(t) < \frac{t}{2}$ for all $t > 0$ and $\phi(0) = 0$.
It is easy to see that for every $\phi \in \Phi$, we can choose $0 < k < \frac{1}{2}$ such that $\phi(t) \leq kt$. For example $\phi(t) = kt$ for $k \in (0, \frac{1}{2})$.

We start our work by proving the following two crucial lemmas.

Lemma 2.1. *Let (X, S) be a S_b-metric space with $b \geq 1$ and suppose that $\{x_n\}$ is a S_b-convergent to x, then we have*

$$\frac{1}{b^2} S(x, x, y) \leq \liminf_{n \longrightarrow \infty} S(x_n, x_n, y) \leq \limsup_{n \longrightarrow \infty} S(x_n, x_n, y) \leq b^2 S(x, x, y).$$

In particular, if $x = y$, then we have $\lim_{n \longrightarrow \infty} S(x_n, x_n, y) = 0$.

Proof. Using the condition $(S_b 3)$ of Definition 1.3 in (X, S), it is easy to see that

$$S(x_n, x_n, y) \leq 2bS(x_n, x_n, x) + b^2 S(x, x, y)$$

and

$$\frac{1}{b^2} S(x, x, y) \leq 2S(x_n, x_n, x) + S(x_n, x_n, y).$$

Taking the upper limit as $n \to \infty$ in the first inequality and the lower limit as $n \to \infty$ in the second inequality we obtain the desired result. □

Lemma 2.2. *Let (X, S) be a S_b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$S(F(x, y), F(u, v), F(z, w)) \leq \phi(S(gx, gu, gz) + S(gy, gv, gw)) \tag{2.1}$$

for some $\phi \in \Phi$ and for all $x, y, z, w, u, v \in X$. Assume that (x, y) is a coupled coincidence point of the mappings F and g. Then

$$F(x, y) = gx = gy = F(y, x).$$

Proof. Since (x, y) is a coupled coincidence point of the mappings F and g, we have $gx = F(x, y)$ and $gy = F(y, x)$. Assume $gx \neq gy$. Then by (2.1), we get

$$\begin{aligned} S(gx, gx, gy) &= S(F(x, y), F(x, y), F(y, x)) \\ &\leq \phi(S(gx, gx, gy) + S(gy, gy, gx)). \end{aligned}$$

Also by (2.1), we have

$$\begin{aligned} S(gy, gy, gx) &= S(F(y, x), F(y, x), F(x, y)) \\ &\leq \phi(S(gy, gy, gx) + S(gx, gx, gy)). \end{aligned}$$

Therefore

$$S(gx, gx, gy) + S(gy, gy, gx) \leq 2\phi(S(gx, gx, gy) + S(gy, gy, gx)).$$

Since $\phi(t) < \frac{t}{2}$, we get

$$S(gx, gx, gy) + S(gy, gy, gx) < S(gx, gx, gy) + S(gy, gy, gx),$$

which is a contradiction. So $gx = gy$, and hence

$$F(x, y) = gx = gy = F(y, x).$$

□

The following is the main result of this section.

Theorem 2.1. *Let (X, S) be a complete S_b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$S(F(x, y), F(u, v), F(z, w)) \leq \frac{1}{b^3}\phi(S(gx, gu, gz) + S(gy, gv, gw)) \qquad (2.2)$$

for all $x, y, z, w, u, v \in X$. Assume that F and g satisfy the following conditions:

(1) *$F(X \times X) \subseteq g(X)$,*
(2) *$g(X)$ is complete, and*
(3) *g is continuous and commutes with F.*

If $\phi \in \Phi$, then there is a unique x in X such that $gx = F(x, x) = x$.

Proof. Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X

such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. For $n \in \mathbb{N} \cup \{\mathbf{0}\}$, by (2.2) we have

$$\begin{aligned}
&S(gx_{n-1}, gx_{n-1}, gx_n) \\
&= S(F(x_{n-2}, y_{n-2}), F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1})) \\
&\le \frac{1}{b^3}\phi(S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})).
\end{aligned}$$

Similarly by (2.2) we have

$$\begin{aligned}
&S(gy_{n-1}, gy_{n-1}, gy_n) \\
&= S(F(y_{n-2}, x_{n-2}), F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1})) \\
&\le \frac{1}{b^3}\phi(S(gy_{n-2}, gy_{n-2}, gy_{n-1}) + S(gx_{n-2}, gx_{n-2}, gx_{n-1})).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
a_n &= S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) \\
&\le \frac{2}{b^3}\phi(S(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S(gy_{n-2}, gy_{n-2}, gy_{n-1})) \\
&= \frac{2}{b^3}\phi(a_{n-1})
\end{aligned}$$

holds for all $n \in \mathbb{N}$. Thus, we get $0 < k < \frac{1}{2}$ such that

$$a_n \le \frac{2}{b^3}\phi(a_{n-1}) \le \frac{2k}{b^3}a_{n-1} \le \frac{2k}{b}a_{n-1} = qa_{n-1},$$

for $q = \frac{2k}{b}$. Hence we have

$$a_n \le \frac{2k}{b}a_{n-1} \le \cdots \le \left(\frac{2k}{b}\right)^n a_0.$$

Let $m, n \in \mathbb{N}$ with $m > n$. By Axioms (S_b3) of Definition 1.3 of S_b-metric spaces, we have

$$\begin{aligned}
&S(gx_{n-1}, gx_{n-1}, gx_m) + S(gy_{n-1}, gy_{n-1}, gy_m) \\
&\le b(2S(gx_{n-1}, gx_{n-1}, gx_n) + S(gx_m, gx_m, gx_n)) \\
&\quad + b(2S(gy_{n-1}, gy_{n-1}, gy_n) + S(gy_m, gy_m, gy_n)) \\
&= 2b(S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)) \\
&\quad + b(S(gx_m, gx_m, gx_n) + S(gy_m, gy_m, gy_n)) \\
&\le \cdots
\end{aligned}$$

$$\le 2(ba_n + b^2a_{n+1} + b^3a_{n+2} + \cdots + b^{m-n}a_{m-1} + b^{m-n}a_m)$$
$$\le 2(bq^n a_0 + b^2q^{n+1}a_0 + \cdots + b^{m-n}q^{m-1}a_0 + b^{m-n}q^m a_0)$$
$$\le 2(bq^n a_0(1 + bq + b^2q^2 + \cdots))$$
$$\le \frac{2bq^n a_0}{1-bq} \longrightarrow 0,$$

since $bq = 2k < 1$. Thus (gx_n) and (gy_n) are S_b-Cauchy in $g(X)$. Since $g(X)$ is complete, we get (gx_n) and (gy_n) are S_b-convergent to some $x \in X$ and $y \in X$ respectively. Since g is continuous, we have (ggx_n) is S_b-convergent to gx and (ggy_n) is S_b-convergent to gy. Also, since g and F are commute, we have

$$ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n)$$

and

$$ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

Thus

$$\begin{aligned} S(ggx_{n+1}, ggx_{n+1}, F(x,y)) &= S(F(gx_n, gy_n), F(gx_n, gy_n), F(x,y)) \\ &\le \frac{1}{b^3}\phi(S(ggx_n, ggx_n, gx) + S(ggy_n, ggy_n, gy)). \end{aligned}$$

Taking $\limsup\limits_{n\longrightarrow\infty}$, and using the Lemma 2.1, we get that

$$\begin{aligned} \frac{1}{b^2}S(gx, gx, F(x,y)) &\le \limsup_{n\longrightarrow\infty} S(F(gx_n, gy_n), F(gx_n, gy_n), F(x,y)) \\ &\le \limsup_{n\longrightarrow\infty} \frac{1}{b^3}\phi(S(ggx_n, ggx_n, gx) + S(ggy_n, ggy_n, gy)) \\ &\le \frac{1}{b^3}\phi(b^2(S(gx, gx, gx) + S(gy, gy, gy)) = 0. \end{aligned}$$

Hence $gx = F(x,y)$. Similarly, we may show that $gy = F(y,x)$. By Lemma 2.2, (x,y) is coupled fixed point of the mappings F and g. So

$$gx = F(x,y) = F(y,x) = gy.$$

Thus, using the Lemma 2.1, we have

$$\begin{aligned} \frac{1}{b^2}S(x, x, gx) &\le \limsup_{n\longrightarrow\infty} S(gx_{n+1}, gx_{n+1}, gx) \\ &= \limsup_{n\longrightarrow\infty} S(F(x_n, y_n), F(x_n, y_n), F(x,y)) \\ &\le \limsup_{n\longrightarrow\infty} \frac{1}{b^3}\phi(S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)) \\ &\le \frac{1}{b^3}\phi(b^2(S(x, x, gx) + S(y, y, gy))). \end{aligned}$$

Hence, we get

$$S(x,x,gx) \leq \frac{1}{b}\phi(b(S(x,x,gx)+S(y,y,gy))).$$

Similarly, we may show that

$$S(y,y,gy) \leq \frac{1}{b}\phi(b(S(y,y,gy)+S(x,x,gx))).$$

Thus

$$\begin{aligned} S(x,x,gx)+S(y,y,gy) &\leq \frac{2}{b}\phi(b(S(x,x,gx)+S(y,y,gy))) \\ &\leq 2kS(x,x,gx)+S(y,y,gy). \end{aligned}$$

Since $2k<1$, the last inequality happened only if $S(x,x,gx)=0$ and $S(y,y,gy)=0$. Hence $x=gx$ and $y=gy$. Thus we get

$$gx = F(x,x) = x.$$

To prove the uniqueness, let $z \in X$ with $z \neq x$ such that

$$z = gz = F(z,z).$$

Then

$$\begin{aligned} S(x,x,z) &= S(F(x,x),F(x,x),F(z,z)) \\ &\leq \frac{1}{b^3}\phi(2S(gx,gx,gz)) \\ &< \frac{1}{b^3}2kS(x,x,z) \leq 2kS(x,x,z). \end{aligned}$$

Since $2k<1$, we get $S(x,x,z)<S(x,x,z)$, which is a contradiction. Thus F and g have a unique common fixed point. □

Corollary 2.1. *Let (X,S) be a S_b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$S(F(x,y),F(x,y),F(u,v)) \leq \frac{k}{b^3}(S(gx,gx,gu)+S(gy,g,gv)) \tag{2.3}$$

for all $x,y,u,v \in X$. Assume F and g satisfy the following conditions:

(1) *$F(X \times X) \subseteq g(X)$,*
(2) *$g(X)$ is complete, and*
(3) *g is continuous and commutes with F.*

If $k \in (0,\frac{1}{2})$, then there is a unique x in X such that $gx = F(x,x) = x$.

Proof. Follows from Theorem 2.1, by taking $z=u$, $v=w$ and $\phi(t)=kt$. □

Corollary 2.2. *Let (X, S) be a complete S_b-metric space. Let $F : X \times X \to X$ be a mapping such that*

$$S(F(x,y), F(x,y), F(u,v)) \leq \frac{k}{b^3}(S(x,x,u) + S(y,y,v))$$

for all $x, y, u, v \in X$. If $k \in [0, \frac{1}{2})$, then there is a unique x in X such that $F(x,x) = x$.

Now, we introduce some examples of our theorem.

Example 2.1. Let $X = [0,1]$. Define $S : X^3 \to \mathbb{R}^+$ by

$$S(x,y,z) = (|y+z-2x| + |y-z|)^2$$

for all $x, y, z \in X$. Then (X, S) is a complete S_b-metric space with $b = 4$, according to Example 1.1. Define a map $F : X \times X \to X$ by $F(x,y) = \dfrac{x}{128} + \dfrac{y}{256}$ for $x, y \in X$. Also, define $g : X \to X$ by $g(x) = \dfrac{x}{4}$ for $x \in X$ and $\phi(t) = \dfrac{t}{4}$ for $t \in R^+$. We have

$$\begin{aligned}
&S(F(x,y), F(u,v), F(z,w)) \\
&= (|F(u,v) + F(z,w) - 2F(x,y)| + |F(u,v) - F(z,w)|)^2 \\
&= \left(\left| \frac{u}{128} + \frac{v}{256} + \frac{z}{128} + \frac{w}{256} - \frac{2x}{128} - \frac{2y}{256} \right| + \left| \frac{u}{128} + \frac{v}{256} - \frac{z}{128} - \frac{w}{256} \right| \right)^2 \\
&\leq \left(\frac{1}{128}|u+z-2x| + \frac{1}{256}|v+w-2y| + \frac{1}{128}|u-z| + \frac{1}{256}|v-w| \right)^2 \\
&= \left(\frac{1}{128}(|u+z-2x| + |u-z|) + \frac{1}{256}(|v+w-2y| + |v-w|) \right)^2 \\
&= \left(\frac{1}{32}\left(\left| \frac{u}{4} + \frac{z}{4} - \frac{2x}{4} \right| + \left| \frac{u}{4} - \frac{z}{4} \right| \right) + \frac{1}{64}\left(\left| \frac{v}{4} + \frac{w}{4} - \frac{2y}{4} \right| + \left| \frac{v}{4} - \frac{w}{4} \right| \right) \right)^2 \\
&\leq \frac{2}{32^2}\left(\left| \frac{u}{4} + \frac{z}{4} - \frac{2x}{4} \right| + \left| \frac{u}{4} - \frac{z}{4} \right| \right)^2 + \frac{2}{64^2}\left(\left| \frac{v}{4} + \frac{w}{4} - \frac{2y}{4} \right| + \left| \frac{v}{4} - \frac{w}{4} \right| \right)^2 \\
&= \frac{2}{32^2} S(gx, gu, gz) + \frac{2}{64^2} S(gy, gv, gw) \\
&\leq \frac{2}{32^2}(S(gx, gu, gz) + S(gy, gv, gw)) \\
&\leq \frac{1}{64} \frac{S(gx, gu, gz) + S(gy, gv, gw)}{4} \\
&= \frac{1}{4^3}\phi(S(gx, gu, gz) + S(gy, gv, gw))
\end{aligned}$$

holds for all $x, y.u, v, z, w \in X$. It is easy to see that F and g satisfy all the hypothesis of Theorem 2.1. Thus F and g have a unique common fixed point. Here $F(0,0) = g(0) = 0$.

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