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GENERALIZED NONLINEAR VARIATIONAL INCLUSION SYSTEMS INVOLVING COCOERCIVELY MONOTONE MAPPINGS

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Abstract. Using the notion of the $H-$monotonicity (also referred to as $H-$maximal monotonicity in literature), the solvability of a system of nonlinear variational inclusion problems involving cocoercively monotone mappings (that unifies most of the existing notions relating to the strong monotonicity and beyond) based on the resolvent operator technique is explored. The obtained results are general in nature.

1. Introduction

Recently, Verma [8] examined the convergence of averaging techniques for relaxation algorithms and for their specializations as projection methods and auxiliary problem principle in the context of solving a class of variational inequalities involving cocoercively monotone mappings. There is a vast literature on globally convergent schemes for variational inequalities involving strongly monotone mappings. The notion of cocoercively monotone mappings is weaker and more inclusive than cocoercive and strongly monotone mappings.

Fang and Huang [1, 2] introduced the notion of the $H-$*monotonicity* in the context of solving some nonlinear inclusion systems. This notion impacted greatly the theory of maximal monotone mappings in several domains of applications. More importantly, $H-$maximal monotonicity can also be applied to first-order evolution equations based on the Yosida approximation. For

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more details on evolution equations as well as evolution inclusions, we recommend [9, 12]. In this communication, our aim is to generalize variational inclusion systems [2] to the case of cocoercively monotone mappings in Hilbert spaces. Moreover, the solvability for our system of nonlinear variational inclusions involving cocoercively monotone mappings is based on the resolvent operator technique. The obtained results seem to be general in nature and do have a wide range of applications, especially to optimization and control theory, and management and decision sciences.

Let X be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$.

Definition 1.1.([1]) Let $H : X \to X$ and $M : X \to 2^X$ be any mappings on X. The map M is said to be $H-$maximal monotone if:
(i) M is monotone.
(ii) $(H + \rho M)(X) = X$ for $\rho > 0$.

Note that if H is strictly monotone and M is $H-$*monotone*, then M is maximal monotone. Let the resolvent operator $J^{\rho}_{H,M} : X \to X$ be defined by

$$J^{\rho}_{H,M}(u) = (H + \rho M)^{-1}(u)\,\forall u \in X.$$

Definition 1.2. Let $T, A : X \to X$ be any mappings on X. The map T is said to be

(i) $(r)-$strongly monotone if there is a positive constant r such that

$$\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2 \;\; \forall x, y \in X.$$

(ii) $(m)-$relaxed monotone if there is a positive constant m such that

$$\langle T(x) - T(y), x - y\rangle \geq -m\|x - y\|^2 \;\; \forall x, y \in X.$$

(iii) (γ, r)-relaxed cocoercive if there exist constants $\gamma, r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq (-\gamma)\|T(x) - T(y)\|^2 + r\|x - y\|^2 \;\; \forall x, y \in X.$$

(iv) $(\gamma)-$relaxed cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq (-\gamma)\|T(x) - T(y)\|^2 \;\forall x, y \in X.$$

(v) $(r)-$strongly monotone (with respect to A) if there is a positive constant r such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq r\|x - y\|^2 \;\; \forall x, y \in X.$$

(vi) $(m)-$relaxed monotone (with respect to A) if there is a positive constant m such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq -m\|x - y\|^2 \;\; \forall x, y \in X.$$

(vii) (γ, r)-relaxed cocoercive (with respect to A) if there exist constants $\gamma, r > 0$ such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq (-\gamma)\|T(x) - T(y)\|^2 + r\|x - y\|^2 \;\; \forall x, y \in X.$$

(viii) $(\gamma)-$relaxed cocoercive (with respect to A) if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq (-\gamma)\|T(x) - T(y)\|^2 \; \forall x, y \in X.$$

Definition 1.3. Let $T, A : X \to X$ be any mappings on X. The map T is said to be:

(i) $(\gamma)-$cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq \gamma\|T(x) - T(y)\|^2 \; \forall \; x, y \in X.$$

(ii) $(\gamma)-$cocoercive (with respect to A) if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq \gamma\|T(x) - T(y)\|^2 \; \forall \; x, y \in X.$$

Definition 1.4.([8]) Let $T, A : X \to X$ be any mappings on X. The map T is said to be:

(i) $(a, b, c)-$cocoercively monotone if there exist positive constants $a, b, c > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq a\|T(x) - T(y)\|^2 - b\|x - y\|^2 + c\|x - y\|^2 \;\; \forall \, x, y \in X.$$

(ii) $(a, b, c)-$cocoercively monotone (with respect to A) if there exist positive constants $a, b, c > 0$ such that

$$\langle T(x)-T(y), A(x)-A(y)\rangle \geq a\|T(x)-T(y)\|^2 - b\|x-y\|^2 + c\|x-y\|^2 \;\; \forall \, x, y \in X.$$

Lemma 1.1.([1]) *Let $H : X \to X$ be $(r)-$strongly monotone and $M : X \to 2^X$ be H-monotone. Then the resolvent operator $J^{\rho}_{H,M} : X \to X$ is $(\frac{1}{r})-$Lipschitz continuous for $r > 0$.*

2. Nonlinear variational inclusion system

Next, let X_1 and X_2 be two real Hilbert spaces. Let $M : X_1 \to 2^{X_1}$ and $N : X_2 \to 2^{X_2}$ be nonlinear mappings. Let $S : X_1 \times X_2 \to X_1$ and $T : X_1 \times X_2 \to X_2$ be any two mappings. Then the problem of finding $(a, b) \in X_1 \times X_2$ such that

$$0 \in S(a, b) + M(a), \tag{1}$$

$$0 \in T(a, b) + N(b), \tag{2}$$

is called the system of nonlinear variational inclusion (abbreviated SNVI) problems.

Lemma 2.1.([1]) *Let X_1 and X_2 be two real Hilbert spaces. Let $H_1 : X_1 \to X_1$ and $H_2 : X_2 \to X_2$ be strictly monotone, $M : X_1 \to 2^{X_1}$ be H_1-monotone and $N : X_2 \to 2^{X_2}$ be H_2-monotone. Let $S : X_1 \times X_2 \to X_1$ and $T : X_1 \times X_2 \to X_2$ be any two mappings. Then a given element $(a, b) \in X_1 \times X_2$ is a solution to the $SNVI\,(1)-(2)$ problem iff (a, b) satisfies*

$$a = J^{\rho}_{H_1,M}(H_1(a) - \rho S(a, b)), \tag{3}$$

$$b = J^{\eta}_{H_2,N}(H_2(b) - \eta T(a, b)), \tag{4}$$

where $\rho, \eta > 0$.

Theorem 2.1. *Let X_1 and X_2 be two real Hilbert spaces. Let $H_1 : X_1 \to X_1$ be $(r_1)-$ strongly monotone and $(\alpha_1)-$ Lipschitz continuous, and $H_2 : X_2 \to X_2$ be $(r_2)-$ strongly monotone and $(\alpha_2)-$ Lipschitz continuous. Let $M : X_1 \to 2^{X_1}$ be H_1-monotone and $N : X_2 \to 2^{X_2}$ be H_2-monotone. Let $S : X_1 \times X_2 \to X_1$ be such that $S(., y)$ is $(a_1, b_1, c_1)-$cocoercively monotone (with respect to H_1) and $(\mu)-$ Lipschitz continuous in the first variable, and $S(x, .)$ is $(\nu)-$ Lipschitz continuous in the second variable for all $(x, y) \in X_1 \times X_2$. Let $T : X_1 \times X_2 \to X_2$ be such that $T(u, .)$ is $(a_2, b_2, c_2)-$ cocoercively monotone (with respect to H_2) and $(\beta)-$ Lipschitz continuous in the second variable, and $T(., v)$ is $(\tau)-$ Lipschitz continuous in the first variable for all $(u, v) \in X_1 \times X_2$. If, in addition, there exist positive constants ρ and η such that*

$$r_2\sqrt{\alpha_1^2 + 2\rho(b_1 - c_1) + (\rho^2 - 2\rho a_1)\mu^2} + r_1\eta\tau < r_1 r_2,$$

$$r_1\sqrt{\alpha_2^2 + 2\rho(b_2 - c_2) + (\eta^2 - 2\eta a_2)\beta^2} + r_2\rho\nu < r_1 r_2,$$

where $\rho^2 - 2\rho a_1 > 0$ and $\eta^2 - 2\rho a_2 > 0$, then the $SNVI\,(1)-(2)$ problem has a unique solution.

Proof. Let us define mappings $S^*(a, b)$ and $T^*(a, b)$, respectively, by

$$S^*(a, b) = J^{\rho}_{H_1,M}[H_1(a) - \rho S(a, b)]$$

and

$$T^*(a, b) = J^{\eta}_{H_2,N}[H_2(b) - \eta T(a, b)].$$

Then for any elements $(u, v), (w, x) \in X_1 \times X_2$, we have from *Lemma* 1.1 that

$$\begin{aligned}\|S^*(u, v) - S^*(w, x)\| \quad &\le \quad \frac{1}{r_1}\|H_1(u) - H_1(w) - \rho[S(u, v) - S(w, v)]\| \\ &\quad + \frac{\rho}{r_1}\|S(w, v) - S(w, x)\|\end{aligned}$$

Next, we estimate

$$\begin{aligned}
& \|H_1(u) - H_1(w) - \rho[S(u,v) - S(w,v)]\|^2 \\
= \; & \|H_1(u) - H_1(w)\|^2 - 2\rho\langle H_1(u) - H_1(w), S(u,v) - S(w,v)\rangle \\
& +\rho^2\|S(u,v) - S(w,v)]\|^2 \\
\leq \; & \|H_1(u) - H_1(w)\|^2 + \rho^2\|S(u,v) - S(w,v)]\|^2 \\
& -2\rho a_1\|S(u,v) - S(w,v)\|^2 + 2\rho(b_1 - c_1)\|u - w\|^2 \\
\leq \; & \alpha_1^2\|u - w\|^2 + (\rho^2 - 2\rho a_1)\|S(u,v) - S(w,v)\|^2 + 2\rho(b_1 - c_1)\|u - w\|^2 \\
= \; & [\alpha_1^2 + 2\rho(b_1 - c_1) + (\rho^2 - 2\rho a_1)\mu^2]\|u - w\|^2,
\end{aligned}$$

where $\rho^2 - 2\rho a_1 > 0$. Thus, we have

$$\begin{aligned}
\|S^*(u,v) - S^*(w,x)\| \; \leq \; & \frac{1}{r_1}\sqrt{\alpha_1^2 + 2\rho(b_1 - c_1) + (\rho^2 - 2\rho a_1)\mu^2}\|u - w\| \\
& +\frac{\rho\nu}{r_1}\|v - x\|.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\|T^*(u,v) - T^*(w,x)\| \; \leq \; & \frac{1}{r_2}\|H_2(v) - H_2(x) - \eta[T(u,v) - T(u,x)]\| \\
& +\frac{\eta}{r_2}\|T(u,x) - T(w,x)\| \\
\leq \; & \frac{1}{r_2}\sqrt{(\alpha_2^2 + 2\rho(b_2 - c_2) + (\eta^2 - 2\eta a_2)\beta^2)}\|v - x\| \\
& +\frac{\eta\tau}{r_2}\|u - w\|.
\end{aligned}$$

It follows from the above arguments that

$$\begin{aligned}
& \|S^*(u,v) - S^*(w,x)\| + \|T^*(u,v) - T^*(w,x)\| \\
\leq \; & \frac{1}{r_1}[\sqrt{\alpha_1^2 + 2\rho(b_1 - c_1) + (\rho^2 - 2\rho a_1)\mu^2}\|u - w\| + \frac{\rho\nu}{r_1}\|v - x\|] \\
& +\frac{1}{r_2}[\sqrt{\alpha_2^2 + 2\rho(b_2 - c_2) + (\eta^2 - 2\eta a_2)\beta^2}\|v - x\| + \frac{\eta\tau}{r_2}\|u - w\|] \\
\leq \; & \max\{\frac{1}{r_1}\sqrt{\alpha_1^2 - 2\rho r + \rho^2\mu^2 + 2\rho\gamma\mu^2} + \frac{\eta\tau}{r_2}, \\
& \frac{1}{r_2}\sqrt{\alpha_2^2 - 2\eta s + \eta^2\beta^2 + 2\eta\lambda\beta^2} + \frac{\rho\nu}{r_1}\} \cdot (\|u - w\| + \|v - x\|).
\end{aligned}$$

Set

$$\begin{aligned}
k \; = \; & max\{\frac{1}{r_1}\sqrt{\alpha_1^2 + 2\rho(b_1 - c_1) + (\rho^2 - 2\rho a_1)\mu^2} + \frac{\eta\tau}{r_2}, \\
& \frac{1}{r_2}\sqrt{\alpha_2^2 - 2\eta s + \eta^2\beta^2 + 2\eta\lambda\beta^2} + \frac{\rho\nu}{r_1}\}.
\end{aligned}$$

Next, we define the norm $\|(u,v)\|^*$ by

$$\|(u,v)\|^* = (\|u\| + \|v\|\ )\,\forall (u,v) \in X_1 \times X_2.$$

Clearly, $X_1 \times X_2$ is a Banach space with the norm $\|(u,v)\|^*$. We define a mapping $U : X_1 \times X_2 \to X_1 \times X_2$ by

$$U(u,v) = (S^*(u,v), T^*(u,v))\forall\,(u,v) \in X_1 \times X_2.$$

Since $0 < k < 1$, it follows that

$$\|U(u,v) - U(w,x)\|^* \leq k\|(u,v) - (w,x)\|^*.$$

Hence, U is a contraction. This implies that there exists a unique element $(a,b) \in X_1 \times X_2$ such that

$$U(a,b) = (a,b),$$

which means,

$$a = J^{\rho}_{H_1,M}(H_1(a) - \rho S(a,b)),$$
$$b = J^{\eta}_{H_2,N}(H_2(b) - \eta T(a,b)),$$

where $\rho, \eta > 0$. □

For $b_1 = c_1$ and $b_2 = c_2$ in Theorem 2.1, we have

Theorem 2.2. *Let X_1 and X_2 be two real Hilbert spaces. Let $H_1 : X_1 \to X_1$ be $(r_1)-$ strongly monotone and $(\alpha_1)-$ Lipschitz continuous, and $H_2 : X_2 \to X_2$ be $(r_2)-$ strongly monotone and $(\alpha_2)-$ Lipschitz continuous. Let $M : X_1 \to 2^{X_1}$ be H_1-monotone and $N : X_2 \to 2^{X_2}$ be H_2-monotone. Let $S : X_1 \times X_2 \to X_1$ be such that $S(.,y)$ is $(a_1)-$cocoercive (with respect to H_1) and $(\mu)-$ Lipschitz continuous in the first variable, and $S(x,.)$ is $(\nu)-$ Lipschitz continuous in the second variable for all $(x,y) \in X_1 \times X_2$. Let $T : X_1 \times X_2 \to X_2$ be such that $T(u,.)$ is $(a_2)-$ cocoercive (with respect to H_2) and $(\beta)-$ Lipschitz continuous in the second variable, and $T(.,v)$ is $(\tau)-$Lipschitz continuous in the first variable for all $(u,v) \in X_1 \times X_2$. If, in addition, there exist positive constants ρ and η such that*

$$r_2\sqrt{\alpha_1^2 + (\rho^2 - 2\rho a_1)\mu^2} + r_1\eta\tau < r_1 r_2,$$
$$r_1\sqrt{\alpha_2^2 + (\eta^2 - 2\eta a_2)\beta^2} + r_2\rho\nu < r_1 r_2,$$

where $\rho^2 - 2\rho a_1 > 0$ and $\eta^2 - 2\rho a_2 > 0$, then the $SNVI\,(1)-(2)$ problem has a unique solution.

For $a_1 = a_2 = b_1 = b_2 = 0$ in Theorem 1, we have

Theorem 2.3.([2]) *Let X_1 and X_2 be two real Hilbert spaces. Let $H_1 : X_1 \to X_1$ be $(r_1)-$ strongly monotone and $(\alpha_1)-$ Lipschitz continuous, and $H_2 : X_2 \to X_2$ be $(r_2)-$ strongly monotone and $(\alpha_2)-$ Lipschitz continuous. Let*

$M : X_1 \to 2^{X_1}$ be H_1−monotone and $N : X_2 \to 2^{X_2}$ be H_2−monotone. Let $S : X_1 \times X_2 \to X_1$ be such that $S(.,y)$ is (c_1)−strongly monotone (with respect to H_1) and $(\mu) -$ Lipschitz continuous in the first variable, and $S(x,.)$ is $(\nu) -$ Lipschitz continuous in the second variable for all $(x,y) \in X_1 \times X_2$. Let $T : X_1 \times X_2 \to X_2$ be such that $T(u,.)$ is (c_2)−strongly monotone (with respect to H_2) and $(\beta) -$ Lipschitz continuous in the second variable, and $T(.,v)$ is $(\tau) -$ Lipschitz continuous in the first variable for all $(u,v) \in X_1 \times X_2$. If, in addition, there exist positive constants ρ and η such that

$$r_2\sqrt{\alpha_1^2 - 2\rho c_1 + \rho^2\mu^2} + r_1\eta\tau < r_1 r_2,$$

$$r_1\sqrt{\alpha_2^2 - 2\rho c_2 + \eta^2\beta^2} + r_2\rho\nu < r_1 r_2,$$

then the $SNVI\,(1) - (2)$ problem has a unique solution.

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