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# ON THE HUT-IRT STABILITY AND SUPERSTABILITY OF A GENERALIZED ADDITIVE FUNCTIONAL EQUATION IN QUASI-BANACH SPACES

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**Abstract.** In this paper, we investigate the generalized Hyers-Ulam type (briefly, HUT) and Isac-Rassias type (briefly, IRT) stability and super stability of the following functional equation

$$\sum_{i=1}^{m} f\left(x_i + \frac{1}{m}\sum_{j=1,j\neq i}^{m} x_j\right) + f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} x_i\right)$$

for a fixed positive integer $m$ with $m \geq 2$ in quasi-Banach spaces.

## 1. Introduction and preliminaries

It is of interest to consider the concept of stability for a functional equation arising when we replace the functional equation by an inequality which acts as a perturbation of the equation.

The first stability problem was raised by Ulam [20] during his talk at the University of Wisconsin in 1940. The stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we would say that the equation is stable.

In 1941, Hyers [9] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces

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such that

$$\|f(x+y)-f(x)-f(y)\| \leq \delta,$$

for all $x, y \in E$ and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x)-T(x)\| \leq \delta,$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then $T$ is linear. Aoki [1], Bourgin [3] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [16] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. It was shown by Gajda [6], as well as by Rassias and Šemrl [19] that one cannot prove a stability theorem of the additive equation for a specific function. Găvruta [7] obtained generalized result of Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function. Isac and Rassias [10] generalized the Hyers' theorem by introducing a mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ subject to the conditions:

(1) $\lim\limits_{t\to\infty} \frac{\psi(t)}{t} = 0$,
(2) $\psi(ts) \leq \psi(t)\psi(s);\ \ s, t > 0$,
(3) $\psi(t) < t;\ \ t > 1$.

These stability results can be applied in stochastic analysis [11], financial and actuarial mathematics, as well as in psychology and sociology.

In 1987, Gajda and Ger [6] showed that one can get analogous stability results for subadditive multifunctions. In 1978, Gruber [8] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. We refer the readers to [5], [10]-[19] and references therein for more detailed results on the stability problems of various functional equations.

We recall some basic facts concerning quasi-Banach space. A quasi-norm is a real-valued function on $X$ satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda.x\| = |\lambda|.\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(3) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|.\|)$ is called a quasi-normed space if $\|.\|$ is a quasi-norm on $X$. A quasi-Banach space is a complete quasi-normed space. A quasi-norm $\|.\|$ is called a $p$-norm $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a $p$-Banach space. Given a $p$-norm, the formula $d(x,y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem (see [2]), each quasi-norm is equivalent to some $p$-norm. Since it is much easier to work with $p$-norms, henceforth we restrict our attention mainly to $p$-norms.

In this paper, we consider the generalized Hyers–Ulam stability of the following functional equation

$$\sum_{i=1}^{m} f\left(x_i + \frac{1}{m}\sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} x_i\right) \tag{1.1}$$

for a fixed positive integer $m$ with $m \geq 2$ in quasi-Banach spaces.

Throughout this paper, assume that $X$ is a quasi-normed space with quasi-norm $\|.\|_X$ and that $Y$ is a $p$-Banach space with $p$-norm $\|.\|_Y$.

## 2. Stability of functional equation (1.1) in quasi Banach spaces

Throught this section, using direct and fixed point methods, we investigate the stability of functional equation (1.1) inquasi Banach spaces.

**Theorem 2.1.** *Let $V$ and $W$ be real vector spaces. A mapping $f : V \to W$ satisfies in* (1.1) *if and only if $f$ is additive.*

*Proof.* Let $f : V \to W$ satisfies (1.1). Substituting $x_1 = x$ and $x_j = 0$ for all $2 \leq j \leq m$ in (1.1), we get

$$f(mx) = mf(x) \tag{2.1}$$

for all $x \in X$. Replacing $x_1 = x$ and $x_j$ by $\frac{y}{m-1}$ for all $2 \leq j \leq m$ in (1.1) and using (2.1), we obtain

$$f(mx + y) + (m-1)f(x + 2y) = (2m-1)f(x + y) \tag{2.2}$$

for all $x, y \in X$. Putting $x_1 = x$, $x_2 = y$ and $x_j = 0$ $(3 \leq j \leq m)$ in (1.1) and using (2.1), we get

$$f(mx + y) + f(x + my) = (m+1)f(x + y) \tag{2.3}$$

for all $x, y \in X$. Therefore, it follows from (2.2) and (2.3) that

$$f(x + my) - (m-1)f(x + 2y) = (2-m)f(x + y) \tag{2.4}$$

for all $x, y \in X$. Replacing $x$ and $y$ by $y$ and $x$, respectively, in (2.2), we have

$$f(x + my) + (m-1)f(2x + y) = (2m-1)f(x + y) \tag{2.5}$$

for all $x, y \in X$. Using (2.4) and (2.5), we obtain

$$3f(x + y) = f(x + 2y) + f(2x + y) \tag{2.6}$$

for all $x, y \in X$. Setting $y = 0$ in (2.6), we get

$$f(2x) = 2f(x) \tag{2.7}$$

for all $x \in X$. Replacing $y$ by $x$ in (2.6) and using (2.7), we get

$$f(3x) = 3f(x) \tag{2.8}$$

for all $x \in X$. Replacing $x$ and $y$ by $\frac{2x-y}{3}$ and $\frac{2y-x}{3}$, respectively, in (2.6) and using (2.8), we get

$$f(x+y) = f(x) + f(y) \tag{2.9}$$

for all $x, y \in X$. So the mapping $f : V \to W$ is additive. Converse is obvious. $\square$

Now, we investigate the generalized Hyers-Ulam stability of functional equation (1.1) in quasi Banach spaces.

**Theorem 2.2.** *Let $\varphi : X^m \to [0, \infty)$ be a function satisfying*

$$\Phi(x) = \sum_{i=1}^{\infty} \left(\frac{1}{m}\right)^{ip} \varphi^p\Big(0, \cdots, \underbrace{m^i x}_{jth}, \cdots, 0\Big) < \infty \tag{2.10}$$

*for all $x \in X$ and*

$$\lim_{n\to\infty} \frac{\varphi(m^n x_1, \cdots, m^n x_m)}{m^n} = 0 \tag{2.11}$$

*for all $x_j \in X \quad (1 \le j \le m)$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\left\|\sum_{i=1}^{m} f\left(x_i + \frac{1}{m}\sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) - 2f\left(\sum_{i=1}^{m} x_i\right)\right\|_Y \le \varphi(x_1, \cdots, x_m), \tag{2.12}$$

*for all $x_j \in X$ $(1 \le j \le m)$. Then there exists a unique additive mapping $L$ defined by*

$$L(x) = \lim_{n\to\infty} \frac{1}{m^n} f(m^n x) \tag{2.13}$$

*for all $x \in X$ and the mapping $L : X \to Y$ satisfies the inequality*

$$\|f(x) - L(x)\| \le [\Phi(x)]^{\frac{1}{p}} \tag{2.14}$$

*for all $x \in X$.*

*Proof.* Putting $x_j = mx$ and $x_i = 0$ $(1 \le i \le m,\ i \neq j)$ in (2.12) and using $f(0) = 0$, we obtain

$$\|f(mx) - mf(x)\| \le \varphi\Big(0, \cdots, \underbrace{mx}_{jth}, \cdots, 0\Big) \tag{2.15}$$

for all $x \in X$. By a simple induction we can prove that

$$\left\| f(x) - \frac{f(m^n x)}{m^n} \right\|^p \leq \sum_{i=1}^{n} \left( \frac{1}{m} \right)^{ip} \varphi^p \Big( 0, \cdots, \underbrace{m^i x}_{jth}, \cdots, 0 \Big) \tag{2.16}$$

for all $x \in X$ and $n \in \mathbb{N}$. Thus

$$\left\| \frac{f(m^l x)}{m^l} - \frac{f(m^{l+n} x)}{m^{n+l}} \right\|^p \leq \sum_{i=1+l}^{n} \left( \frac{1}{m} \right)^{ip} \varphi^p \Big( 0, \cdots, \underbrace{m^i x}_{jth}, \cdots, 0 \Big) \tag{2.17}$$

for all $x \in X$ and all $l \in \mathbb{N}$ $(l \leq n)$. It follows from (2.17) and (2.10) that the sequence $\left\{ \frac{1}{m^n} f(m^n x) \right\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is complete, the sequence $\left\{ \frac{1}{m^n} f(m^n x) \right\}$ converges in $Y$ for all $x \in X$. Hence we can define the mapping $L : X \to Y$ by

$$L(x) = \lim_{n \to \infty} \frac{1}{m^n} f(m^n x) \tag{2.18}$$

for all $x \in X$. Letting $n \to \infty$ in (2.16), we obtain (2.14). Now we show that the mapping $L$ is additive. We conclude from (2.11), (2.12) and (2.18)

$$\begin{aligned}
&\left\| \sum_{i=1}^{m} L \left( x_i + \frac{1}{m} \sum_{j=1, j \neq i}^{m} x_j \right) + L \left( \frac{1}{m} \sum_{i=1}^{m} x_i \right) - 2L \left( \sum_{i=1}^{m} x_i \right) \right\|_Y \\
&= \lim_{n \to \infty} \frac{1}{m^n} \left\| \sum_{i=1}^{m} f \left( m^n x_i + m^{n-1} \sum_{j=1, j \neq i}^{m} x_j \right) \right. \\
&\qquad \left. + f \left( m^{n-1} \sum_{i=1}^{m} x_i \right) - 2f \left( \sum_{i=1}^{m} m^n x_i \right) \right\|_Y \\
&\leq \lim_{n \to \infty} \frac{\varphi(m^n x_1, \cdots, m^n x_m)}{m^n} = 0
\end{aligned}$$

for all $x_j \in G$ $(1 \leq j \leq m)$. So

$$\sum_{i=1}^{m} L \left( x_i + \frac{1}{m} \sum_{j=1, j \neq i}^{m} x_j \right) + L \left( \frac{1}{m} \sum_{i=1}^{m} x_i \right) = 2L \left( \sum_{i=1}^{m} x_i \right).$$

Hence by Theorem 2.1, the mapping $L : X \to Y$ is additive. Now we prove the uniqueness assertion of $L$, by this mean let $L' : X \to Y$ be another mapping satisfies (2.14). It follows from (2.14)

$$\begin{aligned}\|L(x)-L'(x)\|_Y &= \frac{\|L(m^kx)-L'(m^kx)\|_Y}{m^k}\\ &\le \frac{\|T(m^kx)-f(m^kx)\|_Y}{m^k}+\frac{\|f(m^kx)-T'(m^kx)\|_Y}{m^k}\\ &\le \frac{2(\Phi(m^kx))^{\frac{1}{p}}}{m^k}\\ &= 2\frac{1}{m^k}\left(\sum_{i=1}^{\infty}\left(\frac{1}{m}\right)^{ip}\varphi^p\Big(0,\cdots,\underbrace{m^ix}_{jth},\cdots,0\Big)\right)^{\frac{1}{p}}\\ &= 2\sum_{i=1+k}^{\infty}\left(\left(\frac{1}{m}\right)^{ip}\varphi^p\Big(0,\cdots,\underbrace{m^ix}_{jth},\cdots,0\Big)\right)^{\frac{1}{p}}\end{aligned}$$

for all $x\in X$. The right hand side tends to zero as $k\to\infty$, hence $L(x)=L'(x)$ for all $x\in X$. This show the uniqueness of $L$. □

**Corollary 2.3.** *Let $\theta, r_k(1\le k\le m)$ be non-negative real numbers such that $0<r_k<1$. Suppose that a mapping $f:X\to Y$ satisfies the inequality*

$$\begin{aligned}&\left\|\sum_{i=1}^{m} f\left(x_i+\frac{1}{m}\sum_{j=1,j\neq i}^{m}x_j\right)+f\left(\frac{1}{m}\sum_{i=1}^{m}x_i\right)-2f\left(\sum_{i=1}^{m}x_i\right)\right\|_Y\\ &\le \theta\sum_{k=1}^{m}\|x_k\|_X^{r_k}\end{aligned}$$

*for all $x_k\in X$ $(1\le k\le m)$. Then there exists a unique additive mapping $L:X\to Y$ such that*

$$\|f(x)-L(x)\|_Y\le\left\{\frac{m^{(1-r_j)p}}{m^{(1-r_j)p}-1}\right\}^{\frac{1}{p}}\theta\|x\|_X^{r_j}$$

*for all $x\in X$.*

*Proof.* In Theorem 2.2, let

$$\varphi(x_1,x_2,\cdots,x_m):=\theta\sum_{k=1}^{m}\|x_k\|_X^{r_k}$$

for all $x_k\in X$ $(1\le k\le m)$. □

The following corollary is Hyers-Ulam type (briefly, HUT) stability for the functional equation (1.1).

**Corollary 2.4.** *Let $\theta$ be non-negative real number. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{m} f\left(x_i + \frac{1}{m} \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) - 2f\left(\sum_{i=1}^{m} x_i\right)\right\|_Y \leq \theta$$

*for all $x_j \in X \quad (1 \leq j \leq m)$. Then there exists a unique additive mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \theta \left\{\frac{1}{m^p - 1}\right\}^{\frac{1}{p}}$$

*for all $x \in X$.*

*Proof.* In Theorem 2.2, let

$$\varphi(x_1, x_2, \cdots, x_m) := \theta$$

for all $x_i \in X$ $(1 \leq i \leq m)$. □

The following corollary is Isac-Rassias type (briefly, IRT) stability for the functional equation (1.1).

**Corollary 2.5.** *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping such that*

$$\begin{cases} \lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \\ \psi(ts) \leq \psi(t)\psi(s), \quad s, t > 0, \\ \psi(t) < t, \quad t > 1. \end{cases}$$

*Let $\theta, r_k$ $(1 \leq k \leq m)$ be non-negative real numbers. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{m} f\left(x_i + \frac{1}{m} \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) - 2f\left(\sum_{i=1}^{m} x_i\right)\right\|_Y$$
$$\leq \theta \sum_{k=1}^{m} \psi(\|x_k\|_X)$$

*for all $x_k \in X$ $(1 \leq k \leq m)$. Then there exists a unique additive mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{\theta\psi(m)\psi(\|x\|)}{m - \psi(m)}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.2 by taking

$$\varphi(x_1, \cdots, x_m) := \theta \sum_{k=1}^{m} \psi(\|x_k\|_X)$$

for all $x_k \in X$ $(1 \leq k \leq m)$. □

**Remark 2.6.** In Theorem 2.2, if we replace control function by $\theta \prod_{j=1}^{m} \|x_j\|^{r_j}$, then $L = f$. Therefore in this case, $f$ is superstable.

**Theorem 2.7.** *Let $\varphi : X^m \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} m^n \varphi\left(\frac{x_1}{m^n}, \cdots, \frac{x_m}{m^n}\right) = 0 \tag{2.19}$$

*for all $x_j \in X$ $(1 \leq j \leq m)$ and*

$$\Phi(x) = \sum_{i=0}^{\infty} m^{ip} \varphi^p\Big(0, \cdots, \underbrace{\frac{x}{m^i}}_{jth}, \cdots, 0\Big) < \infty \tag{2.20}$$

*for all $x \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\left\|\sum_{i=1}^{m} f\left(x_i + \frac{1}{m} \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) - 2f\left(\sum_{i=1}^{m} x_i\right)\right\|_Y \leq \varphi(x_1, \cdots, x_m) \tag{2.21}$$

*for all $x_j \in X$ $(1 \leq j \leq m)$. Then there exists a unique additive mapping $L$ defined by*

$$L(x) = \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right) \tag{2.22}$$

*for all $x \in X$ and the mapping $L : X \to Y$ satisfies the inequality*

$$\|f(x) - L(x)\| \leq [\Phi(x)]^{\frac{1}{p}} \tag{2.23}$$

*for all $x \in X$.*

*Proof.* Putting $x_j = mx$ and $x_i = 0$ $(1 \leq i \leq m, i \neq j)$ in (2.21) we obtain

$$\|f(mx) - mf(x)\| \leq \varphi\Big(0, \cdots, \underbrace{mx}_{jth}, \cdots, 0\Big) \tag{2.24}$$

for all $x \in X$. Replacing $x$ by $\frac{x}{m^{n+1}}$ in (2.24) and multiplying both sides of (2.24) to $m^n$, we get

$$\left\| m^n f\left(\frac{x}{m^n}\right) - m^{n+1} f\left(\frac{x}{m^{n+1}}\right)\right\| \leq m^n \varphi\Big(0, \cdots, \underbrace{\frac{x}{m^n}}_{jth}, \cdots, 0\Big)$$

for all $x \in X$ and all $n \in \mathbb{N} \cup \{0\}$. Since $Y$ is a $p$-Banach space, we have

$$\begin{aligned} \left\| m^{n+r} f\left(\frac{x}{m^{n+r}}\right) - m^r f\left(\frac{x}{m^r}\right)\right\|^p &\leq \sum_{i=r}^{n} \left\| m^{i+1} f\left(\frac{x}{m^{i+1}}\right) - m^i f\left(\frac{x}{m^i}\right)\right\| \\ &\leq \sum_{i=r}^{n} m^{ip} \varphi^p\Big(0, \cdots, \underbrace{\frac{x}{m^i}}_{jth}, \cdots, 0\Big) \qquad (2.25) \end{aligned}$$

for all $x \in X$ and all non-negative $n$ and $r$ with $n \geq r$. Therefore, we have from (2.20) and (2.25) that the sequence $\left\{m^n f\left(\frac{x}{m^n}\right)\right\}$ is a Cauchy in $Y$ for all $x \in X$. Because of $Y$ is complete, the sequence $\left\{m^n f\left(\frac{x}{m^n}\right)\right\}$ converges for all $x \in X$. Hence we can define the mapping $L : X \to Y$ by

$$L(x) = \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

for all $x \in X$. Putting $r = 0$ and passing the limit $n \to \infty$ in (2.25), we obtain (2.23). Showing the additivity and uniqueness of $L$ is similar to Theorem 2.2, and the proof is complete. □

**Corollary 2.8.** *Let $\theta, r_k$ $(1 \leq k \leq m)$ be non-negative real numbers such that $r_k > 1$ $(1 \leq k \leq m)$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=1}^{m} f\left(x_i + \frac{1}{m} \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) - 2f\left(\sum_{i=1}^{m} x_i\right)\right\|_Y$$
$$\leq \theta \sum_{j=1}^{m} \|x_k\|_X^{r_k},$$

*for all $x_k \in X$ $(1 \leq k \leq m)$. Then there exists a unique additive mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \left\{\frac{m^{(r_j-1)p}}{1 - m^{(r_j-1)p}}\right\}^{\frac{1}{p}} \theta \|x\|^{r_j}$$

*for all $x \in X$.*

*Proof.* In Theorem 2.7, let

$$\varphi(x_1, x_2, \cdots, x_m) := \theta \sum_{k=1}^{m} \|x_k\|_X^{r_k}$$

for all $x_k \in X$ $(1 \le k \le m)$. $\square$

**Remark 2.9.** We can formulate similar statement to Corollaries 2.4 and 2.5 for Theorem 2.7. Moreover, in Theorem 2.7, if we replace control function by $\theta \prod_{j=1}^{m} \|x_j\|^{r_j}$, then $L = f$. Therefore in this case, $f$ is superstable.

Now, we apply a fixed point method and prove the generalized Hyers-Ulam stability of functional equation (1.1).

We recall a fundamental result in fixed point theory.

**Theorem 2.10.** ([4]) *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $J : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschits constant $L \in (0, 1)$. Then, for a given element $x \in \mathcal{X}$, exactly one of the following assertions is true: either*

(1) *$d(J^n x, J^{n+1} x) = \infty$ for all $n \ge 0$*
*or*
(2) *there exists $n_0$ such that $d(J^n x, J^{n+1} x) < \infty$ for all $n \ge n_0$.*

*Actually, if $(a_2)$ holds, then the sequence $J^n x$ is convergent to a fixed point $x^*$ of $J$ and*

(3) *$x^*$ is the unique fixed point of $J$ in $\Lambda := \{y \in \mathcal{X}, d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, x^*) \le \frac{d(y,Jy)}{1-L}$ for all $y \in \Lambda$.*

**Theorem 2.11.** *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^m \to [0, \infty)$ such that*

$$\left\| \sum_{i=1}^{m} f\left( x_i + \frac{1}{m} \sum_{j=1, j \ne i}^{m} x_j \right) + f\left( \frac{1}{m} \sum_{i=1}^{m} x_i \right) - 2f\left( \sum_{i=1}^{m} x_i \right) \right\|_Y \le \varphi(x_1, \cdots, x_m) \tag{2.26}$$

*and*

$$\lim_{n\to\infty} m^n \varphi\left( \frac{x_1}{m^n}, \cdots, \frac{x_m}{m^n} \right) = 0 \tag{2.27}$$

*for all $x_j \in X$ $(1 \le j \le m)$. If there exists an $L < 1$ such that*

$$\varphi\left( \frac{x_1}{m}, \cdots, \frac{x_j}{m} \right) \le \frac{L\varphi(x_1, \cdots, x_j)}{m} \qquad (1 \le j \le m),$$

*then there exists a unique additive mapping $L : X \to Y$ satisfying*

$$\|f(x) - L(x)\| \leq \frac{\varphi(0, \cdots, \overbrace{x}^{jth}, \cdots, 0)}{1 - L} \tag{2.28}$$

*for all $x \in X$.*

*Proof.* Putting $x_j = mx$ and $x_i = 0$ $(1 \leq i \leq m)$, $i \neq j$ in (2.38), we obtain

$$\|f(mx) - mf(x)\| \leq \varphi\Big(0, \cdots, \underbrace{x}_{jth}, \cdots, 0\Big)$$

for all $x \in X$. Hence

$$\Big\|f(x) - mf\left(\frac{x}{m}\right)\Big\| \leq \varphi\Big(0, \cdots, \underbrace{x}_{jth}, \cdots, 0\Big) \tag{2.29}$$

for all $x \in X$. Let $E := \{g : X \to Y\}$. We introduce a generalized metric on $E$ as follows

$$d(g, h) := inf\Big\{\mu \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq \mu\varphi\Big(0, \cdots, \underbrace{x}_{jth}, \cdots, 0\Big), \ \forall x \in X\Big\}.$$

It is easy to show that $(E, d)$ is a generalized complete metric space.
Now we consider the mapping $J : E \to E$ defined by

$$J(g)(x) = mg\left(\frac{x}{m}\right)$$

for all $g \in E$ and all $x \in X$. Let $g, h \in E$ and let $\mu \in \mathbb{R}^+$ ba an arbitrary constant with $d(g, h) \leq \mu$. From the definition of $d$, we have

$$\|g(x) - h(x)\| \leq \mu\varphi(0, \cdots, \underbrace{x}_{jth}, \cdots, 0)$$

for all $x \in X$. By the assumption and last inequality, we have

$$\begin{aligned}
\|J(g)(x) - J(h)(x)\| &= m\Big\|g\left(\frac{x}{m}\right) - h\left(\frac{x}{m}\right)\Big\| \\
&\leq m\mu\varphi\left(0, ..., \overbrace{\frac{x}{m}}^{jth}, ..., 0\right) \\
&\leq \frac{mL\mu\varphi(0, ..., \underbrace{x}_{jth}, ...0)}{m} \\
&= L\mu\varphi(0, ..., \underbrace{x}_{jth}, ...0)
\end{aligned}$$

for all $x \in X$. So $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in E$. It follows from (2.41) that $d(Jf, f) \leq 1$. Therefore, according to Theorem 2.10, the sequence $\{J^n f\}$ converges to a fixed point $L$ of $J$, *i.e.*,

$$L : X \to Y, \qquad L(x) = \lim_{n\to\infty} (J^n f)(x) = \lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$$

and $L(x) = mL\left(\frac{x}{m}\right)$ for all $x \in X$. Also $L$ is the unique fixed point of $J$ in the set $E_\varphi = \{g \in E : \ d(f, g) < \infty\}$ and

$$d(T, f) \leq \frac{1}{1-L} d(Jf, f) \leq \frac{1}{1-L},$$

*i.e.*, inequality (2.30) holds for all $x \in X$. It follows from the definition of $L$, (2.26) and (2.27) that

$$\sum_{i=1}^{m} L\left(x_i + \frac{1}{m}\sum_{j=1, j\neq i}^{m} x_j\right) + L\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) = 2L\left(\sum_{i=1}^{m} x_i\right).$$

Hence by Theorem 2.1, the mapping $L : X \to Y$ is additive. $\square$

**Corollary 2.12.** *Let $r_j \in (1, \infty)$ $(1 \leq j \leq m)$ and $\theta$ be real numbers. Let $f : X \to Y$ such that*

$$\left\|\sum_{i=1}^{m} f\left(x_i + \frac{1}{m}\sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\frac{1}{m}\sum_{i=1}^{m} x_i\right) - 2f\left(\sum_{i=1}^{m} x_i\right)\right\|_Y \leq \theta \sum_{j=1}^{m} \|x_j\|^{r_j}$$

*for all $x_j \in X$ $(1 \leq j \leq m)$. Then there exists a unique additive mapping $L : X \to Y$ satisfies the inequality*

$$\|f(x) - L(x)\| \leq \frac{\theta \|x\|^{r_j}}{m^{r_j} - m}$$

*for all $x \in X$.*

*Proof.* Setting

$$\varphi(x_1, x_2, \cdots, x_m) := \theta \sum_{j=1}^{m} \|x_j\|^{r_j}$$

for all $x_j \in X \quad (1 \leq j \leq m)$ in Theorem 2.11. Then by $L = m^{1-r_j}$, we get the desired result. $\square$

Similarly, we have the following Theorem and we omit the proof.

**Theorem 2.13.** *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^m \to [0, \infty)$ satisfying (2.26) and (2.27). If there exists an $L < 1$ such that*

$$\varphi(mx_1, \cdots, mx_j) \le mL\varphi(x_1, \cdots, x_j) \qquad (1 \le j \le m),$$

*for all $x_j \in X$ $(1 \le j \le m)$. Then there exists a unique additive mapping $L : X \to Y$ satisfying*

$$\|f(x) - L(x)\| \le \frac{L\varphi(0, \cdots, \overbrace{x}^{jth}, \cdots, 0)}{1 - L} \tag{2.30}$$

*for all $x \in X$.*

**Corollary 2.14.** *Let $r_j \in (0,1)$ $(1 \le j \le m)$ and $\theta$ be real numbers. Let $f : X \to Y$ such that*

$$\left\| \sum_{i=1}^{m} f\left( x_i + \frac{1}{m} \sum_{j=1, j\neq i}^{m} x_j \right) + f\left( \frac{1}{m} \sum_{i=1}^{m} x_i \right) - 2f\left( \sum_{i=1}^{m} x_i \right) \right\|_Y$$
$$\le \theta \sum_{j=1}^{m} \|x_j\|^{r_j}$$

*for all $x_j \in X$ $(1 \le j \le m)$. Then there exists a unique additive mapping $L : X \to Y$ satisfies the inequality*

$$\|f(x) - L(x)\| \le \frac{m^{r_j}\theta\|x\|^{r_j}}{m - m^{r_j}}$$

*for all $x \in X$.*

*Proof.* Setting

$$\varphi(x_1, x_2, \cdots, x_m) := \theta \sum_{j=1}^{m} \|x_j\|^{r_j}$$

for all $x_j \in X$ $(1 \le j \le m)$ in Theorem 2.11. Then by $L = m^{r_j - 1}$, we get the desired result. □

**Remark 2.15.** We can formulate similar statements for stability of (1.1) on Banach spaces.

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